

Algebraic Number Theory

(PARI-GP version 2.10.0)

Binary Quadratic Forms

create $ax^2 + bxy + cy^2$ (distance d)	<code>Qfb($a, b, c, \{d\}$)</code>
reduce x ($s = \sqrt{D}$, $l = \lfloor s \rfloor$)	<code>qfbred($x, \{flag\}, \{D\}, \{l\}, \{s\}$)</code>
return $[y, g]$, $g \in \text{SL}_2(\mathbf{Z})$, $y = g \cdot x$ reduced	<code>qfbreds12(x)</code>
composition of forms	$x*y$ or <code>qfbnucomp(x, y, l)</code>
n -th power of form	x^n or <code>qfbnupow(x, n)</code>
composition without reduction	<code>qfbcomprow(x, y)</code>
n -th power without reduction	<code>qfbpowrow(x, n)</code>
prime form of disc. x above prime p	<code>qfbprimeform(x, p)</code>
class number of disc. x	<code>qfbclassno(x)</code>
Hurwitz class number of disc. x	<code>qfbhclassno(x)</code>
Solve $Q(x, y) = p$ in integers, p prime	<code>qfbsolve(Q, p)</code>

Quadratic Fields

quadratic number $\omega = \sqrt{x}$ or $(1 + \sqrt{x})/2$	<code>quadgen(x)</code>
minimal polynomial of ω	<code>quadpoly(x)</code>
discriminant of $\mathbf{Q}(\sqrt{D})$	<code>quaddisc(x)</code>
regulator of real quadratic field	<code>quadregulator(x)</code>
fundamental unit in real $\mathbf{Q}(x)$	<code>quadunit(x)</code>
class group of $\mathbf{Q}(\sqrt{D})$	<code>quadclassunit($D, \{flag\}, \{t\}$)</code>
Hilbert class field of $\mathbf{Q}(\sqrt{D})$	<code>quadhilbert($D, \{flag\}$)</code>
... using specific class invariant ($D < 0$)	<code>polclass($D, \{inv\}$)</code>
ray class field modulo f of $\mathbf{Q}(\sqrt{D})$	<code>quadrays($D, f, \{flag\}$)</code>

General Number Fields: Initializations

The number field $K = \mathbf{Q}[X]/(f)$ is given by irreducible $f \in \mathbf{Q}[X]$. A nf computes a maximal order and allows operations on elements and ideals. A bnf adds class group and units. A bnr is attached to ray class groups and class field theory. A rnf is attached to relative extensions L/K .

init number field structure nf	<code>nfinit($f, \{flag\}$)</code>
known integer basis B	<code>nfinit([f, B])</code>
order maximal at $vp = [p_1, \dots, p_k]$	<code>nfinit([f, vp])</code>
order maximal at all $p \leq P$	<code>nfinit([f, P])</code>
certify maximal order	<code>nfcertify(nf)</code>

nf members:

a monic $F \in \mathbf{Z}[X]$ defining K	<code>nf.pol</code>
number of real/complex places	<code>nf.r1/r2/sign</code>
discriminant of nf	<code>nf.disc</code>
T_2 matrix	<code>nf.t2</code>
complex roots of F	<code>nf.roots</code>
integral basis of \mathbf{Z}_K as powers of θ	<code>nf.zk</code>
different/codifferent	<code>nf.diff, nf.codiff</code>
index $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$	<code>nf.index</code>
recompute nf using current precision	<code>nfnewprec(nf)</code>
init relative rnf $L = K[Y]/(g)$	<code>rnfinit(nf, g)</code>
init bnf structure	<code>bnfinit($f, \{flag\}$)</code>

bnf members: same as nf , plus

underlying nf	<code>bnf.nf</code>
classgroup	<code>bnf.clgp</code>
regulator	<code>bnf.reg</code>
fundamental/torsion units	<code>bnf.fu, bnf.tu</code>
compress a bnf for storage	<code>bnfcompress(bnf)</code>
recover a bnf from compressed $bnfz$	<code>bnfinit($bnfz$)</code>
add S -class group and units, yield $bnfS$	<code>bnfsunit(bnf, S)</code>
init class field structure bnr	<code>bnrinit($bnf, m, \{flag\}$)</code>

bnr members: same as bnf , plus

underlying bnf	<code>bnr.bnf</code>
big ideal structure	<code>bnr.bid</code>
modulus	<code>bnr.mod</code>
structure of $(\mathbf{Z}_K/m)^*$	<code>bnr.zkst</code>

Basic Number Field Arithmetic (nf)

Elements are <code>t_INT</code> , <code>t_FRAC</code> , <code>t_POL</code> , <code>t_POLMOD</code> , or <code>t_COL</code> (on integral basis $nf.zk$). Basic operations (prefix <code>nfelt</code>):	<code>(nfelt)add, mul, pow, div, diveuc, mod, divrem, val, trace, norm</code>
express x on integer basis	<code>nfalgtobasis(nf, x)</code>
express element x as a polmod	<code>nfbasistoalg(nf, x)</code>
complex embeddings of <code>t_POLMOD</code> x	<code>conjvec(x)</code>
reverse polmod $a = A(X) \bmod T(X)$	<code>modreverse(a)</code>
integral basis of field def. by $f = 0$	<code>nfbasis(f)</code>
field discriminant of field $f = 0$	<code>nfdisc(f)</code>
smallest poly defining $f = 0$ (slow)	<code>polredabs($f, \{flag\}$)</code>
small poly defining $f = 0$ (fast)	<code>polredbest($f, \{flag\}$)</code>
random Tschirnhausen transform of f	<code>poltschirnhaus(f)</code>
$\mathbf{Q}[x]/(f) \subset \mathbf{Q}[x]/(g)$? Isomorphic?	<code>nfisincl(f, g), nfisisom</code>
compositum of $\mathbf{Q}[X]/(f)$, $\mathbf{Q}[X]/(g)$	<code>polcompositum($f, g, \{flag\}$)</code>
compositum of $K[X]/(f)$, $K[X]/(g)$	<code>nfcompositum($nf, f, g, \{flag\}$)</code>
splitting field of K (degree divides d)	<code>nfsplitting($nf, \{d\}$)</code>
subfields (of degree d) of nf	<code>nfsubfields($nf, \{d\}$)</code>
d -th degree subfield of $\mathbf{Q}(\zeta_n)$	<code>polsubcyclo($n, d, \{v\}$)</code>
roots of unity in nf	<code>nfrootsof1(nf)</code>
roots of g belonging to nf	<code>nfroots($\{nf\}, g$)</code>
factor g in nf	<code>nfactor(nf, g)</code>
factor g mod prime pr in nf	<code>nfactormod(nf, g, pr)</code>
conjugates of a root θ of nf	<code>nfgaloisconj($nf, \{flag\}$)</code>
apply Galois automorphism s to x	<code>nfgaloisapply(nf, s, x)</code>
quadratic Hilbert symbol (at p)	<code>nfhilbert($nf, a, b, \{p\}$)</code>

Linear and algebraic relations

poly of degree $\leq k$ with root $x \in \mathbf{C}$	<code>algdep(x, k)</code>
alg. dep. with pol. coeffs for series s	<code>seralgdep(s, x, y)</code>
small linear rel. on coords of vector x	<code>lindexp(x)</code>

Dedekind Zeta Function ζ_K , Hecke L series

$R = [c, w, h]$ in initialization means we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w$, $|\Im(s)| < h$; $R = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $R = [1/2, 0, h]$ (critical line up to height h).

ζ_K as Dirichlet series, $N(I) < b$	<code>dirzetak(nf, b)</code>
init $\zeta_K^{(k)}(s)$ for $k \leq n$	<code>L = lfuninit($bnf, R, \{n = 0\}$)</code>
compute $\zeta_K(s)$ (n -th derivative)	<code>lfun($L, s, \{n = 0\}$)</code>
compute $\Lambda_K(s)$ (n -th derivative)	<code>lfunlambda($L, s, \{n = 0\}$)</code>

init $L_K^{(k)}(s, \chi)$ for $k \leq n$	<code>L = lfuninit([bnr, chi], $R, \{n = 0\}$)</code>
compute $L_K(s, \chi)$ (n -th derivative)	<code>lfun($L, s, \{n\}$)</code>
Artin root number of K	<code>bnrrootnumber($bnr, chi, \{flag\}$)</code>
$L(1, \chi)$, for all χ trivial on H	<code>bnrL1($bnr, \{H\}, \{flag\}$)</code>

Class Groups & Units (bnf, bnr)

Class field theory data $a_1, \{a_2\}$ is usually bnr (ray class field), bnr, H (congruence subgroup) or bnr, χ (character on <code>bnr.clgp</code>).	
Any of these define a unique abelian extension of K .	
remove GRH assumption from bnf	<code>bnfcertify(bnf)</code>
expo. of ideal x on class gp	<code>bnfisprincipal($bnf, x, \{flag\}$)</code>
expo. of ideal x on ray class gp	<code>bnrisprincipal($bnr, x, \{flag\}$)</code>
expo. of x on fund. units	<code>bnfisunit(bnf, x)</code>
as above for S -units	<code>bnfissunit($bnfs, x$)</code>

signs of real embeddings of $bnf.fu$	<code>bnfsignunit(bnf)</code>
narrow class group	<code>bnfnarrow(bnf)</code>
Class Field Theory	
ray class number for modulus m	<code>bnrclassno(bnf, m)</code>
discriminant of class field	<code>bnrdisc($a_1, \{a_2\}$)</code>
ray class numbers, l list of moduli	<code>bnrclassolist(bnf, l)</code>
discriminants of class fields	<code>bnrdisclist($bnf, l, \{arch\}, \{flag\}$)</code>
decode output from <code>bnrdisclist</code>	<code>bnfdecodemodule(nf, fa)</code>
is modulus the conductor?	<code>bnrisconductor($a_1, \{a_2\}$)</code>
is class field (bnr, H) Galois over K^G	<code>bnrisgalois(bnr, G, H)</code>
action of automorphism on <code>bnr.gen</code>	<code>bnrgaloismatrix(bnr, aut)</code>
apply <code>bnrgaloismatrix</code> M to H	<code>bnrgaloisapply(bnr, M, H)</code>
characters on <code>bnr.clgp</code> s.t. $\chi(g_i) = e(v_i)$	<code>bnrchar($bnr, g, \{v\}$)</code>
conductor of character χ	<code>bnrconductor(bnr, chi)</code>
conductor of extension	<code>bnrconductor($a_1, \{a_2\}, \{flag\}$)</code>
conductor of extension $K[Y]/(g)$	<code>rnfconductor(bnf, g)</code>
Artin group of extension $K[Y]/(g)$	<code>rnfnormgroup(bnr, g)</code>
subgroups of bnr , index $\leq b$	<code>subgrouplist($bnr, b, \{flag\}$)</code>
rel. eq. for class field def'd by sub	<code>rnfkummer($bnr, sub, \{d\}$)</code>
same, using Stark units (real field)	<code>bnrstark($bnr, sub, \{flag\}$)</code>
is a an n -th power in K_v ?	<code>nfislocalpower(nf, v, a, n)</code>
cyclic L/K satisf. local conditions	<code>nfgrunwaldwang(nf, P, D, pl)</code>

Logarithmic class group

logarithmic ℓ -class group	<code>bnflog(bnf, ℓ)</code>
$[\tilde{e}(F_v/Q_p), \tilde{f}(F_v/Q_p)]$	<code>bnflogef(bnf, pr)</code>
$\exp \deg_F(A)$	<code>bnflogdegree(bnf, A, ℓ)</code>
is ℓ -extension L/K locally cyclotomic	<code>rnfislocalcyclo(rmf)</code>

Ideals: elements, primes, or matrix of generators in HNF

is id an ideal in nf ?	<code>nfisideal(nf, id)</code>
is x principal in bnf ?	<code>bnfisprincipal(bnf, x)</code>
give $[a, b]$, s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$	<code>idealtwoelt($nf, x, \{a\}$)</code>
put ideal a ($a\mathbf{Z}_K + b\mathbf{Z}_K$) in HNF form	<code>idealhnf($nf, a, \{b\}$)</code>
norm of ideal x	<code>idealnrm(nf, x)</code>
minimum of ideal x (direction v)	<code>idealmin(nf, x, v)</code>
LLL-reduce the ideal x (direction v)	<code>idealred($nf, x, \{v\}$)</code>

Ideal Operations

add ideals x and y	<code>idealadd(nf, x, y)</code>
multiply ideals x and y	<code>idealmul($nf, x, y, \{flag\}$)</code>
intersection of ideals x and y	<code>idealintersect($nf, x, y, \{flag\}$)</code>
n -th power of ideal x	<code>idealpow($nf, x, n, \{flag\}$)</code>
inverse of ideal x	<code>idealinv(nf, x)</code>
divide ideal x by y	<code>idealdiv($nf, x, y, \{flag\}$)</code>
Find $(a, b) \in x \times y$, $a + b = 1$	<code>idealaddtoone($nf, x, \{y\}$)</code>
coprime integral A, B such that $x = A/B$	<code>idealnumden(nf, x)</code>

Primes and Multiplicative Structure

factor ideal x in \mathbf{Z}_K	<code>idealfactor(nf, x)</code>
expand ideal factorization in K	<code>idealfactorback($nf, f, \{e\}$)</code>
expand elt factorisation in K	<code>nffactorback($nf, f, \{e\}$)</code>
decomposition of prime p in \mathbf{Z}_K	<code>idealprimedec(nf, p)</code>
valuation of x at prime ideal pr	<code>idealval(nf, x, pr)</code>
weak approximation theorem in nf	<code>idealchinese(nf, x, y)</code>
$a \in K$, s.t. $v_p(a) = v_p(x)$ if $v_p(x) \neq 0$	<code>idealappr(nf, x)</code>
$a \in K$ such that $(a \cdot x, y) = 1$	<code>idealcoprime(nf, x, y)</code>
give bid = structure of $(\mathbf{Z}_K/id)^*$	<code>idealstar($nf, id, \{flag\}$)</code>
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$	<code>idealprincipalunits(nf, pr, k)</code>
discrete log of x in $(\mathbf{Z}_K/bid)^*$	<code>ideallog(nf, x, bid)</code>

Algebraic Number Theory

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idealstar of all ideals of norm $\leq b$ **ideallist**($nf, b, \{flag\}$)
 add Archimedean places **ideallistarch**($nf, b, \{ar\}, \{flag\}$)
 init **modpr** structure **nfmodprinit**(nf, pr)
 project t to \mathbf{Z}_K/pr **nfmodpr**($nf, t, modpr$)
 lift from \mathbf{Z}_K/pr **nfmodprlift**($nf, t, modpr$)

Galois theory over \mathbf{Q}

Galois group of field $\mathbf{Q}[x]/(f)$ **polgalois**(f)
 initializes a Galois group structure G **galoisinit**($pol, \{den\}$)
 action of p in nfgaloisconj form **galoispermopol**($G, \{p\}$)
 identify as abstract group **galoisidentify**(G)
 export a group for GAP/MAGMA **galoisexport**($G, \{flag\}$)
 subgroups of the Galois group G **galoissubgroups**(G)
 is subgroup H normal? **galoisisnormal**(G, H)
 subfields from subgroups **galoissubfields**($G, \{flag\}, \{v\}$)
 fixed field **galoisfixedfield**($G, perm, \{flag\}, \{v\}$)
 Frobenius at maximal ideal P **idealfrobenius**(nf, G, P)
 ramification groups at P **idealramgroups**(nf, G, P)
 is G abelian? **galoisisabelian**($G, \{flag\}$)
 abelian number fields/ \mathbf{Q} **galoissubcyclo**($\mathbf{N}, H, \{flag\}, \{v\}$)
 query the **galpol** package **galoisgetpol**($a, b, \{s\}$)

Relative Number Fields (rnf)

Extension L/K is defined by $T \in K[x]$.
 absolute equation of L **rnfequation**($nf, T, \{flag\}$)
 is L/K abelian? **rnfisabelian**(nf, T)
 relative **nfaltgbasis** **rnfaltgbasis**(rnf, x)
 relative **nfbasistoalg** **rnfbasistoalg**(rnf, x)
 relative **idealhnf** **rnfidealhnf**(rnf, x)
 relative **idealmul** **rnfidealmul**(rnf, x, y)
 relative **idealtwoelt** **rnfidealtwoelt**(rnf, x)

Lifts and Push-downs

absolute \rightarrow relative repres. for x **rnfeltabstorel**(rnf, x)
 relative \rightarrow absolute repres. for x **rnfeltreltoabs**(rnf, x)
 lift x to the relative field **rnfeltup**(rnf, x)
 push x down to the base field **rnfeltdown**(rnf, x)
 idem for x ideal: (**rnfideal**)**reltoabs**, **abstorel**, **up**, **down**

Norms and Trace

relative norm of element $x \in L$ **rnfeltnorm**(rnf, x)
 relative trace of element $x \in L$ **rnfelttrace**(rnf, x)
 absolute norm of ideal x **rnfidealnrmabs**(rnf, x)
 relative norm of ideal x **rnfidealnrmrel**(rnf, x)
 solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$ **bnfisintnorm**(bnf, x)
 is $x \in \mathbf{Q}$ a norm from K ? **bnfisnorm**($bnf, x, \{flag\}$)
 initialize T for norm eq. solver **rnfisnorminit**($K, pol, \{flag\}$)
 is $a \in K$ a norm from L ? **rnfisnorm**($T, a, \{flag\}$)
 initialize t for Thue equation solver **thueinit**(f)
 solve Thue equation $f(x, y) = a$ **thue**($t, a, \{sol\}$)
 characteristic poly. of $a \bmod T$ **rnfcharpoly**($nf, T, a, \{v\}$)

Factorization

factor ideal x in L **rnfidealfactor**(rnf, x)
 $[S, T]: T_{i,j} \mid S_i$; S primes of K above p **rnfidealprimedec**(rnf, p)

Maximal order \mathbf{Z}_L as a \mathbf{Z}_K -module

relative **polredbest** **rnfpolredbest**(nf, T)
 relative Dedekind criterion, prime pr **rnfdedekind**(nf, T, pr)
 discriminant of relative extension **rnfdisc**(nf, T)
 pseudo-basis of \mathbf{Z}_L **rnfpsedobasis**(nf, T)
General \mathbf{Z}_K -modules: $M = [\text{matrix}, \text{vec. of ideals}] \subset L$
 relative HNF / SNF **nfhnf**(nf, M), **nfsnf**
 multiple of det M **nfdetint**(nf, M)
 HNF of M where $d = \text{nfdetint}(M)$ **nfhnfmod**(x, d)
 reduced basis for M **rnflllgram**(nf, T, M)
 determinant of pseudo-matrix M **rnfdet**(nf, M)
 Steinitz class of M **rnfsteinitz**(nf, M)
 \mathbf{Z}_K -basis of M if \mathbf{Z}_K -free, or 0 **rnfhnfbasis**(bnf, M)
 n -basis of M , or $(n+1)$ -generating set **rnfbasis**(bnf, M)
 is M a free \mathbf{Z}_K -module? **rnfisfree**(bnf, M)

Associative Algebras

A is a general associative algebra given by a mult. table mt (over \mathbf{Q} or \mathbf{F}_p); represented by al from **algtbleinit**.
 create al from mt (over \mathbf{F}_p) **algtbleinit**($mt, \{p = 0\}$)
 group algebra $\mathbf{Q}[G]$ (or $\mathbf{F}_p[G]$) **alggroup**($G, \{p = 0\}$)
Properties
 is (mt, p) OK for algtbleinit? **algisassociative**($mt, \{p = 0\}$)
 multiplication table mt **algmtable**(al)
 multiplication table over center **algrelmtable**(al)
 dimension of A over prime subfield **algabsdim**(al)
 characteristic of A **algchar**(al)
 is A commutative? **algiscommutative**(al)
 is A simple? **algissimple**(al)
 is A semi-simple? **algissemisimple**(al)
 is A ramified? (at place v) **algisramified**($al, \{v\}$)
 is A split? (at place v) **algissplit**($al, \{v\}$)
 center of A **algcenter**(al)
 Jacobson radical of A **algradical**(al)
 radical J and simple factors of A/J **algdecomposition**(al)
 simple factors of semi-simple A **algsimpledec**(al)

Operations on algebras

create A/I , I two-sided ideal **algquotient**($al, I, \{flag = 0\}$)
 create $A_1 \otimes A_2$ **algtensor**($al1, al2$)
 create subalgebra from basis B **algsubalg**(al, B)
 \dots from orthogonal central idempotents e **algcentralproj**(al, e)
 prime subalgebra of semi-simple A over \mathbf{F}_p **algprimesubalg**(al)
 lattice generated by cols. of M **alglathnf**(al, M)

Operations on elements

$a + b, a - b, -a$ **algadd**(al, a, b), **algsub**, **algneg**
 $a \times b, a \times a$ **algmul**(al, a, a), **algsqr**
 a^n, a^{-1} **algpow**(al, a, n), **alginv**
 is x invertible? (then set $z = x^{-1}$) **algisinv**($al, x, \{\&z\}$)
 find z such that $x \times z = y$ **algdivl**(al, x, y)
 find z such that $z \times x = y$ **algdivr**(al, x, y)
 does z s.t. $x \times z = y$ exist? (set it) **algisdivl**($al, x, y, \{\&z\}$)
 matrix of $v \mapsto x \cdot v$ **algleftmtable**(al, x)
 absolute norm **algnorm**(al, x)
 absolute trace **algtrace**(al, x)
 absolute char. polynomial **algcharpoly**(al, x)
 given $a \in A$ and polynomial T , return $T(a)$ **algpoleval**(al, T, a)
 random element in a box **algrandom**(al, b)

Central Simple Algebras

A is a central simple algebra over a number field K ; represented by al from **alginitt**; K is given by a nf structure.
 create CSA from data **alginitt**($B, C, \{v\}, \{flag = 0\}$)
 multiplication table over K $B = K, C = mt$
 cyclic algebra $(L/K, \sigma, b)$ $B = rnf, C = [\text{sigma}, b]$
 quaternion algebra $(a, b)_K$ $B = K, C = [a, b]$
 matrix algebra $M_d(K)$ $B = K, C = d$
 local Hasse invariants over K $B = K, C = [d, [PR, HF], HI]$

Properties

type of al (mt , CSA) **algtype**(al)
 is al a division algebra? (at place v) **algisdivision**($al, \{v\}$)
 dimension of al over its center **algdim**(al)
 degree of A ($= \sqrt{\dim}$) **algdegree**(al)
 index of A over K (index at v) **algindex**($al, \{v\}$)
 al a cyclic algebra $(L/K, \sigma, b)$; return σ **algaut**(al)
 \dots return b **algb**(al)
 \dots return L/K , as an rnf **algsplittingfield**(al)
 split A over an extension of K **algsplittingdata**(al)
 splitting field of A as an rnf over center **algsplittingfield**(al)
 places of K at which A ramifies **algramifiedplaces**(al)
 Hasse invariants at finite places of K **alghassef**(al)
 Hasse invariants at infinite places of K **alghassei**(al)
 Hasse invariant at place v **alghasse**(al, v)

Operations on elements

reduced norm **algnorm**(al, x)
 reduced trace **algtrace**(al, x)
 reduced char. polynomial **algcharpoly**(al, x)
 express x on integral basis **algalgtobasis**(al, x)
 convert x to algebraic form **algbasistoalg**(al, x)
 map $x \in A$ to $M_d(L)$, L split. field **algsplittingmatrix**(al, x)

Orders

\mathbf{Z} -basis of order \mathcal{O}_0 **algbasis**(al)
 discriminant of order \mathcal{O}_0 **algdisc**(al)
 \mathbf{Z} -basis of natural order in terms \mathcal{O}_0 's basis **alginvbasis**(al)

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 Send comments and corrections to (Karim.Belabas@math.u-bordeaux.fr)