User's Guide

to

PARI / GP

(version 2.9.2)

The PARI Group

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Chapter 1:
Overview of the PARI system

1.1 Introduction.

PARI/GP is a specialized computer algebra system, primarily aimed at number theorists, but has been put to good use in many other different fields, from topology or numerical analysis to physics.

Although quite an amount of symbolic manipulation is possible, PARI does badly compared to systems like Axiom, Magma, Maple, Mathematica, Maxima, or Reduce on such tasks, e.g. multivariate polynomials, formal integration, etc. On the other hand, the three main advantages of the system are its speed, the possibility of using directly data types which are familiar to mathematicians, and its extensive algebraic number theory module (from the above-mentioned systems, only Magma provides similar features).

Non-mathematical strong points include the possibility to program either in high-level scripting languages or with the PARI library, a mature system (development started in the mid eighties) that was used to conduct and disseminate original mathematical research, while building a large user community, linked by helpful mailing lists and a tradition of great user support from the developers. And, of course, PARI/GP is Free Software, covered by the GNU General Public License, either version 2 of the License or (at your option) any later version.

PARI is used in three different ways:

1) as a library `libpari`, which can be called from an upper-level language application, for instance written in ANSI C or C++;

2) as a sophisticated programmable calculator, named `gp`, whose language `GP` contains most of the control instructions of a standard language like C;

3) the compiler `gp2c` translates GP code to C, and loads it into the `gp` interpreter. A typical script compiled by `gp2c` runs 3 to 10 times faster. The generated C code can be edited and optimized by hand. It may also be used as a tutorial to `libpari` programming.

The present Chapter 1 gives an overview of the PARI/GP system; `gp2c` is distributed separately and comes with its own manual. Chapter 2 describes the `GP` programming language and the `gp` calculator. Chapter 3 describes all routines available in the calculator. Programming in library mode is explained in Chapters 4 and 5 in a separate booklet: *User's Guide to the PARI library* (`libpari.dvi`.

A tutorial for `gp` is provided in the standard distribution: *A tutorial for PARI/GP* (`tutorial.dvi`) and you should read this first. You can then start over and read the more boring stuff which lies ahead. You can have a quick idea of what is available by looking at the `gp` reference card (`refcard.dvi` or `refcard.ps`). In case of need, you can refer to the complete function description in Chapter 3.

How to get the latest version. Everything can be found on PARI's home page:

`http://pari.math.u-bordeaux.fr/`.

From that point you may access all sources, some binaries, version information, the complete mailing list archives, frequently asked questions and various tips. All threaded and fully searchable.

How to report bugs. Bugs are submitted online to our Bug Tracking System, available from PARI's home page, or directly from the URL

`http://pari.math.u-bordeaux.fr/Bugs/`.

Further instructions can be found on that page.

1.2 Multiprecision kernels / Portability.

The PARI multiprecision kernel comes in three non exclusive flavors. See Appendix A for how to set up these on your system; various compilers are supported, but the GNU `gcc` compiler is the definite favorite.

A first version is written entirely in ANSI C, with a C++-compatible syntax, and should be portable without trouble to any 32 or 64-bit computer having no drastic memory constraints. We do not know any example of a computer where a port was attempted and failed.

In a second version, time-critical parts of the kernel are written in inlined assembler. At present this includes

- the whole ix86 family (Intel, AMD, Cyrix) starting at the 386, up to the Xbox gaming console, including the Opteron 64 bit processor.
- three versions for the Sparc architecture: version 7, version 8 with SuperSparc processors, and version 8 with MicroSparc I or II processors. UltraSparcs use the MicroSparc II version;
- the DEC Alpha 64-bit processor;
- the Intel Itanium 64-bit processor;
- the PowerPC equipping old macintoshs (G3, G4, etc.);
- the HPPA processors (both 32 and 64 bit);

A third version uses the GNU MP library to implement most of its multiprecision kernel. It improves significantly on the native one for large operands, say 100 decimal digits of accuracy or more. You *should* enable it if GMP is present on your system. Parts of the first version are still in use within the GMP kernel, but are scheduled to disappear.

A historical version of the PARI/GP kernel, written in 1985, was specific to 680x0 based computers, and was entirely written in MC68020 assembly language. It ran on SUN-3/xx, Sony News, NeXT cubes and on 680x0 based Macs. It is no longer part of the PARI distribution; to run PARI with a 68k assembler micro-kernel, use the GMP kernel!

1.3 The PARI types.

The GP language is not typed in the traditional sense; in particular, variables have no type. In library mode, the type of all PARI objects is `GEN`, a generic type. On the other hand, it is dynamically typed: each object has a specific internal type, depending on the mathematical object it represents.

The crucial word is recursiveness: most of the PARI types are recursive. For example, the basic internal type `t_COMPLEX` exists. However, the components (i.e. the real and imaginary part) of such a "complex number" can be of any type. The only sensible ones are integers (we are then in $\mathbf{Z}[i]$), rational numbers ($\mathbf{Q}[i]$), real numbers ($\mathbf{R}[i] = \mathbf{C}$), or even elements of $\mathbf{Z}/n\mathbf{Z}$ (in $(\mathbf{Z}/n\mathbf{Z})[t]/(t^2+1)$), or p-adic numbers when $p \equiv 3 \bmod 4$ ($\mathbf{Q}_p[i]$). This feature must not be used too rashly in library mode: for example you are in principle allowed to create objects which are "complex numbers of complex numbers". (This is not possible under `gp`.) But do not expect PARI to make sensible use of such objects: you will mainly get nonsense.

On the other hand, it *is* allowed to have components of different, but compatible, types, which can be freely mixed in basic ring operations $+$ or $\times$. For example, taking again complex numbers, the real part could be an integer, and the imaginary part a rational number. On the other hand, if the real part is a real number, the imaginary part cannot be an integer modulo n !

Let us now describe the types. As explained above, they are built recursively from basic types which are as follows. We use the letter T to designate any type; the symbolic names `t_xxx` correspond to the internal representations of the types.

type `t_INT`	$\mathbf{Z}$	Integers (with arbitrary precision)
type `t_REAL`	$\mathbf{R}$	Real numbers (with arbitrary precision)
type `t_INTMOD`	$\mathbf{Z}/n\mathbf{Z}$	Intmods (integers modulo n)
type `t_FRAC`	$\mathbf{Q}$	Rational numbers (in irreducible form)
type `t_FFELT`	$\mathbf{F}_q$	Finite field element
type `t_COMPLEX`	$T[i]$	Complex numbers
type `t_PADIC`	$\mathbf{Q}_p$	p-adic numbers
type `t_QUAD`	$\mathbf{Q}[w]$	Quadratic Numbers (where $[\mathbf{Z}[w] : \mathbf{Z}] = 2$)
type `t_POLMOD`	$T[X]/(P)$	Polmods (polynomials modulo $P \in T[X]$)
type `t_POL`	$T[X]$	Polynomials
type `t_SER`	$T((X))$	Power series (finite Laurent series)
type `t_RFRAC`	$T(X)$	Rational functions (in irreducible form)
type `t_VEC`	T^n	Row (i.e. horizontal) vectors
type `t_COL`	T^n	Column (i.e. vertical) vectors
type `t_MAT`	$\mathcal{M}_{m,n}(T)$	Matrices
type `t_LIST`	T^n	Lists
type `t_STR`		Character strings
type `t_CLOSURE`		Functions
type `t_ERROR`		Error messages
type `t_INFINITY`		$-\infty$ and $+\infty$

and where the types T in recursive types can be different in each component. The first nine basic types, from `t_INT` to `t_POLMOD`, are called scalar types because they essentially occur as coefficients of other more complicated objects. Type `t_POLMOD` is used to define algebraic extensions of a base ring, and as such is a scalar type.

In addition, there exist types `t_QFR` and `t_QFI` for integral binary quadratic forms, and the internal type `t_VECSMALL`. The latter holds vectors of small integers, whose absolute value is bounded

by 2^{31} (resp. 2^{63}) on 32-bit, resp. 64-bit, machines. They are used internally to represent permutations, polynomials or matrices over a small finite field, etc.

Every PARI object (called `GEN` in the sequel) belongs to one of these basic types. Let us have a closer look.

1.3.1 Integers and reals. They are of arbitrary and varying length (each number carrying in its internal representation its own length or precision) with the following mild restrictions (given for 32-bit machines, the restrictions for 64-bit machines being so weak as to be considered nonexistent): integers must be in absolute value less than $2^{536870815}$ (i.e. roughly 161614219 decimal digits). The precision of real numbers is also at most 161614219 significant decimal digits, and the binary exponent must be in absolute value less than 2^{29}, resp. 2^{61}, on 32-bit, resp. 64-bit machines.

Integers and real numbers are non-recursive types.

1.3.2 Intmods, rational numbers, p-adic numbers, polmods, and rational functions. These are recursive, but in a restricted way.

For intmods or polmods, there are two components: the modulus, which must be of type integer (resp. polynomial), and the representative number (resp. polynomial).

For rational numbers or rational functions, there are also only two components: the numerator and the denominator, which must both be of type integer (resp. polynomial).

Finally, p-adic numbers have three components: the prime p, the "modulus" p^k, and an approximation to the p-adic number. Here $\mathbf{Z}_p$ is considered as the projective limit $\varprojlim \mathbf{Z}/p^k\mathbf{Z}$ via its finite quotients, and $\mathbf{Q}_p$ as its field of fractions. Like real numbers, the codewords contain an exponent, giving the p-adic valuation of the number, and also the information on the precision of the number, which is redundant with p^k, but is included for the sake of efficiency.

1.3.3 Finite field elements. The exact internal format depends of the finite field size, but it includes the field characteristic p, an irreducible polynomial $T \in \mathbf{F}_p[X]$ defining the finite field $\mathbf{F}_p[X]/(T)$ and the element expressed as a polynomial in (the class of) X.

1.3.4 Complex numbers and quadratic numbers. Quadratic numbers are numbers of the form $a + bw$, where w is such that $[\mathbf{Z}[w] : \mathbf{Z}] = 2$, and more precisely $w = \sqrt{d}/2$ when $d \equiv 0 \bmod 4$, and $w = (1 + \sqrt{d})/2$ when $d \equiv 1 \bmod 4$, where d is the discriminant of a quadratic order. Complex numbers correspond to the important special case $w = \sqrt{-1}$.

Complex numbers are partially recursive: the two components a and b can be of type `t_INT`, `t_REAL`, `t_INTMOD`, `t_FRAC`, or `t_PADIC`, and can be mixed, subject to the limitations mentioned above. For example, $a+bi$ with a and b p-adic is in $\mathbf{Q}_p[i]$, but this is equal to $\mathbf{Q}_p$ when $p \equiv 1 \bmod 4$, hence we must exclude these p when one explicitly uses a complex p-adic type. Quadratic numbers are more restricted: their components may be as above, except that `t_REAL` is not allowed.

1.3.5 Polynomials, power series, vectors, matrices and lists. They are completely recursive: their components can be of any type, and types can be mixed (however beware when doing operations). Note in particular that a polynomial in two variables is simply a polynomial with polynomial coefficients.

In the present version 2.9.2 of PARI, it is not possible to handle conveniently power series of power series, i.e. power series in several variables. However power series of polynomials (which are power series in several variables of a special type) are OK. This is a difficult design problem: the mathematical problem itself contains some amount of imprecision, and it is not easy to design an intuitive generic interface for such beasts.

1.3.6 Strings. These contain objects just as they would be printed by the gp calculator.

1.3.7 Zero. What is zero? This is a crucial question in all computer systems. The answer we give in PARI is the following. For exact types, all zeros are equivalent and are exact, and thus are usually represented as an integer zero. The problem becomes non-trivial for imprecise types: there are infinitely many distinct zeros of each of these types! For p-adics and power series the answer is as follows: every such object, including 0, has an exponent e. This p-adic or X-adic zero is understood to be equal to $O(p^e)$ or $O(X^e)$ respectively.

Real numbers also have exponents and a real zero is in fact $O(2^e)$ where e is now usually a negative binary exponent. This of course is printed as usual for a floating point number ($0.00\cdots$ or $0.Exx$ depending on the output format) and not with a O symbol as with p-adics or power series. With respect to the natural ordering on the reals we make the following convention: whatever its exponent a real zero is smaller than any positive number, and any two real zeroes are equal.

1.4 The PARI philosophy.

The basic principles which govern PARI is that operations and functions should, firstly, give as exact a result as possible, and secondly, be permitted if they make any kind of sense.

In this respect, we make an important distinction between exact and inexact objects: by definition, types `t_REAL`, `t_PADIC` or `t_SER` are imprecise. A PARI object having one of these imprecise types anywhere in its tree is *inexact*, and *exact* otherwise. No loss of accuracy (rounding error) is involved when dealing with exact objects. Specifically, an exact operation between exact objects will yield an exact object. For example, dividing 1 by 3 does not give $0.333\cdots$, but the rational number $(1/3)$. To get the result as a floating point real number, evaluate `1./3` or `0.+1/3`.

Conversely, the result of operations between imprecise objects, although inexact by nature, will be as precise as possible. Consider for example the addition of two real numbers x and y. The accuracy of the result is *a priori* unpredictable; it depends on the precisions of x and y, on their sizes, and also on the size of $x + y$. From this data, PARI works out the right precision for the result. Even if it is working in calculator mode gp, where there is a notion of default precision, its value is only used to convert exact types to inexact ones.

In particular, if an operation involves objects of different accuracies, some digits will be disregarded by PARI. It is a common source of errors to forget, for instance, that a real number is given as $r + 2^e\varepsilon$ where r is a rational approximation, e a binary exponent and ε is a nondescript real number less than 1 in absolute value. Hence, any number less than 2^e may be treated as an exact zero:

```
? 0.E-28 + 1.E-100
```

```
%1 = 0.E-28
? 0.E100 + 1
%2 = 0.E100
```

As an exercise, if `a = 2^(-100)`, why do `a + 0.` and `a * 1.` differ?

The second principle is that PARI operations are in general quite permissive. For instance taking the exponential of a vector should not make sense. However, it frequently happens that one wants to apply a given function to all elements in a vector. This is easily done using a loop, or using the `apply` built-in function, but in fact PARI assumes that this is exactly what you want to do when you apply a scalar function to a vector. Taking the exponential of a vector will do just that, so no work is necessary. Most transcendental functions work in the same way*.

In the same spirit, when objects of different types are combined they are first automatically mapped to a suitable ring, where the computation becomes meaningful:

```
? 1/3 + Mod(1,5)
%1 = Mod(3, 5)
? I + O(5^9)
%2 = 2 + 5 + 2*5^2 + 5^3 + 3*5^4 + 4*5^5 + 2*5^6 + 3*5^7 + O(5^9)
? Mod(1,15) + Mod(1,10)
%3 = Mod(2, 5)
```

The first example is straightforward: since 3 is invertible mod 5, (1/3) is easily mapped to $\mathbf{Z}/5\mathbf{Z}$. In the second example, `I` stands for the customary square root of -1; we obtain a 5-adic number, 5-adically close to a square root of -1. The final example is more problematic, but there are natural maps from $\mathbf{Z}/15\mathbf{Z}$ and $\mathbf{Z}/10\mathbf{Z}$ to $\mathbf{Z}/5\mathbf{Z}$, and the computation takes place there.

1.5 Operations and functions.

The available operations and functions in PARI are described in detail in Chapter 3. Here is a brief summary:

1.5.1 Standard arithmetic operations.

Of course, the four standard operators `+`, `-`, `*`, `/` exist. We emphasize once more that division is, as far as possible, an exact operation: 4 divided by 3 gives `(4/3)`. In addition to this, operations on integers or polynomials, like `\` (Euclidean division), `%` (Euclidean remainder) exist; for integers, `\/` computes the quotient such that the remainder has smallest possible absolute value. There is also the exponentiation operator `^`, when the exponent is of type integer; otherwise, it is considered as a transcendental function. Finally, the logical operators `!` (`not` prefix operator), `&&` (`and` operator), `||` (`or` operator) exist, giving as results `1` (true) or `0` (false).

1.5.2 Conversions and similar functions.

Many conversion functions are available to convert between different types. For example floor, ceiling, rounding, truncation, etc.... Other simple functions are included like real and imaginary part, conjugation, norm, absolute value, changing precision or creating an intmod or a polmod.

* An ambiguity arises with square matrices. PARI always considers that you want to do componentwise function evaluation in this context, hence to get for example the standard exponential of a square matrix you would need to implement a different function.

1.5.3 Transcendental functions.

They usually operate on any complex number, power series, and some also on p-adics. The list is ever-expanding and of course contains all the elementary functions (exp/log, trigonometric functions), plus many others (modular functions, Bessel functions, polylogarithms...). Recall that by extension, PARI usually allows a transcendental function to operate componentwise on vectors or matrices.

1.5.4 Arithmetic functions.

Apart from a few like the factorial function or the Fibonacci numbers, these are functions which explicitly use the prime factor decomposition of integers. The standard functions are included. A number of factoring methods are used by a rather sophisticated factoring engine (to name a few, Shanks's SQUFOF, Pollard's rho, Lenstra's ECM, the MPQS quadratic sieve). These routines output strong pseudoprimes, which may be certified by the APRCL test.

There is also a large package to work with algebraic number fields. All the usual operations on elements, ideals, prime ideals, etc. are available. More sophisticated functions are also implemented, like solving Thue equations, finding integral bases and discriminants of number fields, computing class groups and fundamental units, computing in relative number field extensions, Galois and class field theory, and also many functions dealing with elliptic curves over $\mathbf{Q}$ or over local fields.

1.5.5 Other functions.

Quite a number of other functions dealing with polynomials (e.g. finding complex or p-adic roots, factoring, etc), power series (e.g. substitution, reversion), linear algebra (e.g. determinant, characteristic polynomial, linear systems), and different kinds of recursions are also included. In addition, standard numerical analysis routines like univariate integration (using the double exponential method), real root finding (when the root is bracketed), polynomial interpolation, infinite series evaluation, and plotting are included.

And now, you should really have a look at the tutorial before proceeding.

Chapter 2:
The gp Calculator

2.1 Introduction.

Originally, `gp` was designed as a debugging device for the PARI system library. Over the years, it has become a powerful user-friendly stand-alone calculator. The mathematical functions available in PARI and `gp` are described in the next chapter. In the present one, we describe the specific use of the `gp` programmable calculator.

EMACS: If you have GNU Emacs and use the PariEmacs package, you can work in a special Emacs shell, described in Section 2.16. Specific features of this Emacs shell are indicated by an EMACS sign in the left margin.

We briefly mention at this point GNU TeXmacs (`http://www.texmacs.org/`), a free wysiwyg editing platform that allows to embed an entire gp session in a document, and provides a nice alternative to PariEmacs.

2.1.1 Startup.

To start the calculator, the general command line syntax is:

`gp [-D` *key=val*`] [`*files*`]`

where items within brackets are optional. The [*files*] argument is a list of files written in the GP scripting language, which will be loaded on startup. There can be any number of arguments of the form `-D` *key=val*, setting some internal parameters of `gp`, or *defaults*: each sets the default *key* to the value *val*. See Section 2.12 below for a list and explanation of all defaults. These defaults can be changed by adding parameters to the input line as above, or interactively during a `gp` session, or in a preferences file also known as `gprc`.

If a preferences file (to be discussed in Section 2.14) is found, `gp` then reads it and executes the commands it contains. This provides an easy way to customize `gp`. The *files* argument is processed right after the `gprc`.

A copyright banner then appears which includes the version number, and a lot of useful technical information. After the copyright, the computer writes the top-level help information, some initial defaults, and then waits after printing its prompt, which is '`? `' by default . Whether extended on-line help and line editing are available or not is indicated in this `gp` banner, between the version number and the copyright message. Consider investigating the matter with the person who installed `gp` if they are not. Do this as well if there is no mention of the GMP kernel.

2.1.2 Getting help.

To get help, type a `?` and hit return. A menu appears, describing the main categories of available functions and how to get more detailed help. If you now type `?`n with $n = 1, 2, \ldots$, you get the list of commands corresponding to category n and simultaneously to Section 3.n of this manual. If you type `?`*functionname* where *functionname* is the name of a PARI function, you will get a short explanation of this function.

If extended help (see Section 2.13.1) is available on your system, you can double or triple the `?` sign to get much more: respectively the complete description of the function (e.g. `??sqrt`), or a list of `gp` functions relevant to your query (e.g. `???"elliptic curve"` or `???"quadratic field"`).

If `gp` was properly installed (see Appendix A), a line editor is available to correct the command line, get automatic completions, and so on. See Section 2.15 or `??readline` for a short summary of the line editor's commands.

If you type `?\` you will get a short description of the metacommands (keyboard shortcuts).

Finally, typing `?.` will return the list of available (pre-defined) member functions. These are functions attached to specific kind of objects, used to retrieve easily some information from complicated structures (you can define your own but they won't be shown here). We will soon describe these commands in more detail.

More generally, commands starting with the symbols `\` or `?`, are not computing commands, but are metacommands which allow you to exchange information with `gp`. The available metacommands can be divided into default setting commands (explained below) and simple commands (or keyboard shortcuts, to be dealt with in Section 2.13).

2.1.3 Input.

Just type in an instruction, e.g. `1 + 1`, or `Pi`. No action is undertaken until you hit the `<Return>` key. Then computation starts, and a result is eventually printed. To suppress printing of the result, end the expression with a `;` sign. Note that many systems use `;` to indicate end of input. Not so in `gp`: a final semicolon means the result should not be printed. (Which is certainly useful if it occupies several screens.)

2.1.4 Interrupt, Quit.

Typing `quit` at the prompt ends the session and exits `gp`. At any point you can type `Ctrl-C` (that is press simultaneously the `Control` and `C` keys): the current computation is interrupted and control given back to you at the `gp` prompt, together with a message like

```
***   at top-level: gcd(a,b)
***                 ^--------
*** gcd: user interrupt after 236 ms.
```

telling you how much time elapsed since the last command was typed in and in which GP function the computation was aborted. It does not mean that that much time was spent in the function, only that the evaluator was busy processing that specific function when you stopped it.

2.2 The general gp input line.

The gp calculator uses a purely interpreted language GP. The structure of this language is reminiscent of LISP with a functional notation, `f(x,y)` rather than `(f x y)`: all programming constructs, such as `if`, `while`, etc... are functions*, and the main loop does not really execute, but rather evaluates (sequences of) expressions. Of course, it is by no means a true LISP, and has been strongly influenced by C and Perl since then.

2.2.1 Introduction. User interaction with a gp session proceeds as follows. First, one types a sequence of characters at the gp prompt; see Section 2.15 for a description of the line editor. When you hit the `<Return>` key, gp gets your input, evaluates it, then prints the result and assigns it to an "history" array.

More precisely, the input is case-sensitive and, outside of character strings, blanks are completely ignored. Inputs are either metacommands or sequences of expressions. Metacommands are shortcuts designed to alter gp's internal state, such as the working precision or general verbosity level; we shall describe them in Section 2.13, and ignore them for the time being.

The evaluation of a sequence of instructions proceeds in two phases: your input is first digested (byte-compiled) to a bytecode suitable for fast evaluation, in particular loop bodies are compiled only once but a priori evaluated many times; then the bytecode is evaluated.

An expression is formed by combining constants, variables, operator symbols, functions and control statements. It is evaluated using the conventions about operator priorities and left to right associativity. An expression always has a value, which can be any PARI object:

```
? 1 + 1
%1 = 2              \\ an ordinary integer
? x
%2 = x              \\ a polynomial of degree 1 in the unknown x
? print("Hello")
Hello               \\ void return value, 'Hello' printed as side effect
? f(x) = x^2
%4 = (x)->x^2       \\ a user function
```

In the third example, `Hello` is printed as a side effect, but is not the return value. The `print` command is a *procedure*, which conceptually returns nothing. But in fact procedures return a special `void` object, meant to be ignored (but which evaluates to 0 in a numeric context, and stored as 0 in the history or results). The final example assigns to the variable `f` the function $x \mapsto x^2$, the alternative form `f = x->x^2` achieving the same effect; the return value of a function definition is, unsurprisingly, a function object (of type `t_CLOSURE`).

Several expressions are combined on a single line by separating them with semicolons (';'). Such an expression sequence will be called a *seq*. A *seq* also has a value, which is the value of the last expression in the sequence. Under gp, the value of the *seq*, and only this last value, becomes an history entry. The values of the other expressions in the *seq* are discarded after the execution of the *seq* is complete, except of course if they were assigned into variables. In addition, the value of the *seq* is printed if the line does not end with a semicolon `;`.

* Not exactly, since not all their arguments need be evaluated. For instance it would be stupid to evaluate both branches of an `if` statement: since only one will apply, only this one is evaluated.

2.2.2 The gp history of results.

This is not to be confused with the history of your *commands*, maintained by `readline`. The `gp` history contains the *results* they produced, in sequence.

The successive elements of the history array are called `%1`, `%2`, ... As a shortcut, the latest computed expression can also be called `%`, the previous one `%‘`, the one before that `%‘‘` and so on.

When you suppress the printing of the result with a semicolon, it is still stored in the history, but its history number will not appear either. It is a better idea to assign it to a variable for later use than to mentally recompute what its number is. Of course, on the next line, you may just use `%`.

The time used to compute that history entry is also stored as part of the entry and can be recovered using the `%#` operator: `%#1`, `%#2`, `%#‘`; `%#` by itself returns the time needed to compute the last result (the one returned by `%`).

Remark. The history "array" is in fact better thought of as a queue: its size is limited to 5000 entries by default, after which `gp` starts forgetting the initial entries. So `%1` becomes unavailable as `gp` prints `%5001`. You can modify the history size using `histsize`.

2.2.3 Special editing characters. A GP program can of course have more than one line. Since your commands are executed as soon as you have finished typing them, there must be a way to tell `gp` to wait for the next line or lines of input before doing anything. There are three ways of doing this.

The first one is to use the backslash character `\` at the end of the line that you are typing, just before hitting `<Return>`. This tells `gp` that what you will write on the next line is the physical continuation of what you have just written. In other words, it makes `gp` forget your newline character. You can type a `\` anywhere. It is interpreted as above only if (apart from ignored whitespace characters) it is immediately followed by a newline. For example, you can type

```
? 3 + \
4
```

instead of typing `3 + 4`.

The second one is a variation on the first, and is mostly useful when defining a user function (see Section 2.7): since an equal sign can never end a valid expression, `gp` disregards a newline immediately following an `=`.

```
? a =
123
%1 = 123
```

The third one is in general much more useful, and uses braces `{` and `}`. An opening brace `{` signals that you are typing a multi-line command, and newlines are ignored until you type a closing brace `}`. There are two important, but easily obeyed, restrictions: first, braces do not nest; second, inside an open brace-close brace pair, all input lines are concatenated, suppressing any newlines. Thus, all newlines should occur after a semicolon (`;`), a comma (`,`) or an operator (for clarity's sake, never split an identifier over two lines in this way). For instance, the following program

```
{
  a = b
  b = c
```

```
}
```

would silently produce garbage, since this is interpreted as `a=bb=c` which assigns the value of `c` to both `bb` and `a`. It should have been written

```
{
  a = b;
  b = c;
}
```

2.3 The PARI types.

We see here how to input values of the different data types known to PARI. Recall that blanks are ignored in any expression which is not a string (see below).

A note on efficiency. The following types are provided for convenience, not for speed: `t_INTMOD`, `t_FRAC`, `t_PADIC`, `t_QUAD`, `t_POLMOD`, `t_RFRAC`. Indeed, they always perform a reduction of some kind after each basic operation, even though it is usually more efficient to perform a single reduction at the end of some complex computation. For instance, in a convolution product $\sum_{i+j=n} x_i y_j$ in $\mathbf{Z}/N\mathbf{Z}$ — common when multiplying polynomials! —, it is quite wasteful to perform n reductions modulo N. In short, basic individual operations on these types are fast, but recursive objects with such components could be handled more efficiently: programming with libpari will save large constant factors here, compared to GP.

2.3.1 Integers (`t_INT`). After an (optional) leading `+` or `-`, type in the decimal digits of your integer. No decimal point!

```
? 1234567
%1 = 1234567
? -3
%2 = -3
? 1.          \\ oops, not an integer
%3 = 1.0000000000000000000000000000
```

Integers can be input in hexadecimal notation by prefixing them with `0x`; hexadecimal digits $(a, \ldots, f)$ can be input either in lowercase or in uppercase:

```
? 0xF
%4 = 15
? 0x1abcd
%5 = 109517
```

Integers can also be input in binary by prefixing them with `0b`:

```
? 0b010101
%6 = 21
```

2.3.2 Real numbers (`t_REAL`).

Real numbers are represented (approximately) in a floating point system, internally in base 2, but converted to base 10 for input / output purposes. A `t_REAL` object has a given *accuracy* (or *precision*) $\ell \geq 0$; it comprises

- a sign s: $+1$, -1 or 0;
- a mantissa m: a multiprecision integer, $0 \leq m < 10^\ell$;
- an exponent e: a small integer in $[-E, E]$, where $E \approx 2^B \log_{10} 2$, and $B = 32$ on a 32-bit machine and 64 otherwise.

This data may represent any real number x such that

$$|x - sm10^e| < 10^{e-\ell}.$$

We consider that a `t_REAL` with sign $s = 0$ has accuracy $\ell = 0$, so that its mantissa is useless, but it still has an exponent e and acts like a machine epsilon for all accuracies $< e$.

After an (optional) leading `+` or `-`, type a number with a decimal point. Leading zeroes may be omitted, up to the decimal point, but trailing zeroes are important: your `t_REAL` is assigned an internal precision, which is the supremum of the input precision, one more than the number of decimal digits input, and the default `realprecision`. For example, if the default precision is 28 digits, typing `2.` yields a precision of 28 digits, but `2.0...0` with 45 zeros gives a number with internal precision at least 45, although less may be printed.

You can also use scientific notation with the letter `E` or `e`. As usual, `e`n is interpreted as $\times 10^n$ for all integers n. Since the result is converted to a `t_REAL`, you may often omit the decimal point in this case: `6.02 E 23` or `1e-5` are fine, but `e10` is not.

By definition, `0.E` n returns a real 0 of exponent n, whereas `0.` returns a real 0 "of default precision" (of exponent $-$`realprecision`), see Section 1.3.7, behaving like the machine epsilon for the current default accuracy: any float of smaller absolute value is indistinguishable from 0.

Note on output formats. A zero real number is printed in `e` format as $0.Exx$ where xx is the (usually negative) *decimal* exponent of the number (cf. Section 1.3.7). This allows the user to check the accuracy of that particular zero.

When the integer part of a real number x is not known exactly because the exponent of x is greater than the internal precision, the real number is printed in `e` format.

Technical note. The internal *precision* is actually expressed in bits and can be viewed and manipulated globally in interactive use via `realprecision` (decimal digits, as explained above; shortcut `\p`) or `realbitprecision` (bits; shortcut `\ps`), the latter allowing finer granularity. See Section 3.3 for details. In programs we advise to leave this global variable alone and adapt precision locally for a given sequence of computations using `localbitprec`.

2.3.3 Intmods (`t_INTMOD`). To create the image of the integer a in $\mathbf{Z}/b\mathbf{Z}$ (for some non-zero integer b), type `Mod(a,b)`; *not* `a%b`. Internally, all operations are done on integer representatives belonging to $[0, b-1]$.

Note that this type is available for convenience, not for speed: each elementary operation involves a reduction modulo b.

If x is a `t_INTMOD` `Mod(a,b)`, the following member function is defined:

`x.mod`: return the modulus `b`.

2.3.4 Rational numbers (`t_FRAC`). All fractions are automatically reduced to lowest terms, so it is impossible to work with reducible fractions. To enter n/m just type it as written. As explained in Section 3.1.5, floating point division is *not* performed, only reduction to lowest terms.

Note that rational computation are almost never the fastest method to proceed: in the PARI implementation, each elementary operation involves computing a gcd. It is generally a little more efficient to cancel denominators and work with integers only:

```
? P = Pol( vector(10^3,i, 1/i) ); \\ big polynomial with small rational coeffs
? P^2
time = 1,392 ms.
? c = content(P); c^2 * (P/c)^2;  \\ same computation in integers
time = 1,116 ms.
```

And much more efficient (but harder to setup) to use homomorphic imaging schemes and modular computations. As the simple example below indicates, if you only need modular information, it is very worthwhile to work with `t_INTMOD`s directly, rather than deal with `t_FRAC`s all the way through:

```
? p = nextprime(10^7);
? sum(i=1, 10^5, 1/i) % p
time = 13,288 ms.
%1 = 2759492
? sum(i=1, 10^5, Mod(1/i, p))
time = 60 ms.
%2 = Mod(2759492, 10000019)
```

2.3.5 Finite field elements (`t_FFELT`). Let $T \in \mathbf{F}_p[X]$ be a monic irreducible polynomial defining your finite field over $\mathbf{F}_p$, for instance obtained using `ffinit`. Then the `ffgen` function creates a generator of the finite field as an $\mathbf{F}_p$-algebra, namely the class of X in $\mathbf{F}_p[X]/(T)$, from which you can build all other elements. For instance, to create the field $\mathbf{F}_{2^8}$, we write

```
? T = ffinit(2, 8);
? y = ffgen(T, 'y);
? y^0    \\ the unit element in the field
%3 = 1
? y^8
%4 = y^6 + y^5 + y^4 + y^3 + y + 1
```

The second (optional) parameter to `ffgen` is only used to display the result; it is customary to use the name of the variable we assign the generator to. If `g` is a `t_FFELT`, the following member functions are defined:

`g.pol`: the polynomial (with reduced integer coefficients) expressing `g` in term of the field generator.

`g.p`: the characteristic of the finite field.

`g.f`: the dimension of the definition field over its prime field; the cardinality of the definition field is thus p^f.

`g.mod`: the minimal polynomial (with reduced integer coefficients) of the field generator.

2.3.6 Complex numbers (t_COMPLEX). To enter $x + iy$, type `x + I*y`. (That's `I`, *not* `i`!) The letter `I` stands for $\sqrt{-1}$. The "real" and "imaginary" parts x and y can be of type `t_INT`, `t_REAL`, `t_INTMOD`, `t_FRAC`, or `t_PADIC`.

2.3.7 p-adic numbers (t_PADIC):. Typing `O(p^k)`, where p and k are integers, yields a p-adic 0 of accuracy k, representing any p-adic number whose valuation is $\geq k$. To input a general non-0 p-adic number, write a suitably precise rational or integer approximation and add `O(p^k)` to it.

Note that it is not checked whether p is indeed prime but results are undefined if this is not the case: you can work on 10-adics if you want, but disasters will happen as soon as you do something non-trivial like taking a square root. Note that `O(25)` is not the same as `O(5^2)`; you want the latter!

For example, you can type in the 7-adic number

```
2*7^(-1) + 3 + 4*7 + 2*7^2 + O(7^3)
```

exactly as shown, or equivalently as `905/7 + O(7^3)`.

If a is a `t_PADIC`, the following member functions are defined:

`a.mod`: returns the modulus p^k.

`a.p`: returns p.

Note that this type is available for convenience, not for speed: internally, `t_PADIC`s are stored as p-adic units modulo some p^k. Each elementary operation involves updating p^k (multiplying or dividing by powers of p) and a reduction mod p^k. In particular, additions are slow.

```
? n = 1+O(2^20);   for (i=1,10^6, n++)
time = 841 ms.
? n = Mod(1,2^20); for (i=1,10^6, n++)
time = 441 ms.
? n = 1;           for (i=1,10^6, n++)
time = 328 ms.
```

The penalty attached to maintaining p^k decreases steeply as p increases (and updates become rare). But `t_INTMOD`s remain at least 25% more efficient. (On the other hand, they do not allow denominators!)

2.3.8 Quadratic numbers (t_QUAD). This type is used to work in the quadratic order of *discriminant* `d`, where `d` is a non-square integer congruent to 0 or 1 (modulo 4). The command

```
w = quadgen(d)
```

assigns to `w` the "canonical" generator for the integer basis of the order of discriminant d, i.e. $w = \sqrt{d}/2$ if $d \equiv 0 \bmod 4$, and $w = (1 + \sqrt{d})/2$ if $d \equiv 1 \bmod 4$. The name `w` is of course just a suggestion, but corresponds to traditional usage. You can use any variable name that you like, but `quadgen(d)` is always printed as `w`, regardless of the discriminant. So beware, two `t_QUAD`s can be printed in the same way and not be equal; however, `gp` will refuse to add or multiply them for example.

Since the order is $\mathbf{Z} + \mathtt{w}\mathbf{Z}$, any other element can be input as `z = x+y*w` for some integers x and y. In fact, you may work in its fraction field $\mathbf{Q}(\sqrt{d})$ and use `t_FRAC` values for x and y.

The member function `z.disc` retrieves the discriminant d; x and y are obtained via `real(z)` and `imag(z)` respectively.

2.3.9 Polmods (`t_POLMOD`). Exactly as for intmods, to enter $x \bmod y$ (where x and y are polynomials), type `Mod(x,y)`, not `x%y`. Note that when y is an irreducible polynomial in one variable, polmods whose modulus is y are simply algebraic numbers in the finite extension defined by the polynomial y. This allows us to work easily in number fields, finite extensions of the p-adic field $\mathbf{Q}_p$, or finite fields.

Note that this type is available for convenience, not for speed: each elementary operation involves a reduction modulo y. If p is a `t_POLMOD`, the following member functions are defined:

`p.pol`: return a representative of the polynomial class of minimal degree.

`p.mod`: return the modulus.

Important remark. Mathematically, the variables occurring in a polmod are not free variables. But internally, a congruence class in $R[t]/(y)$ is represented by its representative of lowest degree, which is a `t_POL` in $R[t]$, and computations occur with polynomials in the variable t. PARI will not recognize that `Mod(y, y^2 + 1)` is "the same" as `Mod(x, x^2 + 1)`, since `x` and `y` are different variables.

To avoid inconsistencies, polmods must use the same variable in internal operations (i.e. between polmods) and variables of lower priority for external operations, typically between a polynomial and a polmod. See Section 2.5.3 for a definition of "priority" and a discussion of (PARI's idea of) multivariate polynomial arithmetic. For instance:

```
? Mod(x, x^2+ 1) + Mod(x, x^2 + 1)
%1 = Mod(2*x, x^2 + 1)     \\ 2i (or -2i), with i^2 = -1
? x + Mod(y, y^2 + 1)
%2 = x + Mod(y, y^2 + 1)   \\ in Q(i)[x]
? y + Mod(x, x^2 + 1)
%3 = Mod(x + y, x^2 + 1)   \\ in Q(y)[i]
```

The first two are straightforward, but the last one may not be what you want: `y` is treated here as a numerical parameter, not as a polynomial variable.

If the main variables are the same, it is allowed to mix `t_POL` and `t_POLMOD`s. The result is the expected `t_POLMOD`. For instance

```
? x + Mod(x, x^2 + 1)
%1 = Mod(2*x, x^2 + 1)
```

2.3.10 Polynomials (`t_POL`). Type the polynomial in a natural way, not forgetting to put a "$*$" between a coefficient and a formal variable;

```
? 1 + 2*x + 3*x^2
%1 = 3*x^2 + 2*x + 1
```

This assumes that `x` is still a "free variable".

```
? x = 1; 1 + 2*x + 3*x^2
%2 = 6
```

generates an integer, not a polynomial! It is good practice to never assign values to polynomial variables to avoid the above problem, but a foolproof construction is available using `'x` instead of `x`: `'x` is a constant evaluating to the free variable with name `x`, independently of the current value of `x`.

```
? x = 1; 1 + 2*'x + 3*'x^2
%3 = 1 + 2*x + 3*x^2
? x = 'x; 1 + 2*x + 3*x^2
%4 = 1 + 2*x + 3*x^2
```

You may also use the functions `Pol` or `Polrev`:

```
? Pol([1,2,3])        \\ Pol creates a polynomial in x by default
%1 = x^2 + 2*x + 3
? Polrev([1,2,3])
%2 = 3*x^2 + 2*x + 1
? Pol([1,2,3], 'y)    \\ we use 'y, safer than y
%3 = y^2 + 2*y + 3
```

The latter two are much more efficient constructors than an explicit summation (the latter is quadratic in the degree, the former linear):

```
? for (i=1, 10^4, Polrev( vector(100, i,i) ) )
time = 124ms
? for (i=1, 10^4, sum(i = 1, 100, (i+1) * 'x^i) )
time = 3,985ms
```

Polynomials are always printed as *univariate* polynomials, with monomials sorted by decreasing degree:

```
? (x+y+1)^2
%1 = x^2 + (2*y + 2)*x + (y^2 + 2*y + 1)
```

(Univariate polynomial in `x` whose coefficients are polynomials in `y`.) See Section 2.5 for valid variable names, and a discussion of multivariate polynomial rings.

2.3.11 Power series (`t_SER`). Typing `O(X^k)`, where k is an integer, yields an X-adic 0 of accuracy k, representing any power series in `X` whose valuation is $\geq k$. Of course, `X` can be replaced by any other variable name! To input a general non-0 power series, type in a polynomial or rational function (in `X`, say), and add `O(X^k)` to it. The discussion in the `t_POL` section about variables remains valid; a constructor `Ser` replaces `Pol` and `Polrev`.

Caveat. Power series with inexact coefficients sometimes have a non-intuitive behavior: if k significant terms are requested, an inexact zero is counted as significant, even if it is the coefficient of lowest degree. This means that useful higher order terms may be disregarded.

If a series with a zero leading coefficient must be inverted, then as a desperation measure that coefficient is discarded, and a warning is issued:

```
? C = 0. + y + O(y^2);
? 1/C
  *** _/_: Warning: normalizing a series with 0 leading term.
%2 = y^-1 + O(1)
```

The last output could be construed as a bug since it is a priori impossible to deduce such a result from the input (0. represents any sufficiently small real number). But it was thought more useful to try and go on with an approximate computation than to raise an early exception.

If the series precision is insufficient, errors may occur (mostly division by 0), which could have been avoided by a better global understanding of the computation:

```
? A = 1/(y + O.); B = 1. + O(y);
? B * denominator(A)
%2 = O.E-28 + O(y)
? A/B
  *** _/_: Warning: normalizing a series with 0 leading term.
%3 = 1.0000000000000000000000000000*y^-1 + O(1)
? A*B
  *** _*_: Warning: normalizing a series with 0 leading term.
%4 = 1.0000000000000000000000000000*y^-1 + O(1)
```

2.3.12 Rational functions (`t_RFRAC`). As for fractions, all rational functions are automatically reduced to lowest terms. All that was said about fractions in Section 2.3.4 remains valid here.

2.3.13 Binary quadratic forms of positive or negative discriminant (`t_QFR` and `t_QFI`). These are input using the function `Qfb`. For example `Qfb(1,2,3)` creates the binary form $q = x^2 + 2xy + 3y^2$. It is imaginary (of internal type `t_QFI`) since its discriminant $2^2 - 4 \times 3 = -8$ is negative. Although imaginary forms could be positive or negative definite, only positive definite forms are implemented.

The discriminant can be retrieved via `poldisc`. The individual components are obtained via either of

```
[a,b,c] = Vec(q);
```

```
a = component(q,1);
b = component(q,2);
c = component(q,3);
```

In the case of forms with positive discriminant (`t_QFR`), you may add an optional fourth component (related to the regulator, more precisely to Shanks and Lenstra's distance), which must be a real number. See also the function `qfbprimeform` which creates a prime form of given discriminant.

2.3.14 Row and column vectors (`t_VEC` and `t_COL`).) To enter a row vector, type the components separated by commas ",", and enclosed between brackets "`[`" and "`]`", e.g. `[1,2,3]`. To enter a column vector, type the vector horizontally, and add a tilde "~" to transpose. `[ ]` yields the empty (row) vector. The function `Vec` can be used to transform any object into a vector (see Chapter 3). The construction $[i..j]$, where $i \leq j$ are two integers returns the vector $[i, i+1, \ldots, j-1, j]$

```
? [1,2,3]
%1 = [1, 2, 3]
? [-2..3]
%2 = [-2, -1, 0, 1, 2, 3]
```

Let the variable v contain a (row or column) vector:

• `v[m]` refers to its m-th entry; you can assign any value to `v[m]`, i.e. write something like $v[m] = expr$.

• `v[i..j]`, where $i \leq j$, returns the vector slice containing elements $v[i], \ldots, v[j]$; you can *not* assign a result to `v[i..j]`.

• `v[^i]` returns the vector whose i-th entry has been removed; you can *not* assign a result to `v[^i]`.

In the last two constructions `v[i..j]` and `v[^i]`, i and j are allowed to be negative integers, in which case, we start counting from the end of the vector: e.g., -1 is the index of the last element.

```
? v = [1,2,3,4];
? v[2..4]
%2 = [2, 3, 4]
? v[^3]
%3 = [1, 2, 4]
? v[^-1]
%3 = [1, 2, 3]
? v[-3..-1]
%4 = [2, 3, 4]
```

Remark. `vector` is the standard constructor for row vectors whose i-th entry is given by a simple function of i; `vectorv` is similar for column vectors:

```
? vector(10, i, i^2+1)
%1 = [2, 5, 10, 17, 26, 37, 50, 65, 82, 101]
```

The functions `Vec` and `Col` convert objects to row and column vectors respectively (as well as `Vecrev` and `Colrev`, which revert the indexing):

```
? T = poltchebi(5)   \\ 5-th Chebyshev polynomial
%1 = 16*x^5 - 20*x^3 + 5*x
? Vec(T)
%2 = [16, 0, -20, 0, 5, 0]  \\ coefficients of T
? Vecrev(T)
%3 = [0, 5, 0, -20, 0, 16]  \\ ... in reverse order
```

Remark. For v a `t_VEC`, `t_COL`, `t_LIST` or `t_MAT`, the alternative set-notations

```
[g(x) | x <- v, f(x)]
[x | x <- v, f(x)]
[g(x) | x <- v]
```

are available as shortcuts for

```
apply(g, select(f, Vec(v)))
select(f, Vec(v))
apply(g, Vec(v))
```

respectively, and may serve as `t_VEC` constructors:

```
? [ p | p <- primes(10), isprime(p+2) ]
%2 = [3, 5, 11, 17, 29]
```

returns the primes p (among the first 10 primes) such that $(p, p+2)$ is a twin pair;

```
? [ p^2 | p <- primes(10), p % 4 == 1 ]
%1 = [25, 169, 289, 841]
```

returns the squares of the primes congruent to 1 modulo 4, where p runs among the first 10 primes.

2.3.15 Matrices (`t_MAT`). To enter a matrix, type the components row by row, the components being separated by commas "`,`", the rows by semicolons "`;`", and everything enclosed in brackets "`[`" and "`]`", e.g. `[x,y; z,t; u,v]`. `[;]` yields an empty (0×0) matrix. The function `Mat` transforms any object into a matrix, and `matrix` creates matrices whose (i, j)-th entry is described by a function $f(i, j)$:

```
? Mat(1)
%1 =
[1]
? matrix(2,2, i,j, 2*i+j)
%2 =
[3 4]

[5 6]
```

Let the variable M contain a matrix, and let i, j, k, l denote four integers:

- `M[i,j]` refers to its (i, j)-th entry; you can assign any result to `M[i,j]`.
- `M[i,]` refers to its i-th row; you can assign a `t_VEC` of the right dimension to `M[i,]`.
- `M[,j]` refers to its j-th column; you can assign a `t_COL` of the right dimension to `M[,j]`.

But `M[i]` is meaningless and triggers an error. The "range" $i..j$ and "caret" `^`c notations are available as for vectors; you can not *assign* to any of these:

- `M[i..j, k..l]`, $i \leq j$, $k \leq l$, returns the submatrix built from the rows i to j and columns k to l of M.
- `M[i..j,]` returns the submatrix built from the rows i to j of M.
- `M[,i..j]` returns the submatrix built from the columns i to j of M.
- `M[i..j, ^k]`, $i \leq j$, returns the submatrix built from the rows i to j and column k removed.
- `M[^k,]` returns the submatrix with row k removed.
- `M[,^k]` returns the submatrix with column k removed.

Finally,

- `M[i..j, k]` returns the `t_COL` built from the k-th column (entries i to j).
- `M[^i, k]` returns the `t_COL` built from the k-th column (entry i removed).
- `M[k, i..j]` returns the `t_VEC` built from the k-th row (entries i to j).
- `M[k, ^i]` returns the `t_VEC` built from the k-th row (entry i removed).

```
? M = [1,2,3;4,5,6;7,8,9];
? M[1..2, 2..3]
%2 =
[2 3]

[5 6]
? M[1..2,]
```

```
%3 =
[1 2 3]

[4 5 6]
? M[,2..3]
%4 =
[2 3]

[5 6]

[8 9]
```

All this is recursive, so if `M` is a matrix of matrices of ..., an expression such as `M[1,1][,3][4] = 1` is perfectly valid (and actually identical to `M[1,1][4,3] = 1`), assuming that all matrices along the way have compatible dimensions.

Technical note (design flaw). Matrices are internally represented as a vector of columns. All matrices with 0 columns are thus represented by the same object (internally, an empty vector), and there is no way to distinguish between them. Thus it is not possible to create or represent matrices with zero columns and an actual nonzero number of rows. The empty matrix `[;]` is handled as though it had an arbitrary number of rows, exactly as many as needed for the current computation to make sense:

```
? [1,2,3; 4,5,6] * [;]
%1 = [;]
```

The empty matrix on the first line is understood as a 3×0 matrix, and the result as a 2×0 matrix. On the other hand, it is possible to create matrices with a given positive number of columns, each of which has zero rows, e.g. using `Mat` as above or using the `matrix` function.

Note that although the internal representation is essentially the same, a row vector of column vectors is *not* a matrix; for example, multiplication will not work in the same way. It is easy to go from one representation to the other using `Vec` / `Mat`, though:

```
? [1,2,3;4,5,6]
%1 =
[1 2 3]

[4 5 6]

? Vec(%)
%2 = [[1, 4]~, [2, 5]~, [3, 6]~]
? Mat(%)
%3 =
[1 2 3]

[4 5 6]
```

2.3.16 Lists (`t_LIST`). Lists can be input directly, as in `List([1,2,3,4])`; but in most cases, one creates an empty list, then appends elements using `listput`:

```
? a = List(); listput(a,1); listput(a,2);
? a
%2 = List([1, 2])
```

Elements can be accessed directly as with the vector types described above.

2.3.17 Strings (`t_STR`). To enter a string, enclose it between double quotes ", like this: `"this is a string"`. The function `Str` can be used to transform any object into a string.

2.3.18 Small vectors (`t_VECSMALL`). This is an internal type, used to code in an efficient way vectors containing only small integers, such as permutations. Most `gp` functions will refuse to operate on these objects.

2.3.19 Functions (`t_CLOSURE`). We will explain this at length in Section 2.7. For the time being, suffice it to say that functions can be assigned to variables, as any other object, and the following equivalent basic forms are available to create new ones

```
f = (x,y) -> x^2 + y^2
f(x,y) = x^2 + y^2
```

2.3.20 Error contexts (`t_ERROR`). An object of this type is created whenever an error occurs: it contains some information about the error and the error context. Usually, an appropriate error is printed immediately, the computation is aborted, and GP enters the "break loop":

```
? 1/0; 1 + 1
  ***   at top-level: 1/0;1+1
  ***                   ^------
  *** _/_: division by a non-invertible object
  ***   Break loop: type 'break' to go back to the GP prompt
```

Here the computation is aborted as soon as we try to evaluate $1/0$, and $1 + 1$ is never executed. Exceptions can be trapped using `iferr`, however: we can evaluate some expression and either recover an ordinary result (no error occurred), or an exception (an error did occur).

```
? i = Mod(6,12); iferr(1/i, E, print(E)); 1 + 1
error("impossible inverse modulo: Mod(6, 12).")
%1 = 2
```

One can ignore the exception, print it as above, or extract non trivial information from the error context:

```
? i = Mod(6,12); iferr(1/i, E, print(component(E,1)));
Mod(6, 12)
```

We can also rethrow the exception: `error(E)`.

2.3.21 Infinity (`t_INFINITY`).

There are only two objects of this type `+oo` and `-oo`, representing $\pm\infty$. This type only contain only two elements `oo` and `-oo`, They are used in functions sur as `intnum` or `polrootsreal`, to encode infinite real intervals. These objects can only be negated and compared to real numbers (`t_INT`, `t_REAL`, `t_FRAC`), but not included in any computation, i.e. `1+oo` is an error, not kbdoo again.

2.4 GP operators.

Loosely speaking, an operator is a function, usually attached to basic arithmetic operations, whose name contains only non-alphanumeric characters. For instance `+` or `-`, but also `=` or `+=`, or even `[ ]` (the selection operator). As all functions, operators take arguments, and return a value; *assignment* operators also have side effects: besides returning a value, they change the value of some variable.

Each operator has a fixed and unchangeable priority, which means that, in a given expression, the operations with the highest priority is performed first. Unless mentioned otherwise, operators at the same priority level are left-associative (performed from left to right), unless they are assignments, in which case they are right-associative. Anything enclosed between parenthesis is considered a complete subexpression, and is resolved recursively, independently of the surrounding context. For instance,

```
a + b + c     -->   (a + b) + c      \\ left-associative
a = b = c     -->   a = (b = c)      \\ right-associative
```

Assuming that op_1, op_2, op_3 are binary operators with increasing priorities (think of `+`, `*`, `^`),

$$x\ op_1\ y\ op_2\ z\ op_2\ x\ op_3\ y$$

is equivalent to

$$x\ op_1\ ((y\ op_2\ z)\ op_2\ (x\ op_3\ y)).$$

GP contains many different operators, either unary (having only one argument) or binary, plus a few special selection operators. Unary operators are defined as either *prefix* or *postfix*, meaning that they respectively precede ($op\ x$) and follow ($x\ op$) their single argument. Some symbols are syntactically correct in both positions, like `!`, but then represent different operators: the `!` symbol represents the negation and factorial operators when in prefix and postfix position respectively. Binary operators all use the (infix) syntax $x\ op\ y$.

Most operators are standard (`+`, `%`, `=`), some are borrowed from the C language (`++`, `<<`), and a few are specific to GP (`\`, `#`). Beware that some GP operators differ slightly from their C counterparts. For instance, GP's postfix `++` returns the *new* value, like the prefix `++` of C, and the binary shifts `<<`, `>>` have a priority which is different from (higher than) that of their C counterparts. When in doubt, just surround everything by parentheses; besides, your code will be more legible.

Here is the list of available operators, ordered by decreasing priority, binary and left-associative unless mentioned otherwise. An expression is an *lvalue* if something can be assigned to it. (The name comes from left-value, to the left of a `=` operator; e.g. `x`, or `v[1]` are lvalues, but `x + 1` is not.)

- Priority 14

  `:` as in `x:small`, is used to indicate to the GP2C compiler that the variable on the left-hand side always contains objects of the type specified on the right hand-side (here, a small integer) in order to produce more efficient or more readable C code. This is ignored by GP.

- Priority 13

  `( )` is the function call operator. If f is a closure and *args* is a comma-separated list of arguments (possibly empty), $f(args)$ evaluates f on those arguments.

- Priority 12

  `++` and `--` (unary, postfix): if x is an `lvalue`, x`++` assigns the value $x + 1$ to x, then returns

the new value of x. This corresponds to the C statement `++`x: there is no prefix `++` operator in GP. x`--` does the same with $x - 1$. These operators are not associative, i.e. `x++++` is invalid, since `x++` is not an lvalue.

• Priority 11

`.`*member* (unary, postfix): x`.`*member* extracts *member* from structure x (see Section 2.8).

`[ ]` is the selection operator. x`[`i`]` returns the i-th component of vector x; x`[`i`,`j`]`, x`[,`j`]` and x`[`i`,]` respectively return the entry of coordinates (i, j), the j-th column, and the i-th row of matrix x. If the assignment operator (`=`) immediately follows a sequence of selections, it assigns its right hand side to the selected component. E.g `x[1][1] = 0` is valid; but beware that `(x[1])[1] = 0` is not (because the parentheses force the complete evaluation of `x[1]`, and the result is not modifiable).

• Priority 10

`'` (unary, postfix): derivative with respect to the main variable. If f is a function (`t_CLOSURE`), f' is allowed and defines a new function, which will perform numerical derivation when evaluated at a scalar x; this is defined as $(f(x + \varepsilon) - f(x - \varepsilon))/2\varepsilon$ for a suitably small epsilon depending on current precision.

```
? (x^2 + y*x + y^2)'   \\ derive with respect to main variable x
%1 = 2*x + y
? SIN = cos'
%2 = cos'
? SIN(Pi/6)            \\ numerical derivation
%3 = -0.50000000000000000000000000000
? cos'(Pi/6)           \\ works directly: no need for intermediate SIN
%4 = -0.50000000000000000000000000000
```

`~` (unary, postfix): vector/matrix transpose.

`!` (unary, postfix): factorial. $x! = x(x-1)\cdots 1$.

`!` (unary, prefix): logical *not*. `!`x returns 1 if x is equal to 0 (specifically, if `gequal0(`x`)==1`), and 0 otherwise.

• Priority 9

`#` (unary, prefix): cardinality; `#`x returns `length(`x`)`.

• Priority 8

`^`: powering. This operator is right associative: `2 ^3^4` is understood as `2 ^(3^4)`.

• Priority 7

`+`, `-` (unary, prefix): `-` toggles the sign of its argument, `+` has no effect whatsoever.

• Priority 6

`*`: multiplication.

`/`: exact division (`3/2` yields 3/2, not 1.5).

`\`, `%`: Euclidean quotient and remainder, i.e. if $x = qy + r$, then `x\y` $= q$, `x%y` $= r$. If x and y are scalars, then q is an integer and r satisfies $0 \le r < |y|$; if x and y are polynomials, then q and r are polynomials such that $\deg r < \deg y$ and the leading terms of r and x have the same sign.

`\/`: rounded Euclidean quotient for integers (rounded towards $+\infty$ when the exact quotient would be a half-integer).

`<<`, `>>`: left and right binary shift. By definition, `x<<n` $= x * 2^n$ if $n > 0$, and `truncate`$(x2^{-n})$ otherwise. Right shift is defined by `x>>n` = `x<<(-n)`.

• Priority 5

`+`, `-`: addition/subtraction.

• Priority 4

`<`, `>`, `<=`, `>=`: the usual comparison operators, returning 1 for `true` and 0 for `false`. For instance, `x<=1` returns 1 if $x \leq 1$ and 0 otherwise.

`<>`, `!=`: test for (exact) inequality.

`==`: test for (exact) equality. `t_QFR` having the same coefficients but a different distance component are tested as equal.

`===`: test whether two objects are identical component-wise. This is stricter than `==`: for instance, the integer 0, a 0 polynomial or a vector with 0 entries, are all tested equal by `==`, but they are not identical.

• Priority 3

`&&`: logical *and*.

`||`: logical (inclusive) *or*. Any sequence of logical *or* and *and* operations is evaluated from left to right, and aborted as soon as the final truth value is known. Thus, for instance,

```
x == 0 || test(1/x)
```

will never produce an error since `test(1/x)` is not even evaluated when the first test is true (hence the final truth value is true). Similarly

```
type(p) == "t_INT" && isprime(p)
```

does not evaluate `isprime(p)` if `p` is not an integer.

• Priority 2

`=` (assignment, *lvalue* `=` *expr*). The result of `x = y` is the value of the expression y, which is also assigned to the variable `x`. This assignment operator is right-associative. This is *not* the equality test operator; a statement like `x = 1` is always true (i.e. non-zero), and sets `x` to 1; the equality test would be `x == 1`. The right hand side of the assignment operator is evaluated before the left hand side.

It is crucial that the left hand-side be an *lvalue* there, it avoids ambiguities in expressions like `1 + x = 1`. The latter evaluates as `1 + (x = 1)`, not as `(1 + x) = 1`, even though the priority of `=` is lower than the priority of `+`: `1 + x` is not an lvalue.

If the expression cannot be parsed in a way where the left hand side is an lvalue, raise an error.

```
? x + 1 = 1
  ***   syntax error, unexpected '=', expecting $end or ';': x+1=1
  ***                                                        ^--
```

op`=`, where *op* is any binary operator among `+`, `-`, `*`, `%`, `/`, `\`, `\/`, `<<`, or `>>` (composed assignment *lvalue op*`=` *expr*). The expression `x` *op*`=` y assigns (`x` *op* y) to `x`, and returns the new value of `x`. The result is *not* an *lvalue*; thus

```
(x += 2) = 3
```

is invalid. These assignment operators are right-associative:

```
? x = 'x; x += x *= 2
%1 = 3*x
```

• Priority 1

`->` (function definition): (*vars*)`->`*expr* returns a function object, of type `t_CLOSURE`.

Remark. Use the *op*`=` operators as often as possible since they make complex assignments more legible: one needs not parse complicated expressions twice to make sure they are indeed identical. Compare

```
v[i+j-1] = v[i+j-1] + 1    -->    v[i+j-1]++
M[i,i+j] = M[i,i+j] * 2    -->    M[i,i+j] *= 2
```

Remark. Less important but still interesting. The `++`, `--` and *op*`=` operators are slightly more efficient:

```
? a = 10^6;
? i = 0; while(i<a, i=i+1)
time = 365 ms.
? i = 0; while(i<a, i++)
time = 352 ms.
```

For the same reason, the shift operators should be preferred to multiplication:

```
? a = 1<<(10^5);
? i = 1; while(i<a, i=i*2);
time = 1,052 ms.
? i = 1; while(i<a, i<<=1);
time = 617 ms.
```

2.5 Variables and symbolic expressions.

In this section we use *variable* in the standard mathematical sense, symbols representing algebraically independent elements used to build rings of polynomials and power series, and explain the all-important concept of *variable priority*. In the next Section 2.6, we shall no longer consider only free variables, but adopt the viewpoint of computer programming and assign values to these symbols: (bound) variables are names attached to values in a given scope.

2.5.1 Variable names. A valid name starts with a letter, followed by any number of keyword characters: `_` or alphanumeric characters (`[A-Za-z0-9]`). The built-in function names are reserved and cannot be used; see the list with `\c`, including the constants `Pi`, `Euler`, `Catalan`, `I` $= \sqrt{-1}$ and `oo` $= \infty$.

GP names are case sensitive. For instance, the symbol `i` is perfectly safe to use, and will not be mistaken for `I` $= \sqrt{-1}$; analogously, `o` is not synonymous to `O`.

In GP you can use up to 16383 variable names (up to 65535 on 64-bit machines). If you ever need thousands of variables and this becomes a serious limitation, you should probably be using vectors instead: e.g. instead of variables `X1`, `X2`, `X3`, ..., you might equally well store their values in `X[1]`, `X[2]`, `X[3]`, ...

2.5.2 Variables and polynomials. The quote operator `'t` registers a new *free variable* with the interpreter, which will be written as `t`, and evaluates to a monomial of degree 1 in the said variable.

Caveat. For reasons of backward compatibility, there is no such thing as an "unbound" (uninitialized) variable in GP. If you use a valid variable name in an expression, `t` say, for the first time *before* assigning a value into it, it is interpreted as `'t` rather than raising an exception. One should not rely on this feature in serious programs, which would otherwise break if some unexpected assignment (e.g. `t = 1`) occurs: use `'t` directly or `t = 't` first, then `t`. A statement like `t = 't` in effect restores `t` as a free variable.

```
? t = 't; t^2 + 1
%1 = t^2 + 1
? t = 2; t^2 + 1
%2 = 5
? %1
%3 = t^2 + 1
? eval(%1)
%4 = 5
```

In the above, we initialize `t` to a monomial, then bind it to 2. Assigning a value to a polynomial variable does not affect previous expressions involving it; to take into account the new variable's value, one must force a new evaluation, using the function `eval` (see Section 3.10.5).

Caveat2. The use of an explicit quote operator avoids the following kind of problems:

```
? t = 't; p = t^2 + 1; subst(p, t, 2)
%1 = 5
? t = 2;
? subst(p, t, 3)     \\ t is no longer free: it evaluates to 2
  ***   at top-level: subst(p,t,3)
  ***                           ^----
  ***   variable name expected.
? subst(p, 't, 3)    \\ OK
%3 = 10
```

2.5.3 Variable priorities, multivariate objects. A multivariate polynomial in PARI is just a polynomial (in one variable), whose coefficients are themselves polynomials, arbitrary but for the fact that they do not involve the main variable. (PARI currently has no sparse representation for polynomials, listing only non-zero monomials.) All computations are then done formally on the coefficients as if the polynomial was univariate.

This is not symmetrical. So if I enter `'x + 'y` in a clean session, what happens? This is understood as

$$x^1 + (y^1 + 0 * y^0) * x^0 \in (\mathbf{Z}[y])[x]$$

but how do we know that x is "more important" than y ? Why not $y^1 + x * y^0$, which is the same mathematical entity after all?

The answer is that variables are ordered implicitly by the interpreter: when a new identifier (e.g x, or y as above) is input, the corresponding variable is registered as having a strictly lower priority than any variable in use at this point*. To see the ordering used by `gp` at any given time, type `variable()`.

* This is not strictly true: the variables x and y are predefined, and satisfy $x > y$. Variables of higher priority than x can be created using `varhigher`.

Given such an ordering, multivariate polynomials are stored so that the variable with the highest priority is the main variable. And so on, recursively, until all variables are exhausted. A different storage pattern (which could only be obtained via `libpari` programming and low-level constructors) would produce an invalid object, and eventually a disaster.

In any case, if you are working with expressions involving several variables and want to have them ordered in a specific manner in the internal representation just described, the simplest is just to write down the variables one after the other under `gp` before starting any real computations. You may also define variables from your `gprc` to have a consistent ordering of common variable names in all your `gp` sessions, e.g read in a file `variables.gp` containing

```
'x; 'y; 'z; 't; 'a;
```

There is no way to change the priority of existing variables, but you may always create new ones with well-defined priorities using `varhigher` or `varlower`.

Important note. PARI allows Euclidean division of multivariate polynomials, but assumes that the computation takes place in the fraction field of the coefficient ring (if it is not an integral domain, the result will a priori not make sense). This can become tricky. For instance assume x has highest priority, then y:

```
? x % y
%1 = 0
? y % x
%2 = y                \\ these two take place in Q(y)[x]
? x * Mod(1,y)
%3 = Mod(1, y)*x      \\ in (Q(y)/yQ(y))[x]  ~  Q[x]
? Mod(x,y)
%4 = 0
```

In the last example, the division by y takes place in $\mathbf{Q}(y)[x]$, hence the `Mod` object is a coset in $(\mathbf{Q}(y)[x])/(y\mathbf{Q}(y)[x])$, which is the null ring since y is invertible! So be very wary of variable ordering when your computations involve implicit divisions and many variables. This also affects functions like `numerator/denominator` or `content`:

```
? denominator(x / y)
%1 = 1
? denominator(y / x)
%2 = x
? content(x / y)
%3 = 1/y
? content(y / x)
%4 = y
? content(2 / x)
%5 = 2
```

Can you see why? Hint: $x/y = (1/y) * x$ is in $\mathbf{Q}(y)[x]$ and denominator is taken with respect to $\mathbf{Q}(y)(x)$; $y/x = (y * x^0)/x$ is in $\mathbf{Q}(y)(x)$ so y is invertible in the coefficient ring. On the other hand, $2/x$ involves a single variable and the coefficient ring is simply $\mathbf{Z}$.

These problems arise because the variable ordering defines an *implicit* variable with respect to which division takes place. This is the price to pay to allow `%` and `/` operators on polynomials instead of requiring a more cumbersome `divrem(`x`, `y`, `var`)` (which also exists). Unfortunately,

in some functions like `content` and `denominator`, there is no way to set explicitly a main variable like in `divrem` and remove the dependence on implicit orderings. This will hopefully be corrected in future versions.

2.5.4 Multivariate power series. Just like multivariate polynomials, power series are fundamentally single-variable objects. It is awkward to handle many variables at once, since PARI's implementation cannot handle multivariate error terms like $O(x^i y^j)$. (It can handle the polynomial $O(y^j) \times x^i$ which is a very different thing, see below.)

The basic assumption in our model is that if variable x has higher priority than y, then y does not depend on x: setting y to a function of x after some computations with bivariate power series does not make sense a priori. This is because implicit constants in expressions like $O(x^i)$ depend on y (whereas in $O(y^j)$ they can not depend on x). For instance

```
? O(x) * y
%1 = O(x)
? O(y) * x
%2 = O(y)*x
```

Here is a more involved example:

```
? A = 1/x^2 + 1 + O(x); B = 1/x + 1 + O(x^3);
? subst(z*A, z, B)
%2 = x^-3 + x^-2 + x^-1 + 1 + O(x)
? B * A
%3 = x^-3 + x^-2 + x^-1 + O(1)
? z * A
%4 = z*x^-2 + z + O(x)
```

The discrepancy between `%2` and `%3` is surprising. Why does `%2` contain a spurious constant term, which cannot be deduced from the input? Well, we ignored the rule that forbids to substitute an expression involving high-priority variables to a low-priority variable. The result `%4` is correct according to our rules since the implicit constant in $O(x)$ may depend on z. It is obviously wrong if z is allowed to have negative valuation in x. Of course, the correct error term should be $O(xz)$, but this is not possible in PARI.

2.6 Variables and Scope.

This section is rather technical, and strives to explain potentially confusing concepts. Skip to the last subsection for practical advice, if the next discussion does not make sense to you. After learning about user functions, study the example in Section 2.7.3 then come back.

Definitions.

A *scope* is an enclosing context where names and values are attached. A user's function body, the body of a loop, an individual command line, all define scopes; the whole program defines the *global* scope. The argument of `eval` is evaluated in the enclosing scope.

Variables are bound to values within a given scope. This is traditionally implemented in two different ways:

• lexical (or static) scoping: the binding makes sense within a given block of program text. The value is private to the block and may not be accessed from outside. Where to find the value is determined at compile time.

• dynamic scoping: introducing a local variable, say `x`, pushes a new value on a stack attached to the name `x` (possibly empty at this point), which is popped out when the control flow leaves the scope. Evaluating `x` in any context, possibly outside of the given block, always yields the top value on this dynamic stack.

GP implements both lexical and dynamic scoping, using the keywords* `my` (lexical) and `local` (dynamic):

```
x = 0;
f() = x
g() =    my(x = 1); f()
h() = local(x = 1); f()
```

The function `g` returns 0 since the global `x` binding is unaffected by the introduction of a private variable of the same name in `g`. On the other hand, `h` returns 1; when it calls `f()`, the binding stack for the `x` identifier contains two items: the global binding to 0, and the binding to 1 introduced in `h`, which is still present on the stack since the control flow has not left `h` yet.

2.6.1 Scoping rules.

Named parameters in a function definition, as well as all loop indices**, have lexical scope within the function body and the loop body respectively.

```
p = 0;
forprime (p = 2, 11, print(p)); p   \\ prints 0 at the end
```

```
x = 0;
f(x) = x++;
f(1)  \\ returns 2, and leave global x unaffected (= 0)
```

If you exit the loop prematurely, e.g. using the `break` statement, you must save the loop index in another variable since its value prior the loop will be restored upon exit. For instance

```
for(i = 1, n,
  if (ok(i), break);
);
if (i > n, return(failure));
```

---

* The names are borrowed from the `Perl` scripting language.

** More generally, in all iterative constructs which use a variable name (`for`, `prod`, `sum`, `vector`, `matrix`, `plot`, etc.) the given variable is lexically scoped to the construct's body.

is incorrect, since the value of i tested by the $(i > n)$ is quite unrelated to the loop index. One ugly workaround is

```
for(i = 1, n,
  if (ok(i), isave = i; break);
);
if (isave > n, return(failure));
```

But it is usually more natural to wrap the loop in a user function and use `return` instead of `break`:

```
try() =
{
  for(i = 1, n,
    if (ok(i), return (i));
  );
  0 \\ failure
}
```

A list of variables can be lexically or dynamically scoped (to the block between the declaration and the end of the innermost enclosing scope) using a `my` or `local` declaration:

```
for (i = 1, 10,
  my(x, y, z, i2 = i^2); \\ temps needed within the loop body
  ...
)
```

Note how the declaration can include (optional) initial values, `i2 = i^2` in the above. Variables for which no explicit default value is given in the declaration are initialized to 0. It would be more natural to initialize them to free variables, but this would break backward compatibility. To obtain this behavior, you may explicitly use the quoting operator:

```
my(x = 'x, y = 'y, z = 'z);
```

A more complicated example:

```
for (i = 1, 3,
  print("main loop");
  my(x = i);          \\ local to the outermost loop
  for (j = 1, 3,
    my (y = x^2);     \\ local to the innermost loop
    print (y + y^2);
    x++;
  )
)
```

When we leave the loops, the values of `x`, `y`, `i`, `j` are the same as before they were started.

Note that `eval` is evaluated in the given scope, and can access values of lexical variables:

```
? x = 1;
? my(x = 0); eval("x")
%2 = 0     \\ we see the local x scoped to this command line, not the global one
```

Variables dynamically scoped using `local` should more appropriately be called *temporary values* since they are in fact local to the function declaring them *and* any subroutine called from

within. In practice, you almost certainly want true private variables, hence should use almost exclusively `my`.

We strongly recommended to explicitly scope (lexically) all variables to the smallest possible block. Should you forget this, in expressions involving such "rogue" variables, the value used will be the one which happens to be on top of the value stack at the time of the call; which depends on the whole calling context in a non-trivial way. This is in general *not* what you want.

2.7 User defined functions.

The most important thing to understand about user-defined functions is that they are ordinary GP objects, bound to variables just like any other object. Those variables are subject to scoping rules as any other: while you can define all your functions in global scope, it is usually possible and cleaner to lexically scope your private helper functions to the block of text where they will be needed.

Whenever gp meets a construction of the form `expr(`*argument list*`)` and the expression `expr` evaluates to a function (an object of type `t_CLOSURE`), the function is called with the proper arguments. For instance, constructions like `funcs[i](x)` are perfectly valid, assuming `funcs` is an array of functions.

2.7.1 Defining a function.

A user function is defined as follows:

`(`*list of formal variables*`) -> `*seq*.

The list of formal variables is a comma-separated list of *distinct* variable names and allowed to be empty. It there is a single formal variable, the parentheses are optional. This list corresponds to the list of parameters you will supply to your function when calling it.

In most cases you want to assign a function to a variable immediately, as in

```
R = (x,y) -> sqrt( x^2+y^2 );
sq = x -> x^2;  \\ or equivalently (x) -> x^2
```

but it is quite possible to define (a priori short-lived) anonymous functions. The trailing semicolon is not part of the definition, but as usual prevents gp from printing the result of the evaluation, i.e. the function object. The construction

`f(`*list of formal variables*`) = `*seq*

is available as an alias for

`f = (`*list of formal variables*`) -> `*seq*

Using that syntax, it is not possible to define anonymous functions (obviously), and the above two examples become:

```
R(x,y) = sqrt( x^2+y^2 );
sq(x) = x^2;
```

The semicolon serves the same purpose as above: preventing the printing of the resulting function object; compare

```
? sq(x) = x^2;  \\ no output
```

```
? sq(x) = x^2    \\ print the result: a function object
%2 = (x)->x^2
```

Of course, the sequence *seq* can be arbitrarily complicated, in which case it will look better written on consecutive lines, with properly scoped variables:

```
{
f(x0, x1, ...) =
  my(t0, t1, ...); \\ variables lexically scoped to the function body
  ...
}
```

Note that the following variant would also work:

```
f(x0, x1, ...) =
{
  my(t0, t1, ...); \\ variables lexically scoped to the function body
  ...
}
```

(the first newline is disregarded due to the preceding = sign, and the others because of the enclosing braces). The `my` statements can actually occur anywhere within the function body, scoping the variables to more restricted blocks than the whole function body.

Arguments are passed by value, not as variables: modifying a function's argument in the function body is allowed, but does not modify its value in the calling scope. In fact, a *copy* of the actual parameter is assigned to the formal parameter when the function is called. Formal parameters are lexically scoped to the function body. It is not allowed to use the same variable name for different parameters of your function:

```
? f(x,x) = 1
  ***   variable declared twice: f(x,x)=1
  ***                                ^----
```

Functions taking an unlimited number of arguments.

A function taking an unlimited number of arguments is called *variadic*. To create such a function, use the syntax

(*list of formal variables*, *var*`[..]`) `->` *seq*

The parameter *var* is replaced by a vector containing all the remaining arguments.

```
? f(c[..]) = sum(i=1,#c,c[i]);
? f(1,2,3)
%1 = 6
? sep(s,v[..]) = for(i=1,#v-1,print1(v[i],s)); if (#v, print(v[#v]));
? sep(":", 1, 2, 3)
1:2:3
```

Finishing touch. You can add a specific help message for your function using `addhelp`, but the online help system already handles it. By default `?`*name* will print the definition of the function *name*: the list of arguments, as well as their default values, the text of *seq* as you input it. Just as `\c` prints the list of all built-in commands, `\u` outputs the list of all user-defined functions.

Backward compatibility (lexical scope). Lexically scoped variables were introduced in version 2.4.2. Before that, the formal parameters were dynamically scoped. If your script depends on this behavior, you may use the following trick: replace the initial `f(x) =` by

```
f(x_orig) = local(x = x_orig)
```

Backward compatibility (disjoint namespaces). Before version 2.4.2, variables and functions lived in disjoint namespaces and it was not possible to have a variable and a function share the same name. Hence the need for a `kill` function allowing to reuse symbols. This is no longer the case.

There is now no distinction between variable and function names: we have PARI objects (functions of type `t_CLOSURE`, or more mundane mathematical entities, like `t_INT`, etc.) and variables bound to them. There is nothing wrong with the following sequence of assignments:

```
? f = 1         \\ assigns the integer 1 to f
%1 = 1;
? f() = 1       \\ a function with a constant value
%2 = ()->1
? f = x^2       \\ f now holds a polynomial
%3 = x^2
? f(x) = x^2  \\ ... and now a polynomial function
%4 = (x)->x^2
? g(fun) = fun(Pi);\\ a function taking a function as argument
? g(cos)
%6 = -1.0000000000000000000000000000
```

Previously used names can be recycled as above: you are just redefining the variable. The previous definition is lost of course.

Important technical note. Built-in functions are a special case since they are read-only (you cannot overwrite their default meaning), and they use features not available to user functions, in particular pointer arguments. In the present version 2.9.2, it is possible to assign a built-in function to a variable, or to use a built-in function name to create an anonymous function, but some special argument combinations may not be available:

```
? issquare(9, &e)
%1 = 1
? e
%2 = 3
? g = issquare;
? g(9)
%4 = 1
? g(9, &e)  \\ pointers are not implemented for user functions
  ***   unexpected &: g(9,&e)
  ***                      ^---
```

2.7.2 Function call, Default arguments.

You may now call your function, as in `f(1,2)`, supplying values for the formal variables. The number of parameters actually supplied may be *less* than the number of formal variables in the function definition. An uninitialized formal variable is given an implicit default value of (the integer) 0, i.e. after the definition

```
f(x, y) = ...
```

you may call `f(1, 2)`, supplying values for the two formal parameters, or for example

```
f(2)      equivalent to      f(2,0),
f()                          f(0,0),
f(,3)                        f(0,3). ("Empty argument" trick)
```

This *implicit* default value of 0, is actually deprecated and setting

```
default(strictargs, 1)
```

allows to disable it (see Section 3.17.41).

The recommended practice is to *explicitly* set a default value: in the function definition, you can append `=`*expr* to a formal parameter, to give that variable a default value. The expression gets evaluated the moment the function is called, and may involve the preceding function parameters: a default value for x_i may involve x_j for $j < i$. For instance, after

```
f(x = 1, y = 2, z = y+1) = ....
```

typing in `f(3,4)` would give you `f(3,4,5)`. In the rare case when you want to set some far away argument, and leave the defaults in between as they stand, use the "empty argument" trick: `f(6,,1)` would yield `f(6,2,1)`. Of course, `f()` by itself yields `f(1,2,3)` as was to be expected.

In short, the argument list is filled with user supplied values, in order. A comma or closing parenthesis, where a value should have been, signals we must use a default value. When no input arguments are left, the defaults are used instead to fill in remaining formal parameters. A final example:

```
f(x, y=2, z=3) = print(x, ":", y, ":", z);
```

defines a function which prints its arguments (at most three of them), separated by colons.

```
? f(6,7)
6:7:3
? f(,5)
0:5:3
? f()
0:2:3
```

If `strictargs` is set (recommended), x is now a mandatory argument, and the above becomes:

```
? default(strictargs,1)
? f(6,7)
6:7:3
? f(,5)
  ***   at top-level: f(,5)
  ***                 ^-----
  ***   in function f: x,y=2,z=3
  ***                  ^---------
  ***   missing mandatory argument 'x' in user function.
```

Example. We conclude with an amusing example, intended to illustrate both user-defined functions and the power of the `sumalt` function. Although the Riemann zeta-function is included (as `zeta`) among the standard functions, let us assume that we want to check other implementations. Since we are highly interested in the critical strip, we use the classical formula

$$(2^{1-s}-1)\zeta(s) = \sum_{n\geq 1}(-1)^n n^{-s}, \qquad \Re s > 0.$$

The implementation is obvious:

```
ZETA(s) = sumalt(n=1, (-1)^n*n^(-s)) / (2^(1-s) - 1)
```

Note that `n` is automatically lexically scoped to the `sumalt` "loop", so that it is unnecessary to add a `my(n)` declaration to the function body. Surprisingly, this gives very good accuracy in a larger region than expected:

```
? check = z -> ZETA(z) / zeta(z);
? check(2)
%1 = 1.0000000000000000000000000000
? check(200)
%2 = 1.0000000000000000000000000000
? check(0)
%3 = 0.99999999999999999999999999994
? check(-5)
%4 = 1.0000000000000000007549266557
? check(-11)
%5 = 0.99997526410478824902660847745
? check(1/2+14.134*I)   \\ very close to a non-trivial zero
%6 = 1.0000000000000000000003747432 + 7.62329066 E-21*I
? check(-1+10*I)
%7 = 1.0000000000000000000000002511 + 2.989950968 E-24*I
```

Now wait a minute; not only are we summing a series which is certainly no longer alternating (it has complex coefficients), but we are also way outside of the region of convergence, and still get decent results! No programming mistake this time: `sumalt` is a "magic" function*, providing very good convergence acceleration; in effect, we are computing the analytic continuation of our original function. To convince ourselves that `sumalt` is a non-trivial implementation, let us try a simpler example:

```
? sum(n=1, 10^7, (-1)^n/n, 0.) / (-log(2)) \\ approximates the well-known formula
time = 7,417 ms.
%1 = 0.99999992786525156228934054 57
? sumalt(n=1, (-1)^n/n) / (-log(2))          \\ accurate and fast
time = 0 ms.
%2 = 1.0000000000000000000000000000
```

No, we are not using a powerful simplification tool here, only numerical computations. Remember, PARI is not a computer algebra system!

---

* `sumalt` is heuristic, but its use can be rigorously justified for a given function, in particular our $\zeta(s)$ formula. Indeed, Peter Borwein (*An efficient algorithm for the Riemann zeta function*, CMS Conf. Proc. **27** (2000), pp. 29–34) proved that the formula used in `sumalt` with n terms computes $(1-2^{1-s})\zeta(s)$ with a relative error of the order of $(3+\sqrt{8})^{-n}|\Gamma(s)|^{-1}$.

2.7.3 Beware scopes. Be extra careful with the scopes of variables. What is wrong with the following definition?

```
FirstPrimeDiv(x) =
{ my(p);
  forprime(p=2, x, if (x%p == 0, break));
  p
}
? FirstPrimeDiv(10)
%1 = 0
```

Hint. The function body is equivalent to

```
{ my(newp = 0);
  forprime(p=2, x, if (x%p == 0, break));
  newp
}
```

Detailed explanation. The index `p` in the `forprime` loop is lexically scoped to the loop and is not visible to the outside world. Hence, it will not survive the `break` statement. More precisely, at this point the loop index is restored to its preceding value. The initial `my(p)`, although well-meant, adds to the confusion: it indeed scopes `p` to the function body, with initial value 0, but the `forprime` loop introduces *another* variable, unfortunately also called `p`, scoped to the loop body, which shadows the one we wanted. So we always return 0, since the value of the `p` scoped to the function body never changes and is initially 0.

To sum up, the routine returns the `p` declared local to it, not the one which was local to `forprime` and ran through consecutive prime numbers. Here is a corrected version:

```
? FirstPrimeDiv(x) = forprime(p=2, x, if (x%p == 0, return(p)))
```

2.7.4 Recursive functions. Recursive functions can easily be written as long as one pays proper attention to variable scope. Here is an example, used to retrieve the coefficient array of a multivariate polynomial (a non-trivial task due to PARI's unsophisticated representation for those objects):

```
coeffs(P, nbvar) =
{
  if (type(P) != "t_POL",
    for (i=1, nbvar, P = [P]);
    return (P)
  );
  vector(poldegree(P)+1, i, coeffs(polcoeff(P, i-1), nbvar-1))
}
```

If P is a polynomial in k variables, show that after the assignment `v = coeffs(P,k)`, the coefficient of $x_1^{n_1} \dots x_k^{n_k}$ in P is given by `v[`n_1`+1][...][`n_k`+1]`.

The operating system automatically limits the recursion depth:

```
? dive(n) = dive(n+1)
? dive(0);
```

```
***   [...] at: dive(n+1)
***             ^---------
***   in function dive: dive(n+1)
***                     ^---------
\\ (last 2 lines repeated 19 times)
***   deep recursion.
```

There is no way to increase the recursion limit (which may be different on your machine) from within `gp`. To increase it before launching `gp`, you can use `ulimit` or `limit`, depending on your shell, and raise the process available stack space (increase `stacksize`).

2.7.5 Function which take functions as parameters. This is done as follows:

```
? calc(f, x) = f(x)
? calc(sin, Pi)
%2 = -5.04870979 E-29
? g(x) = x^2;
? calc(g, 3)
%4 = 9
```

If we do not need `g` elsewhere, we should use an anonymous function here, `calc(x->x^2, 3)`. Here is a variation:

```
? funs = [cos, sin, tan, x->x^3+1]; \\ an array of functions
? call(i, x) = funs[i](x)
```

evaluates the appropriate function on argument `x`, provided $1 \leq i \leq 4$. Finally, a more useful example:

```
APPLY(f, v) = vector(#v, i, f(v[i]))
```

applies the function `f` to every element in the vector `v`. (The built-in function `apply` is more powerful since it also applies to lists and matrices.)

2.7.6 Defining functions within a function. Defining a single function is easy:

```
init(x) = (add = y -> x+y);
```

Basically, we are defining a global variable `add` whose value is the function `y->x+y`. The parentheses were added for clarity and are not mandatory.

```
? init(5);
? add(2)
%2 = 7
```

A more refined approach is to avoid global variables and *return* the function:

```
init(x) = y -> x+y
add = init(5)
```

Then `add(2)` still returns 7, as expected! Of course, if `add` is in global scope, there is no gain, but we can lexically scope it to the place where it is useful:

```
  my ( add = init(5) );
```

How about multiple functions then? We can use the last idea and return a vector of functions, but if we insist on global variables? The first idea

```
init(x) = add(y) = x+y; mul(y) = x*y;
```

does not work since in the construction `f() =` *seq*, the function body contains everything until the end of the expression. Hence executing `init` defines the wrong function `add` (itself defining a function `mul`). The way out is to use parentheses for grouping, so that enclosed subexpressions will be evaluated independently:

```
? init(x) = ( add(y) = x+y ); ( mul(y) = x*y );
? init(5);
? add(2)
%3 = 7
? mul(3)
%4 = 15
```

This defines two global functions which have access to the lexical variables private to `init`! The following would work in exactly the same way:

```
? init5() = my(x = 5); ( add(y) = x+y ); ( mul(y) = x*y );
```

2.7.7 Closures as Objects. Contrary to what you might think after the preceding examples, GP's closures may not be used to simulate true "objects", with private and public parts and methods to access and manipulate them. In fact, closures indeed incorporate an existing context (they may access lexical variables that existed at the time of their definition), but then may not change it. More precisely, they access a copy, which they are welcome to change, but a further function call still accesses the original context, as it existed at the time the function was defined:

```
init() =
{ my(count = 0);
  inc()=count++;
  dec()=count--;
}
? inc()
%1 = 1
? inc()
%2 = 1
? inc()
%3 = 1
```

2.8 Member functions.

Member functions use the 'dot' notation to retrieve information from complicated structures. The built-in structures are *bid*, *ell*, *galois*, *ff*, *nf*, *bnf*, *bnr* and *prid*, which will be described at length in Chapter 3. The syntax `structure.member` is taken to mean: retrieve `member` from `structure`, e.g. `E.j` returns the j-invariant of the elliptic curve `E`, or outputs an error message if `E` is not a proper *ell* structure. To define your own member functions, use the syntax

var`.`*member* `=` *seq*,

where the formal variable *var* is scoped to the function body *seq*. This is of course reminiscent of a user function with a single formal variable *var*. For instance, the current implementation of the `ell` type is a vector, the j-invariant being the thirteenth component. It could be implemented as

```
x.j =
{
  if (type(x) != "t_VEC" || #x < 14, error("not an elliptic curve: " x));
  x[13]
}
```

As for user functions, you can redefine your member functions simply by typing new definitions. On the other hand, as a safety measure, you cannot redefine the built-in member functions, so attempting to redefine `x.j` as above would in fact produce an error; you would have to call it e.g. `x.myj` in order for `gp` to accept it.

Rationale. In most cases, member functions are simple accessors of the form

```
x.a = x[1];
x.b = x[2];
x.c = x[3];
```

where `x` is a vector containing relevant data. There are at least three alternative approaches to the above member functions: 1) hardcode `x[1]`, etc. in the program text, 2) define constant global variables `AINDEX = 1`, `BINDEX = 2` and hardcode `x[AINDEX]`, 3) user functions `a(x) = x[1]` and so on.

Even if 2) improves on 1), these solutions are neither elegant nor flexible, and they scale badly. 3) is a genuine possibility, but the main advantage of member functions is that their namespace is independent from the variables (and functions) namespace, hence we can use very short identifiers without risk. The j-invariant is a good example: it would clearly not be a good idea to define `j(E) = E[13]`, because clashes with loop indices are likely.

Note. Typing `\um` will output all user-defined member functions.

Member function names. A valid name starts with a letter followed by any number of keyword characters: `_` or alphanumeric characters (`[A-Za-z0-9]`). The built-in member function names are reserved and cannot be used (see the list with `?.`). Finally, names starting with `e` or `E` followed by a digit are forbidden, due to a clash with the floating point exponent notation: we understand `1.e2` as 100.000..., not as extracting member `e2` of object `1`.

2.9 Strings and Keywords.

2.9.1 Strings. GP variables can hold values of type character string (internal type `t_STR`). This section describes how they are actually used, as well as some convenient tricks (automatic concatenation and expansion, keywords) valid in string context.

As explained above, the general way to input a string is to enclose characters between quotes `"`. This is the only input construct where whitespace characters are significant: the string will contain the exact number of spaces you typed in. Besides, you can "escape" characters by putting a `\` just before them; the translation is as follows

```
\e: <Escape>
\n: <Newline>
\t: <Tab>
```

For any other character x, $\backslash x$ is expanded to x. In particular, the only way to put a " into a string is to escape it. Thus, for instance, `"\"a\""` would produce the string whose content is "a". This is definitely *not* the same thing as typing `"a"`, whose content is merely the one-letter string a.

You can concatenate two strings using the `concat` function. If either argument is a string, the other is automatically converted to a string if necessary (it will be evaluated first).

```
? concat("ex", 1+1)
%1 = "ex2"
? a = 2; b = "ex"; concat(b, a)
%2 = "ex2"
? concat(a, b)
%3 = "2ex"
```

Some functions expect strings for some of their arguments: `print` would be an obvious example, `Str` is a less obvious but useful one (see the end of this section for a complete list). While typing in such an argument, you will be said to be in *string context*. The rest of this section is devoted to special syntactical tricks which can be used with such arguments (and only here; you will get an error message if you try these outside of string context):

• Writing two strings alongside one another will just concatenate them, producing a longer string. Thus it is equivalent to type in `"a " "b"` or `"a b"`. A little tricky point in the first expression: the first whitespace is enclosed between quotes, and so is part of a string; while the second (before the `"b"`) is completely optional and `gp` actually suppresses it, as it would with any number of whitespace characters at this point (i.e. outside of any string).

• If you insert any expression when a string is expected, it gets "expanded": it is evaluated as a standard GP expression, and the final result (as would have been printed if you had typed it by itself) is then converted to a string, as if you had typed it directly. For instance `"a" 1+1 "b"` is equivalent to `"a2b"`: three strings get created, the middle one being the expansion of `1+1`, and these are then concatenated according to the rule described above. Another tricky point here: assume you did not assign a value to `aaa` in a GP expression before. Then typing `aaa` by itself in a string context will actually produce the correct output (i.e. the string whose content is aaa), but in a fortuitous way. This `aaa` gets expanded to the monomial of degree one in the variable `aaa`, which is of course printed as `aaa`, and thus will expand to the three letters you were expecting.

Warning. Expression involving strings are not handled in a special way; even in string context, the largest possible expression is evaluated, hence `print("a"[1])` is incorrect since `"a"` is not an object whose first component can be extracted. On the other hand `print("a", [1])` is correct (two distinct argument, each converted to a string), and so is `print("a" 1)` (since `"a"1` is not a valid expression, only `"a"` gets expanded, then `1`, and the result is concatenated as explained above).

2.9.2 Keywords. Since there are cases where expansion is not desirable, we now distinguish between "Keywords" and "Strings". String is what has been described so far. Keywords are special relatives of Strings which are automatically assumed to be quoted, whether you actually type in the quotes or not. Thus expansion is never performed on them. They get concatenated, though. The analyzer supplies automatically the quotes you have "forgotten" and treats Keywords just as normal strings otherwise. For instance, if you type `"a"b+b` in Keyword context, you will get the string whose contents are ab+b. In String context, on the other hand, you would get a2*b.

All GP functions have prototypes (described in Chapter 3 below) which specify the types of arguments they expect: either generic PARI objects (GEN), or strings, or keywords, or unevaluated

expression sequences. In the keyword case, only a very small set of words will actually be meaningful (the `default` function is a prominent example).

Reference. The arguments of the following functions are processed in string context:

`Str`
`addhelp` (second argument)
`default` (second argument)
`error`
`extern`
`plotstring` (second argument)
`plotterm` (first argument)
`read` and `readvec`
`system`
all the `print`*xxx* functions
all the `write`*xxx* functions

The arguments of the following functions are processed as keywords:

`alias`
`default` (first argument)
`install` (all arguments but the last)
`trap` (first argument)
`whatnow`

2.9.3 Useful example. The function `Str` converts its arguments into strings and concatenate them. Coupled with `eval`, it is very powerful. The following example creates generic matrices:

```
? genmat(u,v,s="x") = matrix(u,v,i,j, eval( Str(s,i,j) ))
? genmat(2,3) + genmat(2,3,"m")
%1 =
[x11 + m11 x12 + m12 x13 + m13]
[x21 + m21 x22 + m22 x23 + m23]
```

2.10 Errors and error recovery.

2.10.1 Errors. Your input program is first compiled to a more efficient bytecode; then the latter is evaluated, calling appropriate functions from the PARI library. Accordingly, there are two kind of errors: syntax errors produced by the compiler, and runtime errors produced by the PARI library either by the evaluator itself, or in a mathematical function. Both kinds are fatal to your computation: `gp` will report the error and perform some cleanup (restore variables modified while evaluating the erroneous command, close open files, reclaim unused memory, etc.).

At this point, the default is to return to the usual prompt, but if the `recover` option (Section 3.17.36) is off then `gp` exits immediately. This can be useful for batch-mode operation to make untrapped errors fatal.

When reporting a *syntax error*, `gp` gives meaningful context by copying (part of) the expression it was trying to compile, indicating where the error occurred with a caret `^-`, as in

```
? factor()
  ***   too few arguments: factor()
  ***                              ^-
```

```
? 1+
  ***   syntax error, unexpected $end: 1+
  ***                                    ^-
```

possibly enlarged to a full arrow given enough trailing context

```
? if (isprime(1+, do_something())
  ***   syntax error, unexpected ',': if(isprime(1+,do_something()))
  ***                                              ^----------------
```

These error messages may be mysterious, because gp cannot guess what you were trying to do, and the error may occur once gp has been sidetracked. The first error is straightforward: `factor` has one mandatory argument, which is missing.

The other two are simple typos involving an ill-formed addition `1 +` missing its second operand. The error messages differ because the parsing context is slightly different: in the first case we reach the end of input (`$end`) while still expecting a token, and in the second one, we received an unexpected token (the comma).

Here is a more complicated one:

```
? factor(x
  ***   syntax error, unexpected $end, expecting )-> or ',' or ')': factor(x
  ***                                                                      ^-
```

The error is a missing parenthesis, but from gp's point of view, you might as well have intended to give further arguments to `factor` (this is possible and useful, see the description of the function). In fact gp expected either a closing parenthesis, or a second argument separated from the first by a comma. And this is essentially what the error message says: we reached the end of the input (`$end`) while expecting a `')'` or a `','`.

Actually, a third possibility is mentioned in the error message `)->`, which could never be valid in the above context, but a subexpression like `(x)->sin(x)`, defining an inline closure would be valid, and the parser is not clever enough to rule that out, so we get the same message as in

```
? (x
  ***   syntax error, unexpected $end, expecting )-> or ',' or ')': (x
  ***                                                               ^-
```

where all three proposed continuations would be valid.

Runtime errors from the evaluator are nicer because they answer a correctly worded query, otherwise the bytecode compiler would have protested first; here is a slightly pathological case:

```
? if (siN(x) < eps, do_something())
  ***   at top-level: if(siN(x)<eps,do_someth
  ***                    ^--------------------
  ***   not a function in function call
```

(no arrow!) The code is syntactically correct and compiled correctly, even though the `siN` function, a typo for `sin`, was not defined at this point. When trying to evaluate the bytecode, however, it turned out that `siN` is still undefined so we cannot evaluate the function call `siN(x)`.

Library runtime errors are even nicer because they have more mathematical content, which is easier to grasp than a parser's logic:

```
? 1/Mod(2,4)
```

```
***   at top-level: 1/Mod(2,4)
***                  ^---------
*** _/_: impossible inverse in Fp_inv: Mod(2, 4).
```

telling us that a runtime error occurred while evaluating the binary `/` operator (the _ surrounding the operator are placeholders), more precisely the `Fp_inv` library function was fed the argument `Mod(2,4)` and could not invert it. More context is provided if the error occurs deep in the call chain:

```
? f(x) = 1/x;
? g(N) = for(i = -N, N, f(i + O(5)));
? g(10)
  ***   at top-level: g(10)
  ***                 ^-----
  ***   in function g: for(i=-N,N,f(i))
  ***                               ^-----
  ***   in function f: 1/x
  ***                   ^--
  *** _/_: impossible inverse in ginv: O(5).
```

In this example, the debugger reports (at least) 3 enclosed frames: last (innermost) is the body of user function f, the body of g, and the top-level (global scope). In fact, the `for` loop in g's body defines an extra frame, since there exist variables scoped to the loop body.

2.10.2 Error recovery.

It is annoying to wait for a program to finish and find out the hard way that there was a mistake in it (like the division by 0 above), sending you back to the prompt. First you may lose some valuable intermediate data. Also, correcting the error may not be obvious; you might have to change your program, adding a number of extra statements and tests to narrow down the problem.

A different situation, still related to error recovery, is when you actually foresee that some error may occur, are unable to prevent it, but quite capable of recovering from it, given the chance. Examples include lazy factorization, where you knowingly use a pseudo prime N as if it were prime; you may then encounter an "impossible" situation, but this would usually exhibit a factor of N, enabling you to refine the factorization and go on. Or you might run an expensive computation at low precision to guess the size of the output, hence the right precision to use. You can then encounter errors like "precision loss in truncation", e.g when trying to convert `1E1000`, known to 28 digits of accuracy, to an integer; or "division by 0", e.g inverting `0E1000` when all accuracy has been lost, and no significant digit remains. It would be enough to restart part of the computation at a slightly higher precision.

We now describe *error trapping*, a useful mechanism which alleviates much of the pain in the first situation (the break loop debugger), and provides satisfactory ways out of the second one (the `iferr` exception handler).

2.10.3 Break loop.

A *break loop* is a special debugging mode that you enter whenever a user interrupt (`Control-C`) or runtime error occurs, freezing the `gp` state, and preventing cleanup until you get out of the loop. By runtime error, we mean an error from the evaluator, the library or a user error (from `error`), *not* syntax errors. When a break loop starts, a prompt is issued (`break>`). You can type in a `gp` command, which is evaluated when you hit the `<Return>` key, and the result is printed as during the main `gp` loop, except that no history of results is kept. Then the break loop prompt reappears and you can type further commands as long as you do not exit the loop. If you are using readline, the history of commands is kept, and line editing is available as usual. If you type in a command that results in an error, you are sent back to the break loop prompt: errors do *not* terminate the loop.

To get out of a break loop, you can use `next`, `break`, `return`, or type `C-d` (`EOF`), any of which will let `gp` perform its usual cleanup, and send you back to the `gp` prompt. Note that `C-d` is slightly dangerous, since typing it *twice* will not only send you back to the `gp` prompt, but to your shell prompt! (Since `C-d` at the `gp` prompt exits the gp session.)

If the break loop was started by a user interrupt `Control-C`, and not by an error, inputting an empty line, i.e hitting the `<Return>` key at the `break>` prompt, resumes the temporarily interrupted computation. A single empty line has no effect in case of a fatal error, to avoid getting get out of the loop prematurely, thereby losing valuable debugging data. Any of `next`, `break`, `return`, or `C-d` will abort the computation and send you back to the `gp` prompt as above.

Break loops are useful as a debugging tool. You may inspect the values of `gp` variables to understand why an error occurred, or change `gp`'s state in the middle of a computation (increase debugging level, start storing results in a log file, set variables to different values...): hit `C-c`, type in your modifications, then let the computation go on as explained above. A break loop looks like this:

```
? v = 0; 1/v
  ***   at top-level: v=0;1/v
  ***                       ^--
  *** _/_: impossible inverse in gdiv: 0.
  ***   Break loop (type 'break' to go back to the GP prompt)
break>
```

So the standard error message is printed first. The `break>` at the bottom is a prompt, and hitting `v` then `<Return>`, we see:

```
break> v
0
```

explaining the problem. We could have typed any `gp` command, not only the name of a variable, of course. Lexically-scoped variables are accessible to the evaluator during the break loop:

```
? for(v = -2, 2, print(1/v))
-1/2
-1
  ***   at top-level: for(v=-2,2,print(1/v))
  ***                                 ^----
  *** _/_: impossible inverse in gdiv: 0.
  ***   Break loop (type 'break' to go back to the GP prompt)
```

```
break> v
0
```

Even though loop indices are automatically lexically scoped and no longer exist when the break loop is run, enough debugging information is retained in the bytecode to reconstruct the evaluation context. Of course, when the error occurs in a nested chain of user function calls, lexically scoped variables are available only in the corresponding frame:

```
? f(x) = 1/x;
? g(x) = for(i = 1, 10, f(x+i));
? for(j = -5,5, g(j))
  ***   at top-level: for(j=-5,5,g(j))
  ***                               ^-----
  ***   in function g: for(i=1,10,f(x+i))
  ***                                ^-------
  ***   in function f: 1/x
  ***                    ^--
  *** _/_: impossible inverse in gdiv: 0.
  ***   Break loop: type 'break' to go back to GP prompt
break> [i,j,x]     \\  the x in f's body.
[i, j, 0]
break> dbg_up      \\  go up one frame
  ***   at top-level: for(j=-5,5,g(j))
  ***                               ^-----
  ***   in function g: for(i=1,10,f(x+i))
  ***                                ^-------
break> [i,j,x]      \\  the x in g's body, i in the for loop.
[5, j, -5]
```

The following GP commands are available during a break loop to help debugging:

`dbg_up`(n): go up n frames, as seen above.

`dbg_down`(n): go down n frames, cancelling previous `dbg_up`'s.

`dbg_x`(t): examine t, as `\x` but more flexible.

`dbg_err`(): returns the current error context `t_ERROR`. The error components often provide useful additional information:

```
? O(2) + O(3)
  ***   at top-level: O(2)+O(3)
  ***                         ^-----
  *** _+_: inconsistent addition t_PADIC + t_PADIC.
  ***   Break loop: type 'break' to go back to GP prompt
break> E = dbg_err()
error("inconsistent addition t_PADIC + t_PADIC.")
break> Vec(E)
["e_OP", "+", O(2), O(3)]
```

Note. The debugger is enabled by default, and fires up as soon as a runtime error occurs. If you do not like this behavior, you may disable it by setting the default `breakloop` to 0 in for `gprc`. A runtime error will send you back to the prompt. Note that the break loop is automatically disabled when running `gp` in non interactive mode, i.e. when the program's standard input is not attached to a terminal.

Technical Note. When you enter a break loop due to a PARI stack overflow, the PARI stack is reset so that you can run commands. Otherwise the stack would immediately overflow again! Still, as explained above, you do not lose the value of any `gp` variable in the process.

2.10.4 Protecting code. The expression

`iferr(`*statements*`, ERR, `*recovery*`)`

evaluates and returns the value of *statements*, unless an error occurs during the evaluation in which case the value of *recovery* is returned. As in an if/else clause, with the difference that *statements* has been partially evaluated, with possible side effects. We shall give a lot more details about the `ERR` argument shortly; it is the name of a variable, lexically scoped to the *recovery* expression sequence, whose value is set by the exception handler to help the recovery code decide what to do about the error.

For instance one can define a fault tolerant inversion function as follows:

```
? inv(x) = iferr(1/x, ERR, "oo")    \\ ERR is unused...
? for (i=-1,1, print(inv(i)))
-1
oo
1
```

Protected codes can be nested without adverse effect. Let's now see how `ERR` can be used; as written, `inv` is too tolerant:

```
? inv("blah")
%2 = "oo"
```

Let's improve it by checking that we caught a "division by 0" exception, and not an unrelated one like the type error `1 / "blah"`.

```
? inv2(x) = {
  iferr(1/x,
        ERR, if (errname(ERR) != "e_INV", error(ERR)); "oo")
}
? inv2(0)
%3 = "oo"  \\ as before
? inv2("blah")
  ***   at top-level: inv2("blah")
  ***                 ^------------
  ***   in function inv2: ...f(errname(ERR)!="e_INV",error(ERR));"oo")
  ***                                                ^-----------------
  *** error: forbidden division t_INT / t_STR.
```

In the `inv2("blah")` example, the error type was not expected, so we rethrow the exception: `error(ERR)` triggers the original error that we mistakenly trapped. Since the recovery code should

always check whether the error is the one expected, this construction is very common and can be simplified to

```
? inv3(x) = iferr(1/x,
                  ERR, "oo",
                  errname(ERR) == "e_INV")
```

More generally

`iferr(`*statements*`, ERR,` *recovery*`,` *predicate*`)`

only catches the exception if *predicate* (allowed to check various things about `ERR`, not only its name) is non-zero.

Rather than trapping everything, then rethrowing whatever we do not like, we advise to only trap errors of a specific kind, as above. Of course, sometimes, one just want to trap *everything* because we do not know what to expect. The following function check whether `install` works correctly in your `gp`:

```
broken_install() =
{ \\ can we install?
  iferr(install(addii,GG),
        ERR, return ("OS"));
  \\ can we use the installed function?
  iferr(if (addii(1,1) != 2, return("BROKEN")),
        ERR, return("USE"));
  return (0);
}
```

The function returns `OS` if the operating system does not support `install`, `USE` if using an installed function triggers an error, `BROKEN` if the installed function did not behave as expected, and 0 if everything works.

The `ERR` formal parameter contains more useful data than just the error name, which we recovered using `errname(ERR)`. In fact, a `t_ERROR` object usually has extra components, which can be accessed as `component(ERR,1)`, `component(ERR,2)`, and so on. Or globally by casting the error to a `t_VEC`: `Vec(ERR)` returns the vector of all components at once. See Section 3.14.17 for the list of all exception types, and the corresponding contents of `ERR`.

2.11 Interfacing GP with other languages.

The PARI library was meant to be interfaced with C programs. This specific use is dealt with extensively in the *User's guide to the PARI library*. Of course, `gp` itself provides a convenient interpreter to execute rather intricate scripts (see Section 3.14).

Scripts, when properly written, tend to be shorter and clearer than C programs, and are certainly easier to write, maintain or debug. You don't need to deal with memory management, garbage collection, pointers, declarations, and so on. Because of their intrinsic simplicity, they are more robust as well. They are unfortunately somewhat slower. Thus their use will remain complementary: it is suggested that you test and debug your algorithms using scripts, before actually coding them in C if speed is paramount. The GP2C compiler often eases this part.

The `install` command (see Section 3.15.24) efficiently imports foreign functions for use under `gp`, which can of course be written using other libraries than PARI. Thus you may code only critical parts of your program in C, and still maintain most of the program as a GP script.

We are aware of three PARI-related Free Software packages to embed PARI in other languages. We *neither endorse nor support* any of them, but you may want to give them a try if you are familiar with the languages they are based on. The first is the Python-based SAGE system (`http://sagemath.org/`). The second is the `Math::Pari` Perl module (see any CPAN mirror), written by Ilya Zakharevich. Finally, Michael Stoll and Sam Steingold have integrated PARI into `CLISP` (`http://clisp.cons.org/`), a Common Lisp implementation.

These provide interfaces to `gp` functions for use in `python`, `perl`, or `Lisp` programs, respectively.

2.12 Defaults.

There are many internal variables in `gp`, defining how the system will behave in certain situations, unless a specific override has been given. Most of them are a matter of basic customization (colors, prompt) and will be set once and for all in your preferences file (see Section 2.14), but some of them are useful interactively (set timer on, increase precision, etc.).

The function used to manipulate these values is called `default`, which is described in Section 3.15.8. The basic syntax is

`default(`*def*`,` *value*`)`,

which sets the default *def* to *value*. In interactive use, most of these can be abbreviated using `gp` metacommands (mostly, starting with `\`), which we shall describe in the next section.

Available defaults are described in the reference guide, Section 3.17, the most important one being `parisizemax`. Just be aware that typing `default` by itself will list all of them, as well as their current values (see `\d`).

Note. The suffixes `k`, `M` or `G` can be appended to a *value* which is a numeric argument, with the effect of multiplying it by 10^3, 10^6 and 10^9 respectively. Case is not taken into account there, so for instance `30k` and `30K` both stand for 30000. This is mostly useful to modify or set the defaults `parisize` and `parisizemax` which typically involve a lot of trailing zeroes.

(somewhat technical) Note. As we saw in Section 2.9, the second argument to `default` is subject to string context expansion, which means you can use run-time values. In other words, something like

```
a = 3;
default(logfile, "file" a ".log")
```

logs the output in `file3.log`.

Some special defaults, corresponding to file names and prompts, expand further the resulting value at the time they are set. Two kinds of expansions may be performed:

• **time expansion**: the string is sent through the library function `strftime`. This means that `%`*char* combinations have a special meaning, usually related to the time and date. For instance, `%H` = hour (24-hour clock) and `%M` = minute [00,59] (on a Unix system, you can try `man strftime` at your shell prompt to get a complete list). This is applied to `prompt`, `psfile`, and `logfile`. For instance,

```
default(prompt,"(%H:%M) ? ")
```

will prepend the time of day, in the form (*hh*:*mm*) to gp's usual prompt.

• **environment expansion**: When the string contains a sequence of the form $*SOMEVAR*, e.g. `$HOME`, the environment is searched and if *SOMEVAR* is defined, the sequence is replaced by the corresponding value. Also the `~` symbol has the same meaning as in many shells — `~` by itself stands for your home directory, and `~user` is expanded to `user`'s home directory. This is applied to all file names.

2.13 Simple metacommands.

Simple metacommands are meant as shortcuts and should not be used in GP scripts (see Section 3.14). Beware that these, as all of gp input, are *case sensitive*. For example, `\Q` is not identical to `\q`. In the following list, braces are used to denote optional arguments, with their default values when applicable, e.g. $\{n = 0\}$ means that if n is not there, it is assumed to be 0. Whitespace (or spaces) between the metacommand and its arguments and within arguments is optional. (This can cause problems only with `\w`, when you insist on having a file name whose first character is a digit, and with `\r` or `\w`, if the file name itself contains a space. In such cases, just use the underlying `read` or `write` function; see Section 3.15.58).

2.13.1 `?`{*command*}. The gp on-line help interface. If you type `?`n where n is a number from 1 to 11, you will get the list of functions in Section 3.n of the manual (the list of sections being obtained by simply typing `?`).

These names are in general not informative enough. More details can be obtained by typing `?`*function*, which gives a short explanation of the function's calling convention and effects. Of course, to have complete information, read Chapter 3 of this manual (the source code is at your disposal as well, though a trifle less readable).

If the line before the copyright message indicates that extended help is available (this means `perl` is present on your system and the PARI distribution was correctly installed), you can add more `?` signs for extended functionality:

`??` *keyword* yields the function description as it stands in this manual, usually in Chapter 2 or 3. If you're not satisfied with the default chapter chosen, you can impose a given chapter by ending the keyword with `@` followed by the chapter number, e.g. `?? Hello@2` will look in Chapter 2 for section heading `Hello` (which doesn't exist, by the way).

All operators (e.g. `+`, `&&`, etc.) are accepted by this extended help, as well as a few other keywords describing key gp concepts, e.g. `readline` (the line editor), `integer`, `nf` ("number field" as used in most algebraic number theory computations), `ell` (elliptic curves), etc.

In case of conflicts between *function* and *default* names (e.g `log`, `simplify`), the function has higher priority. To get the *default* help, use

```
?? default(log)
?? default(simplify)
```

`???` *pattern* produces a list of sections in Chapter 3 of the manual related to your query. As before, if *pattern* ends by `@` followed by a chapter number, that chapter is searched instead; you also have the option to append a simple `@` (without a chapter number) to browse through the whole manual.

If your query contains dangerous characters (e.g `?` or blanks) it is advisable to enclose it within double quotes, as for GP strings (e.g `??? "elliptic curve"`).

Note that extended help is much more powerful than the short help, since it knows about operators as well: you can type `?? *` or `?? &&`, whereas a single `?` would just yield a not too helpful

```
&&: unknown identifier.}
```

message. Also, you can ask for extended help on section number n in Chapter 3, just by typing `??` n (where `?`n would yield merely a list of functions). Finally, a few key concepts in `gp` are documented in this way: metacommands (e.g `?? "??"`), defaults (e.g `?? psfile`) and type names (e.g `t_INT` or `integer`), as well as various miscellaneous keywords such as `edit` (short summary of line editor commands), `operator`, `member`, `"user defined"`, `nf`, `ell`, ...

Last but not least: `??` without argument will open a `dvi` previewer (`xdvi` by default, `$GPXDVI` if it is defined in your environment) containing the full user's manual. `??tutorial` and `??refcard` do the same with the tutorial and reference card respectively.

Technical note. This functionality is provided by an external `perl` script that you are free to use outside any `gp` session (and modify to your liking, if you are perl-knowledgeable). It is called `gphelp`, lies in the `doc` subdirectory of your distribution (just make sure you run `Configure` first, see Appendix A) and is really two programs in one. The one which is used from within `gp` is `gphelp` which runs TEX on a selected part of this manual, then opens a previewer. `gphelp -detex` is a text mode equivalent, which looks often nicer especially on a colour-capable terminal (see `misc/gprc.dft` for examples). The default `help` selects which help program will be used from within `gp`. You are welcome to improve this help script, or write new ones (and we would like to know about it so that we may include them in future distributions). By the way, outside of `gp` you can give more than one keyword as argument to `gphelp`.

2.13.2 `/*...*/`. A comment. Everything between the stars is ignored by `gp`. These comments can span any number of lines.

2.13.3 `\\`. A one-line comment. The rest of the line is ignored by `gp`.

2.13.4 `\a` $\{n\}$. Prints the object number n (%n) in raw format. If the number n is omitted, print the latest computed object (%).

2.13.5 `\c`. Prints the list of all available hardcoded functions under `gp`, not including operators written as special symbols (see Section 2.4). More information can be obtained using the `?` metacommand (see above). For user-defined functions / member functions, see `\u` and `\um`.

2.13.6 `\d`. Prints the defaults as described in the previous section (shortcut for `default()`, see Section 3.15.8).

2.13.7 `\e` $\{n\}$. Switches the `echo` mode on (1) or off (0). If n is explicitly given, set echo to n.

2.13.8 `\g` $\{n\}$. Sets the debugging level `debug` to the non-negative integer n.

2.13.9 `\gf` $\{n\}$. Sets the file usage debugging level `debugfiles` to the non-negative integer n.

2.13.10 `\gm` {n}. Sets the memory debugging level `debugmem` to the non-negative integer n.

2.13.11 `\h` {m-n}. Outputs some debugging info about the hashtable. If the argument is a number n, outputs the contents of cell n. Ranges can be given in the form m-n (from cell m to cell n, $ = last cell). If a function name is given instead of a number or range, outputs info on the internal structure of the hash cell this function occupies (a `struct entree` in C). If the range is reduced to a dash ('-'), outputs statistics about hash cell usage.

2.13.12 `\l` {*logfile*}. Switches `log` mode on and off. If a *logfile* argument is given, change the default logfile name to *logfile* and switch log mode on.

2.13.13 `\m`. As `\a`, but using prettymatrix format.

2.13.14 `\o` {n}. Sets `output` mode to n (0: raw, 1: prettymatrix, 3: external prettyprint).

2.13.15 `\p` {n}. Sets `realprecision` to n decimal digits. Prints its current value if n is omitted.

2.13.16 `\pb` {n}. Sets `realbitprecision` to n bits. Prints its current value if n is omitted.

2.13.17 `\ps` {n}. Sets `seriesprecision` to n significant terms. Prints its current value if n is omitted.

2.13.18 `\q`. Quits the `gp` session and returns to the system. Shortcut for `quit()` (see Section 3.15.44).

2.13.19 `\r` {*filename*}. Reads into `gp` all the commands contained in the named file as if they had been typed from the keyboard, one line after the other. Can be used in combination with the `\w` command (see below). Related but not equivalent to the function `read` (see Section 3.15.45); in particular, if the file contains more than one line of input, there will be one history entry for each of them, whereas `read` would only record the last one. If *filename* is omitted, re-read the previously used input file (fails if no file has ever been successfully read in the current session). If a `gp binary file` (see Section 3.15.60) is read using this command, it is silently loaded, without cluttering the history.

Assuming `gp` figures how to decompress files on your machine, this command accepts compressed files in `compress`ed (`.Z`) or `gzip`ped (`.gz` or `.z`) format. They will be uncompressed on the fly as `gp` reads them, without changing the files themselves.

2.13.20 `\s`. Prints the state of the PARI *stack* and *heap*. This is used primarily as a debugging device for PARI.

2.13.21 `\t`. Prints the internal longword format of all the PARI types. The detailed bit or byte format of the initial codeword(s) is explained in Chapter 4, but its knowledge is not necessary for a `gp` user.

2.13.22 `\u`. Prints the definitions of all user-defined functions.

2.13.23 `\um`. Prints the definitions of all user-defined member functions.

2.13.24 `\v`. Prints the version number and implementation architecture (680x0, Sparc, Alpha, other) of the `gp` executable you are using.

2.13.25 `\w` {n} {*filename*}. Writes the object number n (%n) into the named file, in raw format. If the number n is omitted, writes the latest computed object (%). If *filename* is omitted, appends to `logfile` (the GP function `write` is a trifle more powerful, as you can have arbitrary file names).

2.13.26 `\x` {n}. Prints the complete tree with addresses and contents (in hexadecimal) of the internal representation of the object number n (%n). If the number n is omitted, uses the latest computed object in `gp`. As for `\s`, this is used primarily as a debugging device for PARI, and the format should be self-explanatory. The underlying GP function `dbg_x` is more versatile, since it can be applied to other objects than history entries.

2.13.27 `\y` {n}. Switches `simplify` on (1) or off (0). If n is explicitly given, set simplify to n.

2.13.28 `#`. Switches the `timer` on or off.

2.13.29 `##`. Prints the time taken by the latest computation. Useful when you forgot to turn on the `timer`.

2.14 The preferences file.

This file, called `gprc` in the sequel, is used to modify or extend `gp` default behavior, in all `gp` sessions: e.g customize `default` values or load common user functions and aliases. `gp` opens the `gprc` file and processes the commands in there, *before* doing anything else, e.g. creating the PARI stack. If the file does not exist or cannot be read, `gp` will proceed to the initialization phase at once, eventually emitting a prompt. If any explicit command line switches are given, they override the values read from the preferences file.

2.14.1 Syntax. The syntax in the `gprc` file (and valid in this file only) is simple-minded, but should be sufficient for most purposes. The file is read line by line; as usual, white space is ignored unless surrounded by quotes and the standard multiline constructions using braces, `\`, or `=` are available (multiline comments between `/* ... */` are also recognized).

2.14.1.1 Preprocessor:. Two types of lines are first dealt with by a preprocessor:

• comments are removed. This applies to all text surrounded by `/* ... */` as well as to everything following `\\` on a given line.

• lines starting with `#if` *boolean* are treated as comments if *boolean* evaluates to `false`, and read normally otherwise. The condition can be negated using either `#if not` (or `#if !`). If the rest of the current line is empty, the test applies to the next line (same behavior as `=` under `gp`). The following tests can be performed:

`EMACS`: `true` if `gp` is running in an Emacs or TeXmacs shell (see Section 2.16).

`READL`: `true` if `gp` is compiled with `readline` support (see Section 2.15).

`VERSION` *op number*: where *op* is in the set $\{>, <, <=, >=\}$, and *number* is a PARI version number of the form *Major.Minor.patch*, where the last two components can be omitted (i.e. 1 is understood as version 1.0.0). This is `true` if `gp`'s version number satisfies the required inequality.

`BITS_IN_LONG ==` *number*: *number* is 32 (resp. 64). This is `true` if `gp` was built for a 32-bit (resp. 64-bit) architecture.

2.14.1.2 Commands:. After preprocessing, the remaining lines are executed as sequence of expressions (as usual, separated by `;` if necessary). Only two kinds of expressions are recognized:

• *default* = *value*, where *default* is one of the available defaults (see Section 2.12), which will be set to *value* on actual startup. Don't forget the quotes around strings (e.g. for `prompt` or `help`).

• `read "`*some_GP_file*`"` where *some_GP_file* is a regular GP script this time, which will be read just before gp prompts you for commands, but after initializing the defaults. In particular, file input is delayed until the `gprc` has been fully loaded. This is the right place to input files containing `alias` commands, or your favorite macros.

For instance you could set your prompt in the following portable way:

```
\\ self modifying prompt looking like (18:03) gp >
prompt   = "(%H:%M) \e[1mgp\e[m > "

\\ readline wants non-printing characters to be braced between ^A/^B pairs
#if READL prompt = "(%H:%M) ^A\e[1m^Bgp^A\e[m^B > "

\\ escape sequences not supported under emacs
#if EMACS prompt = "(%H:%M) gp > "
```

Note that any of the last two lines could be broken in the following way

```
#if EMACS
  prompt = "(%H:%M) gp > "
```

since the preprocessor directive applies to the next line if the current one is empty.

A sample `gprc` file called `misc/gprc.dft` is provided in the standard distribution. It is a good idea to have a look at it and customize it to your needs. Since this file does not use multiline constructs, here is one (note the terminating `;` to separate the expressions):

```
#if VERSION > 2.2.3
{
  read "my_scripts";     \\ syntax errors in older versions
  new_galois_format = 1; \\ default introduced in 2.2.4
}
#if ! EMACS
{
  colors = "9, 5, no, no, 4, 1, 2";
  help   = "gphelp -detex -ch 4 -cb 0 -cu 2";
}
```

2.14.2 The gprc location. When gp is started, it looks for a customization file, or `gprc` in the following places (in this order, only the first one found will be loaded):

• gp checks whether the environment variable `GPRC` is set. On Unix, this can be done with something like:

```
GPRC=/my/dir/anyname; export GPRC   in sh syntax (for instance in your .profile),
setenv GPRC /my/dir/anyname         in csh syntax (in your .login or .cshrc file).
env GPRC=/my/dir/anyname gp         on the command line launching gp.
```

If so, the file named by `$GPRC` is the `gprc`.

• If `GPRC` is not set, and if the environment variable `HOME` is defined, gp then tries

`$HOME/.gprc` on a Unix system

`$HOME\gprc.txt` on a DOS, OS/2, or Windows system.

• If no gprc was found among the user files mentioned above we look for `/etc/gprc` for a system-wide gprc file (you will need root privileges to set up such a file yourself).

• Finally, we look in pari's `datadir` for a file named

`.gprc` on a Unix system

`gprc.txt` on a DOS, OS/2, or Windows system. If you are using our Windows installer, this is where the default preferences file is written.

Note that on Unix systems, the `gprc`'s default name starts with a '.' and thus is hidden to regular `ls` commands; you need to type `ls -a` to list it.

2.15 Using readline.

This very useful library provides line editing and contextual completion to `gp`. You are encouraged to read the `readline` user manual, but we describe basic usage here.

A (too) short introduction to readline. In the following, `C-` stands for "the `Control` key combined with another" and the same for `M-` with the `Meta` key; generally `C-` combinations act on characters, while the `M-` ones operate on words. The `Meta` key might be called `Alt` on some keyboards, will display a black diamond on most others, and can safely be replaced by `Esc` in any case.

Typing any ordinary key inserts text where the cursor stands, the arrow keys enabling you to move in the line. There are many more movement commands, which will be familiar to the Emacs user, for instance `C-a/C-e` will take you to the start/end of the line, `M-b/M-f` move the cursor backward/forward by a word, etc. Just press the `<Return>` key at any point to send your command to `gp`.

All the commands you type at the `gp` prompt are stored in a history, a multiline command being saved as a single concatenated line. The Up and Down arrows (or `C-p/C-n`) will move you through the history, `M-</M->` sending you to the start/end of the history. `C-r/C-s` will start an incremental backward/forward search. You can kill text (`C-k` kills till the end of line, `M-d` to the end of current word) which you can then yank back using the `C-y` key (`M-y` will rotate the kill-ring). `C-_` will undo your last changes incrementally (`M-r` undoes all changes made to the current line). `C-t` and `M-t` will transpose the character (word) preceding the cursor and the one under the cursor.

Keeping the `M-` key down while you enter an integer (a minus sign meaning reverse behavior) gives an argument to your next readline command (for instance `M-- C-k` will kill text back to the start of line). If you prefer Vi–style editing, `M-C-j` will toggle you to Vi mode.

Of course you can change all these default bindings. For that you need to create a file named `.inputrc` in your home directory. For instance (notice the embedding conditional in case you would want specific bindings for `gp`):

```
$if Pari-GP
  set show-all-if-ambiguous
  "\C-h": backward-delete-char
  "\e\C-h": backward-kill-word
```

```
  "\C-xd": dump-functions
  (: "\C-v()\C-b"        # can be annoying when copy-pasting!
  [: "\C-v[]\C-b"
$endif
```

`C-x C-r` will re-read this init file, incorporating any changes made to it during the current session.

Note. By default, `(` and `[` are bound to the function `pari-matched-insert` which, if "electric parentheses" are enabled (default: off) will automatically insert the matching closure (respectively `)` and `]`). This behavior can be toggled on and off by giving the numeric argument -2 to `(` (`M--2(`), which is useful if you want, e.g to copy-paste some text into the calculator. If you do not want a toggle, you can use `M--0` / `M--1` to specifically switch it on or off).

Note. In some versions of readline (2.1 for instance), the `Alt` or `Meta` key can give funny results (output 8-bit accented characters for instance). If you do not want to fall back to the `Esc` combination, put the following two lines in your `.inputrc`:

```
set convert-meta on
set output-meta off
```

Command completion and online help. Hitting `<TAB>` will complete words for you. This mechanism is context-dependent: `gp` will strive to only give you meaningful completions in a given context (it will fail sometimes, but only under rare and restricted conditions).

For instance, shortly after a `~`, we expect a user name, then a path to some file. Directly after `default(` has been typed, we would expect one of the `default` keywords. After a '.', we expect a member keyword. And generally of course, we expect any GP symbol which may be found in the hashing lists: functions (both yours and GP's), and variables.

If, at any time, only one completion is meaningful, `gp` will provide it together with

• an ending comma if we are completing a default,

• a pair of parentheses if we are completing a function name. In that case hitting `<TAB>` again will provide the argument list as given by the online help. (Recall that you can always undo the effect of the preceding keys by hitting `C-_`; this applies here.)

Otherwise, hitting `<TAB>` once more will give you the list of possible completions. Just experiment with this mechanism as often as possible, you will probably find it very convenient. For instance, you can obtain `default(seriesprecision,10)`, just by hitting `def<TAB>se<TAB>10`, which saves 18 keystrokes (out of 27).

Hitting `M-h` will give you the usual short online help concerning the word directly beneath the cursor, `M-H` will yield the extended help corresponding to the `help` default program (usually opens a dvi previewer, or runs a primitive tex-to-ASCII program). None of these disturb the line you were editing.

2.16 GNU Emacs and PariEmacs.

If you install the PariEmacs package (see Appendix A), you may use `gp` as a subprocess in Emacs. You then need to include in your `.emacs` file the following lines:

```
(autoload 'gp-mode "pari" nil t)
(autoload 'gp-script-mode "pari" nil t)
(autoload 'gp "pari" nil t)
(autoload 'gpman "pari" nil t)

(setq auto-mode-alist
  (cons '("\\.gp$" . gp-script-mode) auto-mode-alist))
```

which autoloads functions from the PariEmacs package and ensures that file with the `.gp` suffix are edited in gp-script mode.

Once this is done, under GNU Emacs if you type `M-x gp` (where as usual `M` is the `Meta` key), a special shell will be started launching `gp` with the default stack size and prime limit. You can then work as usual under `gp`, but with all the facilities of an advanced text editor. See the PariEmacs documentation for customizations, menus, etc.

Chapter 3:
Functions and Operations Available in PARI and GP

The functions and operators available in PARI and in the GP/PARI calculator are numerous and ever-expanding. Here is a description of the ones available in version 2.9.2. It should be noted that many of these functions accept quite different types as arguments, but others are more restricted. The list of acceptable types will be given for each function or class of functions. Except when stated otherwise, it is understood that a function or operation which should make natural sense is legal.

On the other hand, many routines list explicit preconditions for some of their argument, e.g. p is a prime number, or q is a positive definite quadratic form. For reasons of efficiency, all trust the user input and only perform minimal sanity checks. When a precondition is not satisfied, any of the following may occur: a regular exception is raised, the PARI stack overflows, a `SIGSEGV` or `SIGBUS` signal is generated, or we enter an infinite loop. The function can also quietly return a mathematically meaningless result: junk in, junk out.

In this chapter, we will describe the functions according to a rough classification. The general entry looks something like:

foo(x, {*flag* = 0}): short description.

The library syntax is `GEN foo(GEN x, long fl = 0)`.

This means that the GP function `foo` has one mandatory argument x, and an optional one, *flag*, whose default value is 0. (The {} should not be typed, it is just a convenient notation we will use throughout to denote optional arguments.) That is, you can type `foo(x,2)`, or `foo(x)`, which is then understood to mean `foo(x,0)`. As well, a comma or closing parenthesis, where an optional argument should have been, signals to GP it should use the default. Thus, the syntax `foo(x,)` is also accepted as a synonym for our last expression. When a function has more than one optional argument, the argument list is filled with user supplied values, in order. When none are left, the defaults are used instead. Thus, assuming that `foo`'s prototype had been

$$\mathbf{foo}(\{x = 1\}, \{y = 2\}, \{z = 3\}),$$

typing in `foo(6,4)` would give you `foo(6,4,3)`. In the rare case when you want to set some far away argument, and leave the defaults in between as they stand, you can use the "empty arg" trick alluded to above: `foo(6,,1)` would yield `foo(6,2,1)`. By the way, `foo()` by itself yields `foo(1,2,3)` as was to be expected.

In this rather special case of a function having no mandatory argument, you can even omit the (): a standalone `foo` would be enough (though we do not recommend it for your scripts, for the sake of clarity). In defining GP syntax, we strove to put optional arguments at the end of the argument list (of course, since they would not make sense otherwise), and in order of decreasing usefulness so that, most of the time, you will be able to ignore them.

Finally, an optional argument (between braces) followed by a star, like $\{x\}*$, means that any number of such arguments (possibly none) can be given. This is in particular used by the various `print` routines.

Flags. A *flag* is an argument which, rather than conveying actual information to the routine, instructs it to change its default behavior, e.g. return more or less information. All such flags are optional, and will be called *flag* in the function descriptions to follow. There are two different kind of flags

• generic: all valid values for the flag are individually described ("If *flag* is equal to 1, then..."),

• binary: use customary binary notation as a compact way to represent many toggles with just one integer. Let $(p_0, \ldots, p_n)$ be a list of switches (i.e. of properties which take either the value 0 or 1), the number $2^3 + 2^5 = 40$ means that p_3 and p_5 are set (that is, set to 1), and none of the others are (that is, they are set to 0). This is announced as "The binary digits of *flag* mean 1: p_0, 2: p_1, 4: p_2", and so on, using the available consecutive powers of 2.

Mnemonics for flags. Numeric flags as mentioned above are obscure, error-prone, and quite rigid: should the authors want to adopt a new flag numbering scheme (for instance when noticing flags with the same meaning but different numeric values across a set of routines), it would break backward compatibility. The only advantage of explicit numeric values is that they are fast to type, so their use is only advised when using the calculator `gp`.

As an alternative, one can replace a numeric flag by a character string containing symbolic identifiers. For a generic flag, the mnemonic corresponding to the numeric identifier is given after it as in

```
fun(x, {flag = 0} ):

  If flag is equal to 1 = AGM, use an agm formula ...
```

which means that one can use indifferently `fun(`x`, 1)` or `fun(`x`, "AGM")`.

For a binary flag, mnemonics corresponding to the various toggles are given after each of them. They can be negated by prepending `no_` to the mnemonic, or by removing such a prefix. These toggles are grouped together using any punctuation character (such as ',' or ';'). For instance (taken from description of `ploth`$(X = a, b, expr, \{flag = 0\}, \{n = 0\})$)

Binary digits of flags mean: 1 = `Parametric`, 2 = `Recursive`, ...

so that, instead of 1, one could use the mnemonic `"Parametric; no_Recursive"`, or simply `"Parametric"` since `Recursive` is unset by default (default value of *flag* is 0, i.e. everything unset). People used to the bit-or notation in languages like C may also use the form `"Parametric | no_Recursive"`.

Pointers. If a parameter in the function prototype is prefixed with a & sign, as in

foo$(x, \&e)$

it means that, besides the normal return value, the function may assign a value to e as a side effect. When passing the argument, the & sign has to be typed in explicitly. As of version 2.9.2, this *pointer* argument is optional for all documented functions, hence the & will always appear between brackets as in `Z_issquare`$(x, \{\&e\})$.

About library programming. The *library* function `foo`, as defined at the beginning of this section, is seen to have two mandatory arguments, x and *flag*: no function seen in the present chapter has been implemented so as to accept a variable number of arguments, so all arguments are mandatory when programming with the library (usually, variants are provided corresponding to the various flag values). We include an `= default value` token in the prototype to signal how a missing argument should be encoded. Most of the time, it will be a `NULL` pointer, or -1 for a variable number. Refer to the *User's Guide to the PARI library* for general background and details.

3.1 Standard monadic or dyadic operators.

3.1.1 `+/-`. The expressions `+`x and `-`x refer to monadic operators (the first does nothing, the second negates x).

The library syntax is `GEN gneg(GEN x)` for `-`x.

3.1.2 `+`. The expression x `+` y is the sum of x and y. Addition between a scalar type x and a `t_COL` or `t_MAT` y returns respectively $[y[1]+x, y[2], \ldots]$ and $y + x\mathrm{Id}$. Other additions between a scalar type and a vector or a matrix, or between vector/matrices of incompatible sizes are forbidden.

The library syntax is `GEN gadd(GEN x, GEN y)`.

3.1.3 `-`. The expression x `-` y is the difference of x and y. Subtraction between a scalar type x and a `t_COL` or `t_MAT` y returns respectively $[y[1]-x, y[2], \ldots]$ and $y - x\mathrm{Id}$. Other subtractions between a scalar type and a vector or a matrix, or between vector/matrices of incompatible sizes are forbidden.

The library syntax is `GEN gsub(GEN x, GEN y)` for x `-` y.

3.1.4 `*`. The expression x `*` y is the product of x and y. Among the prominent impossibilities are multiplication between vector/matrices of incompatible sizes, between a `t_INTMOD` or `t_PADIC` Restricted to scalars, `*` is commutative; because of vector and matrix operations, it is not commutative in general.

Multiplication between two `t_VEC`s or two `t_COL`s is not allowed; to take the scalar product of two vectors of the same length, transpose one of the vectors (using the operator `~` or the function `mattranspose`, see Section 3.11) and multiply a line vector by a column vector:

```
? a = [1,2,3];
? a * a
  ***   at top-level: a*a
  ***                  ^--
  *** _*_: forbidden multiplication t_VEC * t_VEC.
? a * a~
%2 = 14
```

If x, y are binary quadratic forms, compose them; see also `qfbnucomp` and `qfbnupow`. If x, y are `t_VECSMALL` of the same length, understand them as permutations and compose them.

The library syntax is `GEN gmul(GEN x, GEN y)` for x `*` y. Also available is `GEN gsqr(GEN x)` for x `*` x.

3.1.5 `/`. The expression x `/` y is the quotient of x and y. In addition to the impossibilities for multiplication, note that if the divisor is a matrix, it must be an invertible square matrix, and in that case the result is $x * y^{-1}$. Furthermore note that the result is as exact as possible: in particular, division of two integers always gives a rational number (which may be an integer if the quotient is exact) and *not* the Euclidean quotient (see x `\` y for that), and similarly the quotient of two polynomials is a rational function in general. To obtain the approximate real value of the quotient of two integers, add `0.` to the result; to obtain the approximate p-adic value of the quotient of two integers, add `O(p^k)` to the result; finally, to obtain the Taylor series expansion of the quotient of two polynomials, add `O(X^k)` to the result or use the `taylor` function (see Section 3.10.48).

The library syntax is `GEN gdiv(GEN x, GEN y)` for x `/` y.

3.1.6 \. The expression x `\` y is the Euclidean quotient of x and y. If y is a real scalar, this is defined as `floor(`x/y`)` if $y > 0$, and `ceil(`x/y`)` if $y < 0$ and the division is not exact. Hence the remainder x `-` `(`x`\`y`)*`y is in $[0, |y|[$.

Note that when y is an integer and x a polynomial, y is first promoted to a polynomial of degree 0. When x is a vector or matrix, the operator is applied componentwise.

The library syntax is `GEN gdivent(GEN x, GEN y)` for x `\` y.

3.1.7 \/. The expression x `\/` y evaluates to the rounded Euclidean quotient of x and y. This is the same as x `\` y except for scalar division: the quotient is such that the corresponding remainder is smallest in absolute value and in case of a tie the quotient closest to $+\infty$ is chosen (hence the remainder would belong to $]-|y|/2, |y|/2]$).

When x is a vector or matrix, the operator is applied componentwise.

The library syntax is `GEN gdivround(GEN x, GEN y)` for x `\/` y.

3.1.8 %. The expression x `%` y evaluates to the modular Euclidean remainder of x and y, which we now define. When x or y is a non-integral real number, x`%`y is defined as x `-` `(`x`\`y`)*`y. Otherwise, if y is an integer, this is the smallest non-negative integer congruent to x modulo y. (This actually coincides with the previous definition if and only if x is an integer.) If y is a polynomial, this is the polynomial of smallest degree congruent to x modulo y. For instance:

```
? (1/2) % 3
%1 = 2
? 0.5 % 3
%2 = 0.50000000000000000000000000000
? (1/2) % 3.0
%3 = 1/2
```

Note that when y is an integer and x a polynomial, y is first promoted to a polynomial of degree 0. When x is a vector or matrix, the operator is applied componentwise.

The library syntax is `GEN gmod(GEN x, GEN y)` for x `%` y.

3.1.9 ^. The expression x^n is powering.

• If the exponent n is an integer, then exact operations are performed using binary (left-shift) powering techniques. If x is a p-adic number, its precision will increase if $v_p(n) > 0$. Powering a binary quadratic form (types `t_QFI` and `t_QFR`) returns a representative of the class, which is always reduced if the input was. (In particular, `x ^1` returns x itself, whether it is reduced or not.)

PARI is able to rewrite the multiplication $x * x$ of two *identical* objects as x^2, or `sqr(`x`)`. Here, identical means the operands are two different labels referencing the same chunk of memory; no equality test is performed. This is no longer true when more than two arguments are involved.

• If the exponent n is not an integer, powering is treated as the transcendental function $\exp(n \log x)$, and in particular acts componentwise on vector or matrices, even square matrices ! (See Section 3.3.)

• As an exception, if the exponent is a rational number p/q and x an integer modulo a prime or a p-adic number, return a solution y of $y^q = x^p$ if it exists. Currently, q must not have large prime factors. Beware that

```
? Mod(7,19)^(1/2)
%1 = Mod(11, 19) /* is any square root */
? sqrt(Mod(7,19))
%2 = Mod(8, 19)  /* is the smallest square root */
? Mod(7,19)^(3/5)
%3 = Mod(1, 19)
? %3^(5/3)
%4 = Mod(1, 19)  /* Mod(7,19) is just another cubic root */
```

• If the exponent is a negative integer, an inverse must be computed. For non-invertible `t_INTMOD` x, this will fail and implicitly exhibit a non trivial factor of the modulus:

```
? Mod(4,6)^(-1)
  ***   at top-level: Mod(4,6)^(-1)
  ***                           ^-----
  *** _^_: impossible inverse modulo: Mod(2, 6).
```

(Here, a factor 2 is obtained directly. In general, take the gcd of the representative and the modulus.) This is most useful when performing complicated operations modulo an integer N whose factorization is unknown. Either the computation succeeds and all is well, or a factor d is discovered and the computation may be restarted modulo d or N/d.

For non-invertible `t_POLMOD` x, the behaviour is the same:

```
? Mod(x^2, x^3-x)^(-1)
  ***   at top-level: Mod(x^2,x^3-x)^(-1)
  ***                                 ^-----
  *** _^_: impossible inverse in RgXQ_inv: Mod(x^2, x^3 - x).
```

Note that the underlying algorihm (subresultant) assumes the base ring is a domain:

```
? a = Mod(3*y^3+1, 4); b = y^6+y^5+y^4+y^3+y^2+y+1; c = Mod(a,b);
? c^(-1)
  ***   at top-level: Mod(a,b)^(-1)
  ***                           ^-----
  *** _^_: impossible inverse modulo: Mod(2, 4).
```

In fact c is invertible, but $\mathbf{Z}/4\mathbf{Z}$ is not a domain and the algorithm fails. It is possible for the algorithm to succeed in such situations and any returned result will be correct, but chances are an error will occur first. In this specific case, one should work with 2-adics. In general, one can also try the following approach

```
? inversemod(a, b) =
{ my(m, v = variable(b));
  m = polsylvestermatrix(polrecip(a), polrecip(b));
  m = matinverseimage(m, matid(#m)[,1]);
  Polrev(m[1..poldegree(b)], v);
}
? inversemod(a,b)
%2 = Mod(2,4)*y^5 + Mod(3,4)*y^3 + Mod(1,4)*y^2 + Mod(3,4)*y + Mod(2,4)
```

This is not guaranteed to work either since `matinverseimage` must also invert pivots. See Section 3.11.

For a t_MAT x, the matrix is expected to be square and invertible, except in the special case x^(-1) which returns a left inverse if one exists (rectangular x with full column rank).

```
? x = Mat([1;2])
%1 =
[1]

[2]

? x^(-1)
%2 =
[1 0]
```

The library syntax is GEN gpow(GEN x, GEN n, long prec) for $x\hat{}n$.

3.1.10 cmp(x, y). Gives the result of a comparison between arbitrary objects x and y (as -1, 0 or 1). The underlying order relation is transitive, the function returns 0 if and only if $x === y$, and its restriction to integers coincides with the customary one. Besides that, it has no useful mathematical meaning.

In case all components are equal up to the smallest length of the operands, the more complex is considered to be larger. More precisely, the longest is the largest; when lengths are equal, we have matrix > vector > scalar. For example:

```
? cmp(1, 2)
%1 = -1
? cmp(2, 1)
%2 = 1
? cmp(1, 1.0)   \\ note that 1 == 1.0, but (1===1.0) is false.
%3 = -1
? cmp(x + Pi, [])
%4 = -1
```

This function is mostly useful to handle sorted lists or vectors of arbitrary objects. For instance, if v is a vector, the construction vecsort(v, cmp) is equivalent to Set(v).

The library syntax is GEN cmp_universal(GEN x, GEN y).

3.1.11 divrem$(x, y, \{v\})$. Creates a column vector with two components, the first being the Euclidean quotient (x \ y), the second the Euclidean remainder (x - (x\y)*y), of the division of x by y. This avoids the need to do two divisions if one needs both the quotient and the remainder. If v is present, and x, y are multivariate polynomials, divide with respect to the variable v.

Beware that divrem(x,y)[2] is in general not the same as x % y; no GP operator corresponds to it:

```
? divrem(1/2, 3)[2]
%1 = 1/2
? (1/2) % 3
%2 = 2
? divrem(Mod(2,9), 3)[2]
 ***   at top-level: divrem(Mod(2,9),3)[2
 ***                 ^--------------------
 ***   forbidden division t_INTMOD \ t_INT.
```

```
? Mod(2,9) % 6
%3 = Mod(2,3)
```

The library syntax is `GEN divrem(GEN x, GEN y, long v = -1)` where `v` is a variable number. Also available is `GEN gdiventres(GEN x, GEN y)` when v is not needed.

3.1.12 lex(x, y). Gives the result of a lexicographic comparison between x and y (as -1, 0 or 1). This is to be interpreted in quite a wide sense: It is admissible to compare objects of different types (scalars, vectors, matrices), provided the scalars can be compared, as well as vectors/matrices of different lengths. The comparison is recursive.

In case all components are equal up to the smallest length of the operands, the more complex is considered to be larger. More precisely, the longest is the largest; when lengths are equal, we have matrix > vector > scalar. For example:

```
? lex([1,3], [1,2,5])
%1 = 1
? lex([1,3], [1,3,-1])
%2 = -1
? lex([1], [[1]])
%3 = -1
? lex([1], [1]~)
%4 = 0
```

The library syntax is `GEN lexcmp(GEN x, GEN y)`.

3.1.13 max(x, y). Creates the maximum of x and y when they can be compared.

The library syntax is `GEN gmax(GEN x, GEN y)`.

3.1.14 min(x, y). Creates the minimum of x and y when they can be compared.

The library syntax is `GEN gmin(GEN x, GEN y)`.

3.1.15 powers$(x, n, \{x0\})$. For non-negative n, return the vector with $n + 1$ components $[1, x, \ldots, x^n]$ if `x0` is omitted, and $[x_0, x_0 * x, ..., x_0 * x^n]$ otherwise.

```
? powers(Mod(3,17), 4)
%1 = [Mod(1, 17), Mod(3, 17), Mod(9, 17), Mod(10, 17), Mod(13, 17)]
? powers(Mat([1,2;3,4]), 3)
%2 = [[1, 0; 0, 1], [1, 2; 3, 4], [7, 10; 15, 22], [37, 54; 81, 118]]
? powers(3, 5, 2)
%3 = [2, 6, 18, 54, 162, 486]
```

When $n < 0$, the function returns the empty vector `[]`.

The library syntax is `GEN gpowers0(GEN x, long n, GEN x0 = NULL)`. Also available is `GEN gpowers(GEN x, long n)` when `x0` is `NULL`.

3.1.16 shift(x, n). Shifts x componentwise left by n bits if $n \geq 0$ and right by $|n|$ bits if $n < 0$. May be abbreviated as x `<<` n or x `>>` $(-n)$. A left shift by n corresponds to multiplication by 2^n. A right shift of an integer x by $|n|$ corresponds to a Euclidean division of x by $2^{|n|}$ with a remainder of the same sign as x, hence is not the same (in general) as $x\backslash 2^n$.

The library syntax is `GEN gshift(GEN x, long n)`.

3.1.17 shiftmul(x, n). Multiplies x by 2^n. The difference with `shift` is that when $n < 0$, ordinary division takes place, hence for example if x is an integer the result may be a fraction, while for shifts Euclidean division takes place when $n < 0$ hence if x is an integer the result is still an integer.

The library syntax is `GEN gmul2n(GEN x, long n)`.

3.1.18 sign(x). sign (0, 1 or -1) of x, which must be of type integer, real or fraction; `t_QUAD` with positive discriminants and `t_INFINITY` are also supported.

The library syntax is `GEN gsigne(GEN x)`.

3.1.19 vecmax$(x, \{\&v\})$. If x is a vector or a matrix, returns the largest entry of x, otherwise returns a copy of x. Error if x is empty.

If v is given, set it to the index of a largest entry (indirect maximum), when x is a vector. If x is a matrix, set v to coordinates $[i, j]$ such that $x[i, j]$ is a largest entry. This flag is ignored if x is not a vector or matrix.

```
? vecmax([10, 20, -30, 40])
%1 = 40
? vecmax([10, 20, -30, 40], &v); v
%2 = 4
? vecmax([10, 20; -30, 40], &v); v
%3 = [2, 2]
```

The library syntax is `GEN vecmax0(GEN x, GEN *v = NULL)`. When v is not needed, the function `GEN vecmax(GEN x)` is also available.

3.1.20 vecmin$(x, \{\&v\})$. If x is a vector or a matrix, returns the smallest entry of x, otherwise returns a copy of x. Error if x is empty.

If v is given, set it to the index of a smallest entry (indirect minimum), when x is a vector. If x is a matrix, set v to coordinates $[i, j]$ such that $x[i, j]$ is a smallest entry. This is ignored if x is not a vector or matrix.

```
? vecmin([10, 20, -30, 40])
%1 = -30
? vecmin([10, 20, -30, 40], &v); v
%2 = 3
? vecmin([10, 20; -30, 40], &v); v
%3 = [2, 1]
```

The library syntax is `GEN vecmin0(GEN x, GEN *v = NULL)`. When v is not needed, the function `GEN vecmin(GEN x)` is also available.

3.1.21 Comparison and Boolean operators. The six standard comparison operators `<=`, `<`, `>=`, `>`, `==`, `!=` are available in GP. The result is 1 if the comparison is true, 0 if it is false. The operator `==` is quite liberal : for instance, the integer 0, a 0 polynomial, and a vector with 0 entries are all tested equal.

The extra operator `===` tests whether two objects are identical and is much stricter than `==` : objects of different type or length are never identical.

For the purpose of comparison, `t_STR` objects are compared using the standard lexicographic order, and comparing them to objects of a different type raises an exception.

GP accepts `<>` as a synonym for `!=`. On the other hand, `=` is definitely *not* a synonym for `==`: it is the assignment statement.

The standard boolean operators `||` (inclusive or), `&&` (and) and `!` (not) are also available.

3.2 Conversions and similar elementary functions or commands.

Many of the conversion functions are rounding or truncating operations. In this case, if the argument is a rational function, the result is the Euclidean quotient of the numerator by the denominator, and if the argument is a vector or a matrix, the operation is done componentwise. This will not be restated for every function.

3.2.1 Col$(x, \{n\})$. Transforms the object x into a column vector. The dimension of the resulting vector can be optionally specified via the extra parameter n.

If n is omitted or 0, the dimension depends on the type of x; the vector has a single component, except when x is

• a vector or a quadratic form (in which case the resulting vector is simply the initial object considered as a row vector),

• a polynomial or a power series. In the case of a polynomial, the coefficients of the vector start with the leading coefficient of the polynomial, while for power series only the significant coefficients are taken into account, but this time by increasing order of degree. In this last case, `Vec` is the reciprocal function of `Pol` and `Ser` respectively,

• a matrix (the column of row vector comprising the matrix is returned),

• a character string (a vector of individual characters is returned).

In the last two cases (matrix and character string), n is meaningless and must be omitted or an error is raised. Otherwise, if n is given, 0 entries are appended at the end of the vector if $n > 0$, and prepended at the beginning if $n < 0$. The dimension of the resulting vector is $|n|$.

Note that the function `Colrev` does not exist, use `Vecrev`.

The library syntax is `GEN gtocol0(GEN x, long n)`. `GEN gtocol(GEN x)` is also available.

3.2.2 Colrev$(x, \{n\})$. As `Col`$(x, -n)$, then reverse the result. In particular, `Colrev` is the reciprocal function of `Polrev`: the coefficients of the vector start with the constant coefficient of the polynomial and the others follow by increasing degree.

The library syntax is `GEN gtocolrev0(GEN x, long n)`. `GEN gtocolrev(GEN x)` is also available.

3.2.3 List($\{x = []\}$). Transforms a (row or column) vector x into a list, whose components are the entries of x. Similarly for a list, but rather useless in this case. For other types, creates a list with the single element x. Note that, except when x is omitted, this function creates a small memory leak; so, either initialize all lists to the empty list, or use them sparingly.

The library syntax is `GEN gtolist(GEN x = NULL)`. The variant `GEN mklist(void)` creates an empty list.

3.2.4 Map($\{x\}$). A "Map" is an associative array, or dictionary: a data type composed of a collection of (*key*, *value*) pairs, such that each key appears just once in the collection. This function converts the matrix $[a_1, b_1; a_2, b_2; \ldots; a_n, b_n]$ to the map $a_i \mapsto b_i$.

```
? M = Map(factor(13!));
? mapget(M,3)
%2 = 5
```

If the argument x is omitted, creates an empty map, which may be filled later via `mapput`.

The library syntax is `GEN gtomap(GEN x = NULL)`.

3.2.5 Mat($\{x = []\}$). Transforms the object x into a matrix. If x is already a matrix, a copy of x is created. If x is a row (resp. column) vector, this creates a 1-row (resp. 1-column) matrix, *unless* all elements are column (resp. row) vectors of the same length, in which case the vectors are concatenated sideways and the attached big matrix is returned. If x is a binary quadratic form, creates the attached 2×2 matrix. Otherwise, this creates a 1×1 matrix containing x.

```
? Mat(x + 1)
%1 =
[x + 1]
? Vec( matid(3) )
%2 = [[1, 0, 0]~, [0, 1, 0]~, [0, 0, 1]~]
? Mat(%)
%3 =
[1 0 0]

[0 1 0]

[0 0 1]
? Col( [1,2; 3,4] )
%4 = [[1, 2], [3, 4]]~
? Mat(%)
%5 =
[1 2]

[3 4]
? Mat(Qfb(1,2,3))
%6 =
[1 1]

[1 3]
```

The library syntax is `GEN gtomat(GEN x = NULL)`.

3.2.6 Mod(a, b). In its basic form, creates an intmod or a polmod $(a \bmod b)$; b must be an integer or a polynomial. We then obtain a `t_INTMOD` and a `t_POLMOD` respectively:

```
? t = Mod(2,17); t^8
%1 = Mod(1, 17)
? t = Mod(x,x^2+1); t^2
%2 = Mod(-1, x^2+1)
```

If $a\%b$ makes sense and yields a result of the appropriate type (`t_INT` or scalar/`t_POL`), the operation succeeds as well:

```
? Mod(1/2, 5)
%3 = Mod(3, 5)
? Mod(7 + O(3^6), 3)
%4 = Mod(1, 3)
? Mod(Mod(1,12), 9)
%5 = Mod(1, 3)
? Mod(1/x, x^2+1)
%6 = Mod(-1, x^2+1)
? Mod(exp(x), x^4)
%7 = Mod(1/6*x^3 + 1/2*x^2 + x + 1, x^4)
```

If a is a complex object, "base change" it to $\mathbf{Z}/b\mathbf{Z}$ or $K[x]/(b)$, which is equivalent to, but faster than, multiplying it by `Mod(1,b)`:

```
? Mod([1,2;3,4], 2)
%8 =
[Mod(1, 2) Mod(0, 2)]

[Mod(1, 2) Mod(0, 2)]
? Mod(3*x+5, 2)
%9 = Mod(1, 2)*x + Mod(1, 2)
? Mod(x^2 + y*x + y^3, y^2+1)
%10 = Mod(1, y^2 + 1)*x^2 + Mod(y, y^2 + 1)*x + Mod(-y, y^2 + 1)
```

This function is not the same as x `%` y, the result of which has no knowledge of the intended modulus y. Compare

```
? x = 4 % 5; x + 1
%1 = 5
? x = Mod(4,5); x + 1
%2 = Mod(0,5)
```

Note that such "modular" objects can be lifted via `lift` or `centerlift`. The modulus of a `t_INTMOD` or `t_POLMOD` z can be recovered via z`.mod`.

The library syntax is `GEN gmodulo(GEN a, GEN b)`.

3.2.7 Pol$(t, \{v =' x\})$. Transforms the object t into a polynomial with main variable v. If t is a scalar, this gives a constant polynomial. If t is a power series with non-negative valuation or a rational function, the effect is similar to `truncate`, i.e. we chop off the $O(X^k)$ or compute the Euclidean quotient of the numerator by the denominator, then change the main variable of the result to v.

The main use of this function is when t is a vector: it creates the polynomial whose coefficients are given by t, with $t[1]$ being the leading coefficient (which can be zero). It is much faster to evaluate `Pol` on a vector of coefficients in this way, than the corresponding formal expression $a_nX^n + \ldots + a_0$, which is evaluated naively exactly as written (linear versus quadratic time in n). `Polrev` can be used if one wants $x[1]$ to be the constant coefficient:

```
? Pol([1,2,3])
%1 = x^2 + 2*x + 3
? Polrev([1,2,3])
%2 = 3*x^2 + 2*x + 1
```

The reciprocal function of `Pol` (resp. `Polrev`) is `Vec` (resp. `Vecrev`).

```
? Vec(Pol([1,2,3]))
%1 = [1, 2, 3]
? Vecrev( Polrev([1,2,3]) )
%2 = [1, 2, 3]
```

Warning. This is *not* a substitution function. It will not transform an object containing variables of higher priority than v.

```
? Pol(x + y, y)
  ***   at top-level: Pol(x+y,y)
  ***                 ^----------
  *** Pol: variable must have higher priority in gtopoly.
```

The library syntax is `GEN gtopoly(GEN t, long v = -1)` where `v` is a variable number.

3.2.8 Polrev$(t, \{v =' x\})$. Transform the object t into a polynomial with main variable v. If t is a scalar, this gives a constant polynomial. If t is a power series, the effect is identical to `truncate`, i.e. it chops off the $O(X^k)$.

The main use of this function is when t is a vector: it creates the polynomial whose coefficients are given by t, with $t[1]$ being the constant term. `Pol` can be used if one wants $t[1]$ to be the leading coefficient:

```
? Polrev([1,2,3])
%1 = 3*x^2 + 2*x + 1
? Pol([1,2,3])
%2 = x^2 + 2*x + 3
```

The reciprocal function of `Pol` (resp. `Polrev`) is `Vec` (resp. `Vecrev`).

The library syntax is `GEN gtopolyrev(GEN t, long v = -1)` where `v` is a variable number.

3.2.9 Qfb$(a, b, c, \{D = 0.\})$. Creates the binary quadratic form $ax^2 + bxy + cy^2$. If $b^2 - 4ac > 0$, initialize Shanks' distance function to D. Negative definite forms are not implemented, use their positive definite counterpart instead.

The library syntax is `GEN Qfb0(GEN a, GEN b, GEN c, GEN D = NULL, long prec)`. Also available are `GEN qfi(GEN a, GEN b, GEN c)` (assumes $b^2 - 4ac < 0$) and `GEN qfr(GEN a, GEN b, GEN c, GEN D)` (assumes $b^2 - 4ac > 0$).

3.2.10 Ser$(s, \{v =' x\}, \{d = seriesprecision\})$. Transforms the object s into a power series with main variable v (x by default) and precision (number of significant terms) equal to $d \geq 0$ ($d =$ `seriesprecision` by default). If s is a scalar, this gives a constant power series in v with precision `d`. If s is a polynomial, the polynomial is truncated to d terms if needed

```
? Ser(1, 'y, 5)
%1 = 1 + O(y^5)
? Ser(x^2,, 5)
%2 = x^2 + O(x^7)
? T = polcyclo(100)
%3 = x^40 - x^30 + x^20 - x^10 + 1
? Ser(T, 'x, 11)
%4 = 1 - x^10 + O(x^11)
```

The function is more or less equivalent with multiplication by $1 + O(v^d)$ in theses cases, only faster.

If s is a vector, on the other hand, the coefficients of the vector are understood to be the coefficients of the power series starting from the constant term (as in `Polrev`(x)), and the precision d is ignored: in other words, in this case, we convert `t_VEC` / `t_COL` to the power series whose significant terms are exactly given by the vector entries. Finally, if s is already a power series in v, we return it verbatim, ignoring d again. If d significant terms are desired in the last two cases, convert/truncate to `t_POL` first.

```
? v = [1,2,3]; Ser(v, t, 7)
%5 = 1 + 2*t + 3*t^2 + O(t^3)  \\ 3 terms: 7 is ignored!
? Ser(Polrev(v,t), t, 7)
%6 = 1 + 2*t + 3*t^2 + O(t^7)
? s = 1+x+O(x^2); Ser(s, x, 7)
%7 = 1 + x + O(x^2)  \\ 2 terms: 7 ignored
? Ser(truncate(s), x, 7)
%8 = 1 + x + O(x^7)
```

The warning given for `Pol` also applies here: this is not a substitution function.

The library syntax is `GEN gtoser(GEN s, long v = -1, long precdl)` where `v` is a variable number.

3.2.11 Set($\{x = []\}$). Converts x into a set, i.e. into a row vector, with strictly increasing entries with respect to the (somewhat arbitrary) universal comparison function `cmp`. Standard container types `t_VEC`, `t_COL`, `t_LIST` and `t_VECSMALL` are converted to the set with corresponding elements. All others are converted to a set with one element.

```
? Set([1,2,4,2,1,3])
%1 = [1, 2, 3, 4]
? Set(x)
%2 = [x]
? Set(Vecsmall([1,3,2,1,3]))
%3 = [1, 2, 3]
```

The library syntax is `GEN gtoset(GEN x = NULL)`.

3.2.12 Str($\{x\}*$). Converts its argument list into a single character string (type `t_STR`, the empty string if x is omitted). To recover an ordinary `GEN` from a string, apply `eval` to it. The arguments of `Str` are evaluated in string context, see Section 2.9.

```
? x2 = 0; i = 2; Str(x, i)
%1 = "x2"
? eval(%)
%2 = 0
```

This function is mostly useless in library mode. Use the pair `strtoGEN`/`GENtostr` to convert between `GEN` and `char*`. The latter returns a malloced string, which should be freed after usage.

3.2.13 Strchr(x). Converts x to a string, translating each integer into a character.

```
? Strchr(97)
%1 = "a"
? Vecsmall("hello world")
%2 = Vecsmall([104, 101, 108, 108, 111, 32, 119, 111, 114, 108, 100])
? Strchr(%)
%3 = "hello world"
```

The library syntax is `GEN Strchr(GEN x)`.

3.2.14 Strexpand($\{x\}*$). Converts its argument list into a single character string (type `t_STR`, the empty string if x is omitted). Then perform environment expansion, see Section 2.12. This feature can be used to read environment variable values.

```
? Strexpand("$HOME/doc")
%1 = "/home/pari/doc"
```

The individual arguments are read in string context, see Section 2.9.

3.2.15 Strtex($\{x\}*$). Translates its arguments to TeX format, and concatenates the results into a single character string (type `t_STR`, the empty string if x is omitted).

The individual arguments are read in string context, see Section 2.9.

3.2.16 Vec$(x, \{n\})$. Transforms the object x into a row vector. The dimension of the resulting vector can be optionally specified via the extra parameter n.

If n is omitted or 0, the dimension depends on the type of x; the vector has a single component, except when x is

- a vector or a quadratic form: returns the initial object considered as a row vector,

- a polynomial or a power series: returns a vector consisting of the coefficients. In the case of a polynomial, the coefficients of the vector start with the leading coefficient of the polynomial, while for power series only the significant coefficients are taken into account, but this time by increasing order of degree. `Vec` is the reciprocal function of `Pol` for a polynomial and of `Ser` for a power series,

- a matrix: returns the vector of columns comprising the matrix,

- a character string: returns the vector of individual characters,

- a map: returns the vector of the domain of the map,

- an error context (`t_ERROR`): returns the error components, see `iferr`.

In the last four cases (matrix, character string, map, error), n is meaningless and must be omitted or an error is raised. Otherwise, if n is given, 0 entries are appended at the end of the vector if $n > 0$, and prepended at the beginning if $n < 0$. The dimension of the resulting vector is $|n|$. Variant: `GEN gtovec(GEN x)` is also available.

The library syntax is `GEN gtovec0(GEN x, long n)`.

3.2.17 Vecrev$(x, \{n\})$. As `Vec`$(x, -n)$, then reverse the result. In particular, `Vecrev` is the reciprocal function of `Polrev`: the coefficients of the vector start with the constant coefficient of the polynomial and the others follow by increasing degree.

The library syntax is `GEN gtovecrev0(GEN x, long n)`. `GEN gtovecrev(GEN x)` is also available.

3.2.18 Vecsmall$(x, \{n\})$. Transforms the object x into a row vector of type `t_VECSMALL`. The dimension of the resulting vector can be optionally specified via the extra parameter n.

This acts as `Vec`(x, n), but only on a limited set of objects: the result must be representable as a vector of small integers. If x is a character string, a vector of individual characters in ASCII encoding is returned (`Strchr` yields back the character string).

The library syntax is `GEN gtovecsmall0(GEN x, long n)`. `GEN gtovecsmall(GEN x)` is also available.

3.2.19 binary(x). Outputs the vector of the binary digits of $|x|$. Here x can be an integer, a real number (in which case the result has two components, one for the integer part, one for the fractional part) or a vector/matrix.

```
? binary(10)
%1 = [1, 0, 1, 0]

? binary(3.14)
%2 = [[1, 1], [0, 0, 1, 0, 0, 0, [...]]

? binary([1,2])
%3 = [[1], [1, 0]]
```

By convention, 0 has no digits:

```
? binary(0)
%4 = []
```

The library syntax is `GEN binaire(GEN x)`.

3.2.20 bitand(x, y). Bitwise `and` of two integers x and y, that is the integer

$$\sum_i (x_i \text{ and } y_i) 2^i$$

Negative numbers behave 2-adically, i.e. the result is the 2-adic limit of `bitand`(x_n, y_n), where x_n and y_n are non-negative integers tending to x and y respectively. (The result is an ordinary integer, possibly negative.)

```
? bitand(5, 3)
%1 = 1
? bitand(-5, 3)
%2 = 3
? bitand(-5, -3)
%3 = -7
```

The library syntax is `GEN gbitand(GEN x, GEN y)`. Also available is `GEN ibitand(GEN x, GEN y)`, which returns the bitwise *and* of $|x|$ and $|y|$, two integers.

3.2.21 bitneg$(x, \{n = -1\})$. bitwise negation of an integer x, truncated to n bits, $n \geq 0$, that is the integer

$$\sum_{i=0}^{n-1} \text{not}(x_i) 2^i.$$

The special case $n = -1$ means no truncation: an infinite sequence of leading 1 is then represented as a negative number.

See Section 3.2.20 for the behavior for negative arguments.

The library syntax is `GEN gbitneg(GEN x, long n)`.

3.2.22 bitnegimply(x, y). Bitwise negated imply of two integers x and y (or `not` $(x \Rightarrow y)$), that is the integer

$$\sum (x_i \texttt{ andnot } (y_i))2^i$$

See Section 3.2.20 for the behavior for negative arguments.

The library syntax is `GEN gbitnegimply(GEN x, GEN y)`. Also available is `GEN ibitnegimply(GEN x, GEN y)`, which returns the bitwise negated imply of $|x|$ and $|y|$, two integers.

3.2.23 bitor(x, y). bitwise (inclusive) `or` of two integers x and y, that is the integer

$$\sum (x_i \texttt{ or } y_i)2^i$$

See Section 3.2.20 for the behavior for negative arguments.

The library syntax is `GEN gbitor(GEN x, GEN y)`. Also available is `GEN ibitor(GEN x, GEN y)`, which returns the bitwise *ir* of $|x|$ and $|y|$, two integers.

3.2.24 bitprecision$(x, \{n\})$. The function behaves differently according to whether n is present and positive or not. If n is missing, the function returns the (floating point) precision in bits of the PARI object x. If x is an exact object, the function returns `+oo`.

```
? bitprecision(exp(1e-100))
%1 = 512                  \\ 512 bits
? bitprecision( [ exp(1e-100), 0.5 ] )
%2 = 128                  \\ minimal accuracy among components
? bitprecision(2 + x)
%3 = +oo                   \\ exact object
```

If n is present and positive, the function creates a new object equal to x with the new bitprecision roughly n. In fact, the smallest multiple of 64 (resp. 32 on a 32-bit machine) larger than or equal to n.

For x a vector or a matrix, the operation is done componentwise; for series and polynomials, the operation is done coefficientwise. For real x, n is the number of desired significant *bits*. If n is smaller than the precision of x, x is truncated, otherwise x is extended with zeros. For exact or non-floating point types, no change.

```
? bitprecision(Pi, 10)    \\ actually 64 bits ~ 19 decimal digits
%1 = 3.141592653589793239
? bitprecision(1, 10)
%2 = 1
? bitprecision(1 + O(x), 10)
%3 = 1 + O(x)
? bitprecision(2 + O(3^5), 10)
%4 = 2 + O(3^5)
```

The library syntax is `GEN bitprecision0(GEN x, long n)`.

3.2.25 bittest(x, n). Outputs the n^{th} bit of x starting from the right (i.e. the coefficient of 2^n in the binary expansion of x). The result is 0 or 1.

```
? bittest(7, 0)
%1 = 1 \\ the bit 0 is 1
? bittest(7, 2)
%2 = 1 \\ the bit 2 is 1
? bittest(7, 3)
%3 = 0 \\ the bit 3 is 0
```

See Section 3.2.20 for the behavior at negative arguments.

The library syntax is `GEN gbittest(GEN x, long n)`. For a `t_INT` x, the variant `long bittest(GEN x, long n)` is generally easier to use, and if furthermore $n \geq 0$ the low-level function `ulong int_bit(GEN x, long n)` returns `bittest(abs(x),n)`.

3.2.26 bitxor(x, y). Bitwise (exclusive) `or` of two integers x and y, that is the integer

$$\sum (x_i \texttt{ xor } y_i) 2^i$$

See Section 3.2.20 for the behavior for negative arguments.

The library syntax is `GEN gbitxor(GEN x, GEN y)`. Also available is `GEN ibitxor(GEN x, GEN y)`, which returns the bitwise *xor* of $|x|$ and $|y|$, two integers.

3.2.27 ceil(x). Ceiling of x. When x is in $\mathbf{R}$, the result is the smallest integer greater than or equal to x. Applied to a rational function, `ceil`(x) returns the Euclidean quotient of the numerator by the denominator.

The library syntax is `GEN gceil(GEN x)`.

3.2.28 centerlift$(x, \{v\})$. Same as `lift`, except that `t_INTMOD` and `t_PADIC` components are lifted using centered residues:

• for a `t_INTMOD` $x \in \mathbf{Z}/n\mathbf{Z}$, the lift y is such that $-n/2 < y \leq n/2$.

• a `t_PADIC` x is lifted in the same way as above (modulo $p^{\texttt{padicprec(x)}}$) if its valuation v is non-negative; if not, returns the fraction p^v `centerlift`(xp^{-v}); in particular, rational reconstruction is not attempted. Use `bestappr` for this.

For backward compatibility, `centerlift(x,'v)` is allowed as an alias for `lift(x,'v)`.

The library syntax is **centerlift**`(GEN x)`.

3.2.29 characteristic(x). Returns the characteristic of the base ring over which x is defined (as defined by `t_INTMOD` and `t_FFELT` components). The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components.

```
? characteristic(Mod(1,24)*x + Mod(1,18)*y)
%1 = 6
```

The library syntax is `GEN characteristic(GEN x)`.

3.2.30 component(x, n). Extracts the n^{th}-component of x. This is to be understood as follows: every PARI type has one or two initial code words. The components are counted, starting at 1, after these code words. In particular if x is a vector, this is indeed the n^{th}-component of x, if x is a matrix, the n^{th} column, if x is a polynomial, the n^{th} coefficient (i.e. of degree $n-1$), and for power series, the n^{th} significant coefficient.

For polynomials and power series, one should rather use `polcoeff`, and for vectors and matrices, the `[]` operator. Namely, if x is a vector, then `x[n]` represents the n^{th} component of x. If x is a matrix, `x[m,n]` represents the coefficient of row `m` and column `n` of the matrix, `x[m,]` represents the m^{th} *row* of x, and `x[,n]` represents the n^{th} *column* of x.

Using of this function requires detailed knowledge of the structure of the different PARI types, and thus it should almost never be used directly. Some useful exceptions:

```
? x = 3 + O(3^5);
? component(x, 2)
%2 = 81   \\ p^(p-adic accuracy)
? component(x, 1)
%3 = 3    \\ p
? q = Qfb(1,2,3);
? component(q, 1)
%5 = 1
```

The library syntax is `GEN compo(GEN x, long n)`.

3.2.31 conj(x). Conjugate of x. The meaning of this is clear, except that for real quadratic numbers, it means conjugation in the real quadratic field. This function has no effect on integers, reals, intmods, fractions or p-adics. The only forbidden type is polmod (see `conjvec` for this).

The library syntax is `GEN gconj(GEN x)`.

3.2.32 conjvec(z). Conjugate vector representation of z. If z is a polmod, equal to `Mod`(a, T), this gives a vector of length degree(T) containing:

• the complex embeddings of z if T has rational coefficients, i.e. the $a(r[i])$ where $r =$ `polroots`(T);

• the conjugates of z if T has some intmod coefficients;

if z is a finite field element, the result is the vector of conjugates $[z, z^p, z^{p^2}, \ldots, z^{p^{n-1}}]$ where $n =$ degree(T).

If z is an integer or a rational number, the result is z. If z is a (row or column) vector, the result is a matrix whose columns are the conjugate vectors of the individual elements of z.

The library syntax is `GEN conjvec(GEN z, long prec)`.

3.2.33 denominator(x). Denominator of x. The meaning of this is clear when x is a rational number or function. If x is an integer or a polynomial, it is treated as a rational number or function, respectively, and the result is equal to 1. For polynomials, you probably want to use

```
denominator( content(x) )
```

instead. As for modular objects, `t_INTMOD` and `t_PADIC` have denominator 1, and the denominator of a `t_POLMOD` is the denominator of its (minimal degree) polynomial representative.

If x is a recursive structure, for instance a vector or matrix, the lcm of the denominators of its components (a common denominator) is computed. This also applies for `t_COMPLEX`s and `t_QUAD`s.

Warning. Multivariate objects are created according to variable priorities, with possibly surprising side effects (x/y is a polynomial, but y/x is a rational function). See Section 2.5.3.

The library syntax is `GEN denom(GEN x)`.

3.2.34 digits$(x, \{b = 10\})$. Outputs the vector of the digits of $|x|$ in base b, where x and b are integers ($b = 10$ by default). See `fromdigits` for the reverse operation.

```
? digits(123)
%1 = [1, 2, 3, 0]

? digits(10, 2) \\ base 2
%2 = [1, 0, 1, 0]
```

By convention, 0 has no digits:

```
? digits(0)
%3 = []
```

The library syntax is `GEN digits(GEN x, GEN b = NULL)`.

3.2.35 floor(x). Floor of x. When x is in $\mathbf{R}$, the result is the largest integer smaller than or equal to x. Applied to a rational function, `floor`(x) returns the Euclidean quotient of the numerator by the denominator.

The library syntax is `GEN gfloor(GEN x)`.

3.2.36 frac(x). Fractional part of x. Identical to $x - \mathrm{floor}(x)$. If x is real, the result is in $[0, 1[$.

The library syntax is `GEN gfrac(GEN x)`.

3.2.37 fromdigits$(x, \{b = 10\})$. Gives the integer formed by the elements of x seen as the digits of a number in base b ($b = 10$ by default). This is the reverse of `digits`:

```
? digits(1234,5)
%1 = [1,4,4,1,4]
? fromdigits([1,4,4,1,4],5)
%2 = 1234
```

By convention, 0 has no digits:

```
? fromdigits([])
%3 = 0
```

The library syntax is `GEN fromdigits(GEN x, GEN b = NULL)`.

3.2.38 hammingweight(x). If x is a `t_INT`, return the binary Hamming weight of $|x|$. Otherwise x must be of type `t_POL`, `t_VEC`, `t_COL`, `t_VECSMALL`, or `t_MAT` and the function returns the number of non-zero coefficients of x.

```
? hammingweight(15)
%1 = 4
? hammingweight(x^100 + 2*x + 1)
%2 = 3
? hammingweight([Mod(1,2), 2, Mod(0,3)])
%3 = 2
? hammingweight(matid(100))
%4 = 100
```

The library syntax is `long hammingweight(GEN x)`.

3.2.39 imag(x). Imaginary part of x. When x is a quadratic number, this is the coefficient of ω in the "canonical" integral basis $(1, \omega)$.

The library syntax is `GEN gimag(GEN x)`.

3.2.40 length(x). Length of x; `#`x is a shortcut for `length`(x). This is mostly useful for

- vectors: dimension (0 for empty vectors),
- lists: number of entries (0 for empty lists),
- matrices: number of columns,
- character strings: number of actual characters (without trailing `\0`, should you expect it from *C* `char*`).

```
? #"a string"
%1 = 8
? #[3,2,1]
%2 = 3
? #[]
%3 = 0
? #matrix(2,5)
%4 = 5
? L = List([1,2,3,4]); #L
%5 = 4
```

The routine is in fact defined for arbitrary GP types, but is awkward and useless in other cases: it returns the number of non-code words in x, e.g. the effective length minus 2 for integers since the `t_INT` type has two code words.

The library syntax is `long glength(GEN x)`.

3.2.41 lift$(x, \{v\})$. If v is omitted, lifts intmods from $\mathbf{Z}/n\mathbf{Z}$ in $\mathbf{Z}$, p-adics from $\mathbf{Q}_p$ to $\mathbf{Q}$ (as `truncate`), and polmods to polynomials. Otherwise, lifts only polmods whose modulus has main variable v. `t_FFELT` are not lifted, nor are List elements: you may convert the latter to vectors first, or use `apply(lift,L)`. More generally, components for which such lifts are meaningless (e.g. character strings) are copied verbatim.

```
? lift(Mod(5,3))
%1 = 2
? lift(3 + O(3^9))
%2 = 3
? lift(Mod(x,x^2+1))
%3 = x
? lift(Mod(x,x^2+1))
%4 = x
```

Lifts are performed recursively on an object components, but only by *one level*: once a `t_POLMOD` is lifted, the components of the result are *not* lifted further.

```
? lift(x * Mod(1,3) + Mod(2,3))
%4 = x + 2
? lift(x * Mod(y,y^2+1) + Mod(2,3))
%5 = y*x + Mod(2, 3)    \\ do you understand this one?
? lift(x * Mod(y,y^2+1) + Mod(2,3), 'x)
%6 = Mod(y, y^2 + 1)*x + Mod(Mod(2, 3), y^2 + 1)
? lift(%, y)
%7 = y*x + Mod(2, 3)
```

To recursively lift all components not only by one level, but as long as possible, use `liftall`. To lift only `t_INTMOD`s and `t_PADIC`s components, use `liftint`. To lift only `t_POLMOD`s components, use `liftpol`. Finally, `centerlift` allows to lift `t_INTMOD`s and `t_PADIC`s using centered residues (lift of smallest absolute value).

The library syntax is `GEN lift0(GEN x, long v = -1)` where `v` is a variable number. Also available is `GEN lift(GEN x)` corresponding to `lift0(x,-1)`.

3.2.42 liftall(x). Recursively lift all components of x from $\mathbf{Z}/n\mathbf{Z}$ to $\mathbf{Z}$, from $\mathbf{Q}_p$ to $\mathbf{Q}$ (as `truncate`), and polmods to polynomials. `t_FFELT` are not lifted, nor are List elements: you may convert the latter to vectors first, or use `apply(liftall,L)`. More generally, components for which such lifts are meaningless (e.g. character strings) are copied verbatim.

```
? liftall(x * (1 + O(3)) + Mod(2,3))
%1 = x + 2
? liftall(x * Mod(y,y^2+1) + Mod(2,3)*Mod(z,z^2))
%2 = y*x + 2*z
```

The library syntax is `GEN liftall(GEN x)`.

3.2.43 liftint(x). Recursively lift all components of x from $\mathbf{Z}/n\mathbf{Z}$ to $\mathbf{Z}$ and from $\mathbf{Q}_p$ to $\mathbf{Q}$ (as `truncate`). `t_FFELT` are not lifted, nor are List elements: you may convert the latter to vectors first, or use `apply(liftint,L)`. More generally, components for which such lifts are meaningless (e.g. character strings) are copied verbatim.

```
? liftint(x * (1 + O(3)) + Mod(2,3))
%1 = x + 2
? liftint(x * Mod(y,y^2+1) + Mod(2,3)*Mod(z,z^2))
%2 = Mod(y, y^2 + 1)*x + Mod(Mod(2*z, z^2), y^2 + 1)
```

The library syntax is `GEN liftint(GEN x)`.

3.2.44 liftpol(x). Recursively lift all components of x which are polmods to polynomials. `t_FFELT` are not lifted, nor are List elements: you may convert the latter to vectors first, or use `apply(liftpol,L)`. More generally, components for which such lifts are meaningless (e.g. character strings) are copied verbatim.

```
? liftpol(x * (1 + O(3)) + Mod(2,3))
%1 = (1 + O(3))*x + Mod(2, 3)
? liftpol(x * Mod(y,y^2+1) + Mod(2,3)*Mod(z,z^2))
%2 = y*x + Mod(2, 3)*z
```

The library syntax is `GEN liftpol(GEN x)`.

3.2.45 norm(x). Algebraic norm of x, i.e. the product of x with its conjugate (no square roots are taken), or conjugates for polmods. For vectors and matrices, the norm is taken componentwise and hence is not the L^2-norm (see `norml2`). Note that the norm of an element of $\mathbf{R}$ is its square, so as to be compatible with the complex norm.

The library syntax is `GEN gnorm(GEN x)`.

3.2.46 numerator(x). Numerator of x. The meaning of this is clear when x is a rational number or function. If x is an integer or a polynomial, it is treated as a rational number or function, respectively, and the result is x itself. For polynomials, you probably want to use

```
numerator( content(x) )
```

instead.

In other cases, `numerator(x)` is defined to be `denominator(x)*x`. This is the case when x is a vector or a matrix, but also for `t_COMPLEX` or `t_QUAD`. In particular since a `t_PADIC` or `t_INTMOD` has denominator 1, its numerator is itself.

Warning. Multivariate objects are created according to variable priorities, with possibly surprising side effects (x/y is a polynomial, but y/x is a rational function). See Section 2.5.3.

The library syntax is `GEN numer(GEN x)`.

3.2.47 numtoperm(n, k). Generates the k-th permutation (as a row vector of length n) of the numbers 1 to n. The number k is taken modulo $n!$, i.e. inverse function of `permtonum`. The numbering used is the standard lexicographic ordering, starting at 0.

The library syntax is `GEN numtoperm(long n, GEN k)`.

3.2.48 oo. Returns an object meaning $+\infty$, for use in functions such as `intnum`. It can be negated (`-oo` represents $-\infty$), and compared to real numbers (`t_INT`, `t_FRAC`, `t_REAL`), with the expected meaning: $+\infty$ is greater than any real number and $-\infty$ is smaller.

The library syntax is `GEN mkoo()`.

3.2.49 padicprec(x, p). Returns the absolute p-adic precision of the object x; this is the minimum precision of the components of x. The result is `+oo` if x is an exact object (as a p-adic):

```
? padicprec((1 + O(2^5)) * x + (2 + O(2^4)), 2)
%1 = 4
? padicprec(x + 2, 2)
%2 = +oo
? padicprec(2 + x + O(x^2), 2)
%3 = +oo
```

The function raises an exception if it encounters an object incompatible with p-adic computations:

```
? padicprec(O(3), 2)
 ***   at top-level: padicprec(O(3),2)
 ***                 ^-----------------
 *** padicprec: inconsistent moduli in padicprec: 3 != 2
? padicprec(1.0, 2)
 ***   at top-level: padicprec(1.0,2)
 ***                 ^----------------
 *** padicprec: incorrect type in padicprec (t_REAL).
```

The library syntax is `GEN gppadicprec(GEN x, GEN p)`. Also available is the function `long padicprec(GEN x, GEN p)`, which returns `LONG_MAX` if $x = 0$ and the p-adic precision as a `long` integer.

3.2.50 permtonum(x). Given a permutation x on n elements, gives the number k such that $x =$ `numtoperm(n,k)`, i.e. inverse function of `numtoperm`. The numbering used is the standard lexicographic ordering, starting at 0.

The library syntax is `GEN permtonum(GEN x)`.

3.2.51 precision$(x, \{n\})$. The function behaves differently according to whether n is present and positive or not. If n is missing, the function returns the precision in decimal digits of the PARI object x. If x is an exact object, the function returns `+oo`.

```
? precision(exp(1e-100))
%1 = 154                \\ 154 significant decimal digits
? precision(2 + x)
%2 = +oo                \\ exact object
? precision(0.5 + O(x))
%3 = 38                 \\ floating point accuracy, NOT series precision
? precision( [ exp(1e-100), 0.5 ] )
%4 = 38                 \\ minimal accuracy among components
```

If n is present, the function creates a new object equal to x with a new floating point precision n: n is the number of desired significant *decimal* digits. If n is smaller than the precision of a `t_REAL`

component of x, it is truncated, otherwise it is extended with zeros. For exact or non-floating point types, no change.

The library syntax is `GEN precision0(GEN x, long n)`. Also available are `GEN gprec(GEN x, long n)` and `long precision(GEN x)`. In both, the accuracy is expressed in *words* (32-bit or 64-bit depending on the architecture).

3.2.52 random($\{N = 2^{31}\}$). Returns a random element in various natural sets depending on the argument N.

• `t_INT`: returns an integer uniformly distributed between 0 and $N-1$. Omitting the argument is equivalent to `random(2^31)`.

• `t_REAL`: returns a real number in $[0, 1[$ with the same accuracy as N (whose mantissa has the same number of significant words).

• `t_INTMOD`: returns a random intmod for the same modulus.

• `t_FFELT`: returns a random element in the same finite field.

• `t_VEC` of length 2, $N = [a, b]$: returns an integer uniformly distributed between a and b.

• `t_VEC` generated by `ellinit` over a finite field k (coefficients are `t_INTMOD`s modulo a prime or `t_FFELT`s): returns a "random" k-rational *affine* point on the curve. More precisely if the curve has a single point (at infinity!) we return it; otherwise we return an affine point by drawing an abscissa uniformly at random until `ellordinate` succeeds. Note that this is definitely not a uniform distribution over $E(k)$, but it should be good enough for applications.

• `t_POL` return a random polynomial of degree at most the degree of N. The coefficients are drawn by applying `random` to the leading coefficient of N.

```
? random(10)
%1 = 9
? random(Mod(0,7))
%2 = Mod(1, 7)
? a = ffgen(ffinit(3,7), 'a); random(a)
%3 = a^6 + 2*a^5 + a^4 + a^3 + a^2 + 2*a
? E = ellinit([3,7]*Mod(1,109)); random(E)
%4 = [Mod(103, 109), Mod(10, 109)]
? E = ellinit([1,7]*a^0); random(E)
%5 = [a^6 + a^5 + 2*a^4 + 2*a^2, 2*a^6 + 2*a^4 + 2*a^3 + a^2 + 2*a]
? random(Mod(1,7)*x^4)
%6 = Mod(5, 7)*x^4 + Mod(6, 7)*x^3 + Mod(2, 7)*x^2 + Mod(2, 7)*x + Mod(5, 7)
```

These variants all depend on a single internal generator, and are independent from your operating system's random number generators. A random seed may be obtained via `getrand`, and reset using `setrand`: from a given seed, and given sequence of `random`s, the exact same values will be generated. The same seed is used at each startup, reseed the generator yourself if this is a problem. Note that internal functions also call the random number generator; adding such a function call in the middle of your code will change the numbers produced.

Technical note. Up to version 2.4 included, the internal generator produced pseudo-random numbers by means of linear congruences, which were not well distributed in arithmetic progressions. We now use Brent's XORGEN algorithm, based on Feedback Shift Registers, see `http://wwwmaths.anu.edu.au/~brent/random.html`. The generator has period $2^{4096} - 1$, passes the Crush battery of statistical tests of L'Ecuyer and Simard, but is not suitable for cryptographic purposes: one can reconstruct the state vector from a small sample of consecutive values, thus predicting the entire sequence.

The library syntax is `GEN genrand(GEN N = NULL)`.

Also available: `GEN ellrandom(GEN E)` and `GEN ffrandom(GEN a)`.

3.2.53 real(x). Real part of x. In the case where x is a quadratic number, this is the coefficient of 1 in the "canonical" integral basis $(1, \omega)$.

The library syntax is `GEN greal(GEN x)`.

3.2.54 round$(x, \{\&e\})$. If x is in $\mathbf{R}$, rounds x to the nearest integer (rounding to $+\infty$ in case of ties), then and sets e to the number of error bits, that is the binary exponent of the difference between the original and the rounded value (the "fractional part"). If the exponent of x is too large compared to its precision (i.e. $e > 0$), the result is undefined and an error occurs if e was not given.

Important remark. Contrary to the other truncation functions, this function operates on every coefficient at every level of a PARI object. For example

$$\text{truncate}\left(\frac{2.4 * X^2 - 1.7}{X}\right) = 2.4 * X,$$

whereas

$$\text{round}\left(\frac{2.4 * X^2 - 1.7}{X}\right) = \frac{2 * X^2 - 2}{X}.$$

An important use of `round` is to get exact results after an approximate computation, when theory tells you that the coefficients must be integers.

The library syntax is `GEN round0(GEN x, GEN *e = NULL)`. Also available are `GEN grndtoi(GEN x, long *e)` and `GEN ground(GEN x)`.

3.2.55 serprec(x, v). Returns the absolute precision of x with respect to power series in the variable v; this is the minimum precision of the components of x. The result is `+oo` if x is an exact object (as a series in v):

```
? serprec(x + O(y^2), y)
%1 = 2
? serprec(x + 2, x)
%2 = +oo
? serprec(2 + x + O(x^2), y)
%3 = +oo
```

The library syntax is `GEN gpserprec(GEN x, long v)` where `v` is a variable number. Also available is `long serprec(GEN x, GEN p)`, which returns `LONG_MAX` if $x = 0$ and the series precision as a `long` integer.

3.2.56 simplify(x). This function simplifies x as much as it can. Specifically, a complex or quadratic number whose imaginary part is the integer 0 (i.e. not `Mod(0,2)` or `0.E-28`) is converted to its real part, and a polynomial of degree 0 is converted to its constant term. Simplifications occur recursively.

This function is especially useful before using arithmetic functions, which expect integer arguments:

```
? x = 2 + y - y
%1 = 2
? isprime(x)
  ***   at top-level: isprime(x)
  ***                 ^----------
  *** isprime: not an integer argument in an arithmetic function
? type(x)
%2 = "t_POL"
? type(simplify(x))
%3 = "t_INT"
```

Note that GP results are simplified as above before they are stored in the history. (Unless you disable automatic simplification with `\y`, that is.) In particular

```
? type(%1)
%4 = "t_INT"
```

The library syntax is `GEN simplify(GEN x)`.

3.2.57 sizebyte(x). Outputs the total number of bytes occupied by the tree representing the PARI object x.

The library syntax is `long gsizebyte(GEN x)`. Also available is `long gsizeword(GEN x)` returning a number of *words*.

3.2.58 sizedigit(x). This function is DEPRECATED, essentially meaningless, and provided for backwards compatibility only. Don't use it!

outputs a quick upper bound for the number of decimal digits of (the components of) x, off by at most 1. More precisely, for a positive integer x, it computes (approximately) the ceiling of

$$\texttt{floor}(1 + \log_2 x) \log_{10} 2,$$

To count the number of decimal digits of a positive integer x, use `#digits(x)`. To estimate (recursively) the size of x, use `normlp(x)`.

The library syntax is `long sizedigit(GEN x)`.

3.2.59 truncate$(x, \{\&e\})$. Truncates x and sets e to the number of error bits. When x is in $\mathbf{R}$, this means that the part after the decimal point is chopped away, e is the binary exponent of the difference between the original and the truncated value (the "fractional part"). If the exponent of x is too large compared to its precision (i.e. $e > 0$), the result is undefined and an error occurs if e was not given. The function applies componentwise on vector / matrices; e is then the maximal number of error bits. If x is a rational function, the result is the "integer part" (Euclidean quotient of numerator by denominator) and e is not set.

Note a very special use of `truncate`: when applied to a power series, it transforms it into a polynomial or a rational function with denominator a power of X, by chopping away the $O(X^k)$. Similarly, when applied to a p-adic number, it transforms it into an integer or a rational number by chopping away the $O(p^k)$.

The library syntax is `GEN trunc0(GEN x, GEN *e = NULL)`. The following functions are also available: `GEN gtrunc(GEN x)` and `GEN gcvtoi(GEN x, long *e)`.

3.2.60 valuation(x, p). Computes the highest exponent of p dividing x. If p is of type integer, x must be an integer, an intmod whose modulus is divisible by p, a fraction, a q-adic number with $q = p$, or a polynomial or power series in which case the valuation is the minimum of the valuation of the coefficients.

If p is of type polynomial, x must be of type polynomial or rational function, and also a power series if x is a monomial. Finally, the valuation of a vector, complex or quadratic number is the minimum of the component valuations.

If $x = 0$, the result is `+oo` if x is an exact object. If x is a p-adic numbers or power series, the result is the exponent of the zero. Any other type combinations gives an error.

The library syntax is `GEN gpvaluation(GEN x, GEN p)`. Also available is `long gvaluation(GEN x, GEN p)`, which returns `LONG_MAX` if $x = 0$ and the valuation as a `long` integer.

3.2.61 varhigher$(name, \{v\})$. Return a variable *name* whose priority is higher than the priority of v (of all existing variables if v is omitted). This is a counterpart to `varlower`.

```
? Pol([x,x], t)
 ***   at top-level: Pol([x,x],t)
 ***                 ^------------
 *** Pol: incorrect priority in gtopoly: variable x <= t
? t = varhigher("t", x);
? Pol([x,x], t)
%3 = x*t + x
```

This routine is useful since new GP variables directly created by the interpreter always have lower priority than existing GP variables. When some basic objects already exist in a variable that is incompatible with some function requirement, you can now create a new variable with a suitable priority instead of changing variables in existing objects:

```
? K = nfinit(x^2+1);
? rnfequation(K,y^2-2)
 ***   at top-level: rnfequation(K,y^2-2)
 ***                 ^--------------------
 *** rnfequation: incorrect priority in rnfequation: variable y >= x
? y = varhigher("y", x);
```

```
? rnfequation(K, y^2-2)
%3 = y^4 - 2*y^2 + 9
```

Caution 1. The *name* is an arbitrary character string, only used for display purposes and need not be related to the GP variable holding the result, nor to be a valid variable name. In particular the *name* can not be used to retrieve the variable, it is not even present in the parser's hash tables.

```
? x = varhigher("#");
? x^2
%2 = #^2
```

Caution 2. There are a limited number of variables and if no existing variable with the given display name has the requested priority, the call to `varhigher` uses up one such slot. Do not create new variables in this way unless it's absolutely necessary, reuse existing names instead and choose sensible priority requirements: if you only need a variable with higher priority than x, state so rather than creating a new variable with highest priority.

```
\\ quickly use up all variables
? n = 0; while(1,varhigher("tmp"); n++)
 ***   at top-level: n=0;while(1,varhigher("tmp");n++)
 ***                               ^-------------------
 *** varhigher: no more variables available.
 ***   Break loop: type 'break' to go back to GP prompt
break> n
65510
\\ infinite loop: here we reuse the same 'tmp'
? n = 0; while(1,varhigher("tmp", x); n++)
```

The library syntax is `GEN varhigher(const char *name, long v = -1)` where `v` is a variable number.

3.2.62 variable($\{x\}$). Gives the main variable of the object x (the variable with the highest priority used in x), and p if x is a p-adic number. Return 0 if x has no variable attached to it.

```
? variable(x^2 + y)
%1 = x
? variable(1 + O(5^2))
%2 = 5
? variable([x,y,z,t])
%3 = x
? variable(1)
%4 = 0
```

The construction

```
if (!variable(x),...)
```

can be used to test whether a variable is attached to x.

If x is omitted, returns the list of user variables known to the interpreter, by order of decreasing priority. (Highest priority is initially x, which come first until `varhigher` is used.) If `varhigher` or `varlower` are used, it is quite possible to end up with different variables (with different priorities) printed in the same way: they will then appear multiple times in the output:

```
? varhigher("y");
? varlower("y");
? variable()
%4 = [y, x, y]
```

Using `v = variable()` then `v[1]`, `v[2]`, etc. allows to recover and use existing variables.

The library syntax is `GEN gpolvar(GEN x = NULL)`. However, in library mode, this function should not be used for x non-`NULL`, since `gvar` is more appropriate. Instead, for x a p-adic (type `t_PADIC`), p is $gel(x, 2)$; otherwise, use `long gvar(GEN x)` which returns the variable number of x if it exists, `NO_VARIABLE` otherwise, which satisfies the property `varncmp(NO_VARIABLE,` v`)` > 0 for all valid variable number v, i.e. it has lower priority than any variable.

3.2.63 variables$(\{x\})$. Returns the list of all variables occuring in object x (all user variables known to the interpreter if x is omitted), sorted by decreasing priority.

```
? variables([x^2 + y*z + O(t), a+x])
%1 = [x, y, z, t, a]
```

The construction

```
if (!variables(x),...)
```

can be used to test whether a variable is attached to x.

If `varhigher` or `varlower` are used, it is quite possible to end up with different variables (with different priorities) printed in the same way: they will then appear multiple times in the output:

```
? y1 = varhigher("y");
? y2 = varlower("y");
? variables(y*y1*y2)
%4 = [y, y, y]
```

The library syntax is `GEN variables_vec(GEN x = NULL)`.

Also available is `GEN variables_vecsmall(GEN x)` which returns the (sorted) variable numbers instead of the attached monomials of degree 1.

3.2.64 varlower$(name, \{v\})$. Return a variable *name* whose priority is lower than the priority of v (of all existing variables if v is omitted). This is a counterpart to `varhigher`.

New GP variables directly created by the interpreter always have lower priority than existing GP variables, but it is not easy to check whether an identifier is currently unused, so that the corresponding variable has the expected priority when it's created! Thus, depending on the session history, the same command may fail or succeed:

```
? t; z;  \\ now t > z
? rnfequation(t^2+1,z^2-t)
 ***   at top-level: rnfequation(t^2+1,z^
 ***                 ^--------------------
 *** rnfequation: incorrect priority in rnfequation: variable t >= t
```

Restart and retry:

```
? z; t;  \\ now z > t
? rnfequation(t^2+1,z^2-t)
```

```
%2 = z^4 + 1
```

It is quite annoying for package authors, when trying to define a base ring, to notice that the package may fail for some users depending on their session history. The safe way to do this is as follows:

```
? z; t;  \\ In new session: now z > t
...
? t = varlower("t", 'z);
? rnfequation(t^2+1,z^2-2)
%2 = z^4 - 2*z^2 + 9
? variable()
%3 = [x, y, z, t]
```

```
? t; z;  \\ In new session: now t > z
...
? t = varlower("t", 'z); \\ create a new variable, still printed "t"
? rnfequation(t^2+1,z^2-2)
%2 = z^4 - 2*z^2 + 9
? variable()
%3 = [x, y, t, z, t]
```

Now both constructions succeed. Note that in the first case, `varlower` is essentially a no-op, the existing variable t has correct priority. While in the second case, two different variables are displayed as `t`, one with higher priority than z (created in the first line) and another one with lower priority (created by `varlower`).

Caution 1. The *name* is an arbitrary character string, only used for display purposes and need not be related to the GP variable holding the result, nor to be a valid variable name. In particular the *name* can not be used to retrieve the variable, it is not even present in the parser's hash tables.

```
? x = varlower("#");
? x^2
%2 = #^2
```

Caution 2. There are a limited number of variables and if no existing variable with the given display name has the requested priority, the call to `varlower` uses up one such slot. Do not create new variables in this way unless it's absolutely necessary, reuse existing names instead and choose sensible priority requirements: if you only need a variable with higher priority than x, state so rather than creating a new variable with highest priority.

```
\\ quickly use up all variables
? n = 0; while(1,varlower("x"); n++)
 ***   at top-level: n=0;while(1,varlower("x");n++)
 ***                               ^-------------------
 *** varlower: no more variables available.
 ***   Break loop: type 'break' to go back to GP prompt
break> n
65510
\\ infinite loop: here we reuse the same 'tmp'
? n = 0; while(1,varlower("tmp", x); n++)
```

The library syntax is `GEN varlower(const char *name, long v = -1)` where `v` is a variable number.

3.3 Transcendental functions.

Since the values of transcendental functions cannot be exactly represented, these functions will always return an inexact object: a real number, a complex number, a p-adic number or a power series. All these objects have a certain finite precision.

As a general rule, which of course in some cases may have exceptions, transcendental functions operate in the following way:

• If the argument is either a real number or an inexact complex number (like `1.0 + I` or `Pi*I` but not `2 - 3*I`), then the computation is done with the precision of the argument. In the example below, we see that changing the precision to 50 digits does not matter, because x only had a precision of 19 digits.

```
? \p 15
   realprecision = 19 significant digits (15 digits displayed)
? x = Pi/4
%1 = 0.785398163397448
? \p 50
   realprecision = 57 significant digits (50 digits displayed)
? sin(x)
%2 = 0.70710678118654752440
```

Note that even if the argument is real, the result may be complex (e.g. acos(2.0) or acosh(0.0)). See each individual function help for the definition of the branch cuts and choice of principal value.

• If the argument is either an integer, a rational, an exact complex number or a quadratic number, it is first converted to a real or complex number using the current precision, which can be view and manipulated using the defaults `realprecision` (in decimal digits) or `realbitprecision` (in bits). This precision can be changed indifferently

• in decimal digits: use `\p` or `default(realprecision,...)`.

• in bits: use `\pb` or `default(realbitprecision,...)`.

After this conversion, the computation proceeds as above for real or complex arguments.

In library mode, the `realprecision` does not matter; instead the precision is taken from the `prec` parameter which every transcendental function has. As in `gp`, this `prec` is not used when the argument to a function is already inexact. Note that the argument *prec* stands for the length in words of a real number, including codewords. Hence we must have $prec \geq 3$. (Some functions allow a `bitprec` argument instead which allow finer granularity.)

Some accuracies attainable on 32-bit machines cannot be attained on 64-bit machines for parity reasons. For example the default `gp` accuracy is 28 decimal digits on 32-bit machines, corresponding to *prec* having the value 5, but this cannot be attained on 64-bit machines.

• If the argument is a polmod (representing an algebraic number), then the function is evaluated for every possible complex embedding of that algebraic number. A column vector of results is returned, with one component for each complex embedding. Therefore, the number of components equals the degree of the `t_POLMOD` modulus.

• If the argument is an intmod or a p-adic, at present only a few functions like `sqrt` (square root), `sqr` (square), `log`, `exp`, powering, `teichmuller` (Teichmüller character) and `agm` (arithmetic-geometric mean) are implemented.

Note that in the case of a 2-adic number, `sqr`(x) may not be identical to $x * x$: for example if $x = 1 + O(2^5)$ and $y = 1 + O(2^5)$ then $x * y = 1 + O(2^5)$ while `sqr`$(x) = 1 + O(2^6)$. Here, $x * x$ yields the same result as `sqr`(x) since the two operands are known to be *identical*. The same statement holds true for p-adics raised to the power n, where $v_p(n) > 0$.

Remark. If we wanted to be strictly consistent with the PARI philosophy, we should have $x * y = (4 \bmod 8)$ and `sqr`$(x) = (4 \bmod 32)$ when both x and y are congruent to 2 modulo 4. However, since intmod is an exact object, PARI assumes that the modulus must not change, and the result is hence (0 mod 4) in both cases. On the other hand, p-adics are not exact objects, hence are treated differently.

• If the argument is a polynomial, a power series or a rational function, it is, if necessary, first converted to a power series using the current series precision, held in the default `seriesprecision`. This precision (the number of significant terms) can be changed using `\ps` or `default(seriesprecision,...)`. Then the Taylor series expansion of the function around $X = 0$ (where X is the main variable) is computed to a number of terms depending on the number of terms of the argument and the function being computed.

Under `gp` this again is transparent to the user. When programming in library mode, however, it is *strongly* advised to perform an explicit conversion to a power series first, as in `x = gtoser(x, seriesprec)`, where the number of significant terms `seriesprec` can be specified explicitly. If you do not do this, a global variable `precdl` is used instead, to convert polynomials and rational functions to a power series with a reasonable number of terms; tampering with the value of this global variable is *deprecated* and strongly discouraged.

• If the argument is a vector or a matrix, the result is the componentwise evaluation of the function. In particular, transcendental functions on square matrices, which are not implemented in the present version 2.9.2, will have a different name if they are implemented some day.

3.3.1 `^`. If y is not of type integer, `x^y` has the same effect as `exp(y*log(x))`. It can be applied to p-adic numbers as well as to the more usual types.

The library syntax is `GEN gpow(GEN x, GEN n, long prec)` for $x\hat{}n$.

3.3.2 Catalan. Catalan's constant $G = \sum_{n>=0} \frac{(-1)^n}{(2n+1)^2} = 0.91596\cdots$. Note that `Catalan` is one of the few reserved names which cannot be used for user variables.

The library syntax is `GEN mpcatalan(long prec)`.

3.3.3 Euler. Euler's constant $\gamma = 0.57721\cdots$. Note that `Euler` is one of the few reserved names which cannot be used for user variables.

The library syntax is `GEN mpeuler(long prec)`.

3.3.4 I. The complex number $\sqrt{-1}$.

The library syntax is `GEN gen_I()`.

3.3.5 Pi. The constant π $(3.14159\cdots)$. Note that `Pi` is one of the few reserved names which cannot be used for user variables.

The library syntax is `GEN mppi(long prec)`.

3.3.6 abs(x). Absolute value of x (modulus if x is complex). Rational functions are not allowed. Contrary to most transcendental functions, an exact argument is *not* converted to a real number before applying `abs` and an exact result is returned if possible.

```
? abs(-1)
%1 = 1
? abs(3/7 + 4/7*I)
%2 = 5/7
? abs(1 + I)
%3 = 1.4142135623730950488016887242
```

If x is a polynomial, returns $-x$ if the leading coefficient is real and negative else returns x. For a power series, the constant coefficient is considered instead.

The library syntax is `GEN gabs(GEN x, long prec)`.

3.3.7 acos(x). Principal branch of $\cos^{-1}(x) = -i\log(x + i\sqrt{1-x^2})$. In particular, $\Re(\text{acos}(x)) \in [0,\pi]$ and if $x \in \mathbf{R}$ and $|x| > 1$, then $\text{acos}(x)$ is complex. The branch cut is in two pieces: $]-\infty,-1]$, continuous with quadrant II, and $[1,+\infty[$, continuous with quadrant IV. We have $\text{acos}(x) = \pi/2 - \text{asin}(x)$ for all x.

The library syntax is `GEN gacos(GEN x, long prec)`.

3.3.8 acosh(x). Principal branch of $\cosh^{-1}(x) = 2\log(\sqrt{(x+1)/2} + \sqrt{(x-1)/2})$. In particular, $\Re(\text{acosh}(x)) \geq 0$ and $\Im(\text{acosh}(x)) \in]-\pi,\pi]$; if $x \in \mathbf{R}$ and $x < 1$, then $\text{acosh}(x)$ is complex.

The library syntax is `GEN gacosh(GEN x, long prec)`.

3.3.9 agm(x,y). Arithmetic-geometric mean of x and y. In the case of complex or negative numbers, the optimal AGM is returned (the largest in absolute value over all choices of the signs of the square roots). p-adic or power series arguments are also allowed. Note that a p-adic agm exists only if x/y is congruent to 1 modulo p (modulo 16 for $p = 2$). x and y cannot both be vectors or matrices.

The library syntax is `GEN agm(GEN x, GEN y, long prec)`.

3.3.10 arg(x). Argument of the complex number x, such that $-\pi < \arg(x) \leq \pi$.

The library syntax is `GEN garg(GEN x, long prec)`.

3.3.11 asin(x). Principal branch of $\sin^{-1}(x) = -i\log(ix + \sqrt{1-x^2})$. In particular, $\Re(\text{asin}(x)) \in [-\pi/2,\pi/2]$ and if $x \in \mathbf{R}$ and $|x| > 1$ then $\text{asin}(x)$ is complex. The branch cut is in two pieces: $]-\infty,-1]$, continuous with quadrant II, and $[1,+\infty[$ continuous with quadrant IV. The function satisfies $i\text{asin}(x) = \text{asinh}(ix)$.

The library syntax is `GEN gasin(GEN x, long prec)`.

3.3.12 asinh(x). Principal branch of $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$. In particular $\Im(\text{asinh}(x)) \in [-\pi/2,\pi/2]$. The branch cut is in two pieces: $]-i\infty,-i]$, continuous with quadrant III and $[+i,+i\infty[$, continuous with quadrant I.

The library syntax is `GEN gasinh(GEN x, long prec)`.

3.3.13 atan(x). Principal branch of $\tan^{-1}(x) = \log((1+ix)/(1-ix))/2i$. In particular the real part of atan(x) belongs to $]-\pi/2, \pi/2[$. The branch cut is in two pieces: $]-i\infty, -i[$, continuous with quadrant IV, and $]i, +i\infty[$ continuous with quadrant II. The function satisfies atan$(x) = -i$atanh(ix) for all $x \neq \pm i$.

The library syntax is `GEN gatan(GEN x, long prec)`.

3.3.14 atanh(x). Principal branch of $\tanh^{-1}(x) = \log((1+x)/(1-x))/2$. In particular the imaginary part of atanh(x) belongs to $[-\pi/2, \pi/2]$; if $x \in \mathbf{R}$ and $|x| > 1$ then atanh(x) is complex.

The library syntax is `GEN gatanh(GEN x, long prec)`.

3.3.15 bernfrac(x). Bernoulli number B_x, where $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$,..., expressed as a rational number. The argument x should be of type integer.

The library syntax is `GEN bernfrac(long x)`.

3.3.16 bernpol$(n, \{v =' x\})$. Bernoulli polynomial B_n in variable v.

```
? bernpol(1)
%1 = x - 1/2
? bernpol(3)
%2 = x^3 - 3/2*x^2 + 1/2*x
```

The library syntax is `GEN bernpol(long n, long v = -1)` where `v` is a variable number.

3.3.17 bernreal(x). Bernoulli number B_x, as `bernfrac`, but B_x is returned as a real number (with the current precision).

The library syntax is `GEN bernreal(long x, long prec)`.

3.3.18 bernvec(x). This routine is obsolete, kept for backward compatibility only.

The library syntax is `GEN bernvec(long x)`.

3.3.19 besselh1(nu, x). H^1-Bessel function of index nu and argument x.

The library syntax is `GEN hbessel1(GEN nu, GEN x, long prec)`.

3.3.20 besselh2(nu, x). H^2-Bessel function of index nu and argument x.

The library syntax is `GEN hbessel2(GEN nu, GEN x, long prec)`.

3.3.21 besseli(nu, x). I-Bessel function of index nu and argument x. If x converts to a power series, the initial factor $(x/2)^\nu/\Gamma(\nu+1)$ is omitted (since it cannot be represented in PARI when ν is not integral).

The library syntax is `GEN ibessel(GEN nu, GEN x, long prec)`.

3.3.22 besselj(nu, x). J-Bessel function of index nu and argument x. If x converts to a power series, the initial factor $(x/2)^\nu/\Gamma(\nu+1)$ is omitted (since it cannot be represented in PARI when ν is not integral).

The library syntax is `GEN jbessel(GEN nu, GEN x, long prec)`.

3.3.23 besseljh(n, x). J-Bessel function of half integral index. More precisely, `besseljh`(n, x) computes $J_{n+1/2}(x)$ where n must be of type integer, and x is any element of **C**. In the present version 2.9.2, this function is not very accurate when x is small.

The library syntax is `GEN jbesselh(GEN n, GEN x, long prec)`.

3.3.24 besselk(nu, x). K-Bessel function of index nu and argument x.

The library syntax is `GEN kbessel(GEN nu, GEN x, long prec)`.

3.3.25 besseln(nu, x). N-Bessel function of index nu and argument x.

The library syntax is `GEN nbessel(GEN nu, GEN x, long prec)`.

3.3.26 cos(x). Cosine of x.

The library syntax is `GEN gcos(GEN x, long prec)`.

3.3.27 cosh(x). Hyperbolic cosine of x.

The library syntax is `GEN gcosh(GEN x, long prec)`.

3.3.28 cotan(x). Cotangent of x.

The library syntax is `GEN gcotan(GEN x, long prec)`.

3.3.29 cotanh(x). Hyperbolic cotangent of x.

The library syntax is `GEN gcotanh(GEN x, long prec)`.

3.3.30 dilog(x). Principal branch of the dilogarithm of x, i.e. analytic continuation of the power series $\log_2(x) = \sum_{n\geq 1} x^n/n^2$.

The library syntax is `GEN dilog(GEN x, long prec)`.

3.3.31 eint1$(x, \{n\})$. Exponential integral $\int_x^\infty \frac{e^{-t}}{t}\,dt =$ `incgam`$(0, x)$, where the latter expression extends the function definition from real $x > 0$ to all complex $x \neq 0$.

If n is present, we must have $x > 0$; the function returns the n-dimensional vector [`eint1`$(x), \ldots,$ `eint1`(nx)]. Contrary to other transcendental functions, and to the default case (n omitted), the values are correct up to a bounded *absolute*, rather than relative, error 10^{-n}, where n is `precision`(x) if x is a `t_REAL` and defaults to `realprecision` otherwise. (In the most important application, to the computation of L-functions via approximate functional equations, those values appear as weights in long sums and small individual relative errors are less useful than controlling the absolute error.) This is faster than repeatedly calling `eint1(`i `* x)`, but less precise.

The library syntax is `GEN veceint1(GEN x, GEN n = NULL, long prec)`. Also available is `GEN eint1(GEN x, long prec)`.

3.3.32 erfc(x). Complementary error function, analytic continuation of $(2/\sqrt{\pi})\int_x^\infty e^{-t^2}\,dt =$ `incgam`$(1/2, x^2)/\sqrt{\pi}$, where the latter expression extends the function definition from real x to all complex $x \neq 0$.

The library syntax is `GEN gerfc(GEN x, long prec)`.

3.3.33 eta$(z, \{flag = 0\})$. Variants of Dedekind's η function. If *flag* $= 0$, return $\prod_{n=1}^{\infty}(1-q^n)$, where q depends on x in the following way:

• $q = e^{2i\pi x}$ if x is a *complex number* (which must then have positive imaginary part); notice that the factor $q^{1/24}$ is missing!

• $q = x$ if x is a `t_PADIC`, or can be converted to a *power series* (which must then have positive valuation).

If *flag* is non-zero, x is converted to a complex number and we return the true η function, $q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, where $q = e^{2i\pi x}$.

The library syntax is `GEN eta0(GEN z, long flag, long prec)`.

Also available is `GEN trueeta(GEN x, long prec)` (*flag* $= 1$).

3.3.34 exp(x). Exponential of x. p-adic arguments with positive valuation are accepted.

The library syntax is `GEN gexp(GEN x, long prec)`. For a `t_PADIC` x, the function `GEN Qp_exp(GEN x)` is also available.

3.3.35 expm1(x). Return $\exp(x)-1$, computed in a way that is also accurate when the real part of x is near 0. A naive direct computation would suffer from catastrophic cancellation; PARI's direct computation of $\exp(x)$ alleviates this well known problem at the expense of computing $\exp(x)$ to a higher accuracy when x is small. Using `expm1` is recommended instead:

```
? default(realprecision, 10000); x = 1e-100;
? a = expm1(x);
time = 4 ms.
? b = exp(x)-1;
time = 28 ms.
? default(realprecision, 10040); x = 1e-100;
? c = expm1(x);  \\ reference point
? abs(a-c)/c  \\ relative error in expm1(x)
%7 = 0.E-10017
? abs(b-c)/c  \\ relative error in exp(x)-1
%8 = 1.7907031188259675794 E-9919
```

As the example above shows, when x is near 0, `expm1` is both faster and more accurate than `exp(x)-1`.

The library syntax is `GEN gexpm1(GEN x, long prec)`.

3.3.36 gamma(s). For s a complex number, evaluates Euler's gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} \exp(-t)\, dt.$$

Error if s is a non-positive integer, where Γ has a pole.

For s a `t_PADIC`, evaluates the Morita gamma function at s, that is the unique continuous p-adic function on the p-adic integers extending $\Gamma_p(k) = (-1)^k \prod'_{j<k} j$, where the prime means that p does not divide j.

```
? gamma(1/4 + O(5^10))
%1= 1 + 4*5 + 3*5^4 + 5^6 + 5^7 + 4*5^9 + O(5^10)
? algdep(%,4)
%2 = x^4 + 4*x^2 + 5
```

The library syntax is `GEN ggamma(GEN s, long prec)`. For a `t_PADIC` x, the function `GEN Qp_gamma(GEN x)` is also available.

3.3.37 gammah(x). Gamma function evaluated at the argument $x + 1/2$.

The library syntax is `GEN ggammah(GEN x, long prec)`.

3.3.38 gammamellininv$(G, t, \{m = 0\})$. Returns the value at t of the inverse Mellin transform G initialized by `gammamellininvinit`.

```
? G = gammamellininvinit([0]);
? gammamellininv(G, 2) - 2*exp(-Pi*2^2)
%2 = -4.484155085839414627 E-44
```

The alternative shortcut

```
gammamellininv(A,t,m)
```

for

```
gammamellininv(gammamellininvinit(A,m), t)
```

is available.

The library syntax is `GEN gammamellininv(GEN G, GEN t, long m, long bitprec)`.

3.3.39 gammamellininvasymp$(A, n, \{m = 0\})$. Return the first n terms of the asymptotic expansion at infinity of the m-th derivative $K^{(m)}(t)$ of the inverse Mellin transform of the function

$$f(s) = \Gamma_{\mathbf{R}}(s + a_1) \dots \Gamma_{\mathbf{R}}(s + a_d)\ ,$$

where `A` is the vector $[a_1, \dots, a_d]$ and $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ (Euler's `gamma`). The result is a vector $[M[1]...M[n]]$ with M[1]=1, such that

$$K^{(m)}(t) = \sqrt{2^{d+1}/d}t^{a+m(2/d-1)} e^{-d\pi t^{2/d}} \sum_{n\geq 0} M[n+1](\pi t^{2/d})^{-n}$$

with $a = (1 - d + \sum_{1\leq j\leq d} a_j)/d$.

The library syntax is `GEN gammamellininvasymp(GEN A, long precdl, long n)`.

3.3.40 gammamellininvinit$(A, \{m = 0\})$. Initialize data for the computation by `gammamellininv` of the m-th derivative of the inverse Mellin transform of the function

$$f(s) = \Gamma_{\mathbf{R}}(s + a_1) \dots \Gamma_{\mathbf{R}}(s + a_d)$$

where `A` is the vector $[a_1, \dots, a_d]$ and $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ (Euler's `gamma`). This is the special case of Meijer's G functions used to compute L-values via the approximate functional equation.

Caveat. Contrary to the PARI convention, this function guarantees an *absolute* (rather than relative) error bound.

For instance, the inverse Mellin transform of $\Gamma_{\mathbf{R}}(s)$ is $2\exp(-\pi z^2)$:

```
? G = gammamellininvinit([0]);
? gammamellininv(G, 2) - 2*exp(-Pi*2^2)
%2 = -4.484155085839414627 E-44
```

The inverse Mellin transform of $\Gamma_{\mathbf{R}}(s + 1)$ is $2z\exp(-\pi z^2)$, and its second derivative is $4\pi z\exp(-\pi z^2)(2\pi z^2 - 3)$:

```
? G = gammamellininvinit([1], 2);
? a(z) = 4*Pi*z*exp(-Pi*z^2)*(2*Pi*z^2-3);
? b(z) = gammamellininv(G,z);
? t(z) = b(z) - a(z);
? t(3/2)
%3 = -1.4693679385278593850 E-39
```

The library syntax is `GEN gammamellininvinit(GEN A, long m, long bitprec)`.

3.3.41 hyperu(a, b, x). U-confluent hypergeometric function with parameters a and b. The parameters a and b can be complex but the present implementation requires x to be positive.

The library syntax is `GEN hyperu(GEN a, GEN b, GEN x, long prec)`.

3.3.42 incgam$(s, x, \{g\})$. Incomplete gamma function $\int_x^\infty e^{-t}t^{s-1}\,dt$, extended by analytic continuation to all complex x, s not both 0. The relative error is bounded in terms of the precision of s (the accuracy of x is ignored when determining the output precision). When g is given, assume that $g = \Gamma(s)$. For small $|x|$, this will speed up the computation.

The library syntax is `GEN incgam0(GEN s, GEN x, GEN g = NULL, long prec)`. Also available is `GEN incgam(GEN s, GEN x, long prec)`.

3.3.43 incgamc(s, x). Complementary incomplete gamma function. The arguments x and s are complex numbers such that s is not a pole of Γ and $|x|/(|s|+1)$ is not much larger than 1 (otherwise the convergence is very slow). The result returned is $\int_0^x e^{-t}t^{s-1}\,dt$.

The library syntax is `GEN incgamc(GEN s, GEN x, long prec)`.

3.3.44 lambertw(y). Lambert W function, solution of the implicit equation $xe^x = y$, for $y > 0$.

The library syntax is `GEN glambertW(GEN y, long prec)`.

3.3.45 lngamma(x). Principal branch of the logarithm of the gamma function of x. This function is analytic on the complex plane with non-positive integers removed, and can have much larger arguments than `gamma` itself.

For x a power series such that $x(0)$ is not a pole of `gamma`, compute the Taylor expansion. (PARI only knows about regular power series and can't include logarithmic terms.)

```
? lngamma(1+x+O(x^2))
%1 = -0.57721566490153286060651209008240243104*x + O(x^2)
? lngamma(x+O(x^2))
 ***   at top-level: lngamma(x+O(x^2))
 ***                 ^-----------------
 *** lngamma: domain error in lngamma: valuation != 0
? lngamma(-1+x+O(x^2))
 *** lngamma: Warning: normalizing a series with 0 leading term.
 ***   at top-level: lngamma(-1+x+O(x^2))
 ***                 ^--------------------
 *** lngamma: domain error in intformal: residue(series, pole) != 0
```

The library syntax is `GEN glngamma(GEN x, long prec)`.

3.3.46 log(x). Principal branch of the natural logarithm of $x \in \mathbf{C}^*$, i.e. such that $\Im(\log(x)) \in]-\pi, \pi]$. The branch cut lies along the negative real axis, continuous with quadrant 2, i.e. such that $\lim_{b\to 0^+} \log(a+bi) = \log a$ for $a \in \mathbf{R}^*$. The result is complex (with imaginary part equal to π) if $x \in \mathbf{R}$ and $x < 0$. In general, the algorithm uses the formula

$$\log(x) \approx \frac{\pi}{2\mathrm{agm}(1, 4/s)} - m\log 2,$$

if $s = x2^m$ is large enough. (The result is exact to B bits provided $s > 2^{B/2}$.) At low accuracies, the series expansion near 1 is used.

p-adic arguments are also accepted for x, with the convention that $\log(p) = 0$. Hence in particular $\exp(\log(x))/x$ is not in general equal to 1 but to a $(p-1)$-th root of unity (or ± 1 if $p = 2$) times a power of p.

The library syntax is `GEN glog(GEN x, long prec)`. For a `t_PADIC` x, the function `GEN Qp_log(GEN x)` is also available.

3.3.47 polylog$(m, x, \{flag = 0\})$. One of the different polylogarithms, depending on *flag*:

If *flag* $= 0$ or is omitted: m^{th} polylogarithm of x, i.e. analytic continuation of the power series $\mathrm{Li}_m(x) = \sum_{n\geq 1} x^n/n^m$ $(x < 1)$. Uses the functional equation linking the values at x and $1/x$ to restrict to the case $|x| \leq 1$, then the power series when $|x|^2 \leq 1/2$, and the power series expansion in $\log(x)$ otherwise.

Using *flag*, computes a modified m^{th} polylogarithm of x. We use Zagier's notations; let $\Re_m$ denote $\Re$ or $\Im$ depending on whether m is odd or even:

If *flag* $= 1$: compute $\tilde{D}_m(x)$, defined for $|x| \leq 1$ by

$$\Re_m\left(\sum_{k=0}^{m-1} \frac{(-\log|x|)^k}{k!}\mathrm{Li}_{m-k}(x) + \frac{(-\log|x|)^{m-1}}{m!}\log|1-x|\right).$$

If *flag* = 2: compute $D_m(x)$, defined for $|x| \leq 1$ by

$$\Re_m\left(\sum_{k=0}^{m-1}\frac{(-\log|x|)^k}{k!}\mathrm{Li}_{m-k}(x) - \frac{1}{2}\frac{(-\log|x|)^m}{m!}\right).$$

If *flag* = 3: compute $P_m(x)$, defined for $|x| \leq 1$ by

$$\Re_m\left(\sum_{k=0}^{m-1}\frac{2^kB_k}{k!}(\log|x|)^k\mathrm{Li}_{m-k}(x) - \frac{2^{m-1}B_m}{m!}(\log|x|)^m\right).$$

These three functions satisfy the functional equation $f_m(1/x) = (-1)^{m-1}f_m(x)$.

The library syntax is `GEN polylog0(long m, GEN x, long flag, long prec)`. Also available is `GEN gpolylog(long m, GEN x, long prec)` (*flag*= 0).

3.3.48 psi(x). The ψ-function of x, i.e. the logarithmic derivative $\Gamma'(x)/\Gamma(x)$.

The library syntax is `GEN gpsi(GEN x, long prec)`.

3.3.49 sin(x). Sine of x.

The library syntax is `GEN gsin(GEN x, long prec)`.

3.3.50 sinc(x). Cardinal sine of x, i.e. $\sin(x)/x$ if $x \neq 0$, 1 otherwise. Note that this function also allows to compute

$$(1-\cos(x))/x^2 = \texttt{sinc}(x/2)^2/2$$

accurately near $x = 0$.

The library syntax is `GEN gsinc(GEN x, long prec)`.

3.3.51 sinh(x). Hyperbolic sine of x.

The library syntax is `GEN gsinh(GEN x, long prec)`.

3.3.52 sqr(x). Square of x. This operation is not completely straightforward, i.e. identical to $x*x$, since it can usually be computed more efficiently (roughly one-half of the elementary multiplications can be saved). Also, squaring a 2-adic number increases its precision. For example,

```
? (1 + O(2^4))^2
%1 = 1 + O(2^5)
? (1 + O(2^4)) * (1 + O(2^4))
%2 = 1 + O(2^4)
```

Note that this function is also called whenever one multiplies two objects which are known to be *identical*, e.g. they are the value of the same variable, or we are computing a power.

```
? x = (1 + O(2^4)); x * x
%3 = 1 + O(2^5)
? (1 + O(2^4))^4
%4 = 1 + O(2^6)
```

(note the difference between `%2` and `%3` above).

The library syntax is `GEN gsqr(GEN x)`.

3.3.53 sqrt(x). Principal branch of the square root of x, defined as $\sqrt{x} = \exp(\log x/2)$. In particular, we have $\mathrm{Arg}(\mathrm{sqrt}(x)) \in]-\pi/2, \pi/2]$, and if $x \in \mathbf{R}$ and $x < 0$, then the result is complex with positive imaginary part.

Intmod a prime p, `t_PADIC` and `t_FFELT` are allowed as arguments. In the first 2 cases (`t_INTMOD`, `t_PADIC`), the square root (if it exists) which is returned is the one whose first p-adic digit is in the interval $[0, p/2]$. For other arguments, the result is undefined.

The library syntax is `GEN gsqrt(GEN x, long prec)`. For a `t_PADIC` x, the function `GEN Qp_sqrt(GEN x)` is also available.

3.3.54 sqrtn$(x, n, \{\&z\})$. Principal branch of the nth root of x, i.e. such that $\mathrm{Arg}(\mathrm{sqrtn}(x)) \in]-\pi/n, \pi/n]$. Intmod a prime and p-adics are allowed as arguments.

If z is present, it is set to a suitable root of unity allowing to recover all the other roots. If it was not possible, z is set to zero. In the case this argument is present and no nth root exist, 0 is returned instead of raising an error.

```
? sqrtn(Mod(2,7), 2)
%1 = Mod(3, 7)
? sqrtn(Mod(2,7), 2, &z); z
%2 = Mod(6, 7)
? sqrtn(Mod(2,7), 3)
  ***   at top-level: sqrtn(Mod(2,7),3)
  ***                 ^-----------------
  *** sqrtn: nth-root does not exist in gsqrtn.
? sqrtn(Mod(2,7), 3,  &z)
%2 = 0
? z
%3 = 0
```

The following script computes all roots in all possible cases:

```
sqrtnall(x,n)=
{ my(V,r,z,r2);
  r = sqrtn(x,n, &z);
  if (!z, error("Impossible case in sqrtn"));
  if (type(x) == "t_INTMOD" || type(x)=="t_PADIC",
    r2 = r*z; n = 1;
    while (r2!=r, r2*=z;n++));
  V = vector(n); V[1] = r;
  for(i=2, n, V[i] = V[i-1]*z);
  V
}
addhelp(sqrtnall,"sqrtnall(x,n):compute the vector of nth-roots of x");
```

The library syntax is `GEN gsqrtn(GEN x, GEN n, GEN *z = NULL, long prec)`. If x is a `t_PADIC`, the function `GEN Qp_sqrtn(GEN x, GEN n, GEN *z)` is also available.

3.3.55 tan(x). Tangent of x.

The library syntax is `GEN gtan(GEN x, long prec)`.

3.3.56 tanh(x). Hyperbolic tangent of x.

The library syntax is `GEN gtanh(GEN x, long prec)`.

3.3.57 teichmuller$(x, \{tab\})$. Teichmüller character of the p-adic number x, i.e. the unique $(p-1)$-th root of unity congruent to $x/p^{v_p(x)}$ modulo p. If x is of the form $[p, n]$, for a prime p and integer n, return the lifts to $\mathbf{Z}$ of the images of $i + O(p^n)$ for $i = 1, \ldots, p-1$, i.e. all roots of 1 ordered by residue class modulo p. Such a vector can be fed back to `teichmuller`, as the optional argument `tab`, to speed up later computations.

```
? z = teichmuller(2 + O(101^5))
%1 = 2 + 83*101 + 18*101^2 + 69*101^3 + 62*101^4 + O(101^5)
? z^100
%2 = 1 + O(101^5)
? T = teichmuller([101, 5]);
? teichmuller(2 + O(101^5), T)
%4 = 2 + 83*101 + 18*101^2 + 69*101^3 + 62*101^4 + O(101^5)
```

As a rule of thumb, if more than

$$p \,/\, 2(\log_2(p) + \texttt{hammingweight}(p))$$

values of `teichmuller` are to be computed, then it is worthwile to initialize:

```
? p = 101; n = 100; T = teichmuller([p,n]); \\ instantaneous
? for(i=1,10^3, vector(p-1, i, teichmuller(i+O(p^n), T)))
time = 60 ms.
? for(i=1,10^3, vector(p-1, i, teichmuller(i+O(p^n))))
time = 1,293 ms.
? 1 + 2*(log(p)/log(2) + hammingweight(p))
%8 = 22.316[...]
```

Here the precompuation induces a speedup by a factor $1293/60 \approx 21.5$.

Caveat. If the accuracy of `tab` (the argument n above) is lower than the precision of x, the *former* is used, i.e. the cached value is not refined to higher accuracy. It the accuracy of `tab` is larger, then the precision of x is used:

```
? Tlow = teichmuller([101, 2]); \\ lower accuracy !
? teichmuller(2 + O(101^5), Tlow)
%10 = 2 + 83*101 + O(101^5)  \\ no longer a root of 1
? Thigh = teichmuller([101, 10]); \\ higher accuracy
? teichmuller(2 + O(101^5), Thigh)
%12 = 2 + 83*101 + 18*101^2 + 69*101^3 + 62*101^4 + O(101^5)
```

The library syntax is `GEN teichmuller(GEN x, GEN tab = NULL)`.

Also available are the functions `GEN teich(GEN x)` (`tab` is `NULL`) as well as `GEN teichmullerinit(long p, long n)`.

3.3.58 theta(q,z). Jacobi sine theta-function

$$\theta_1(z,q) = 2q^{1/4}\sum_{n\geq 0}(-1)^n q^{n(n+1)}\sin((2n+1)z).$$

The library syntax is `GEN theta(GEN q, GEN z, long prec)`.

3.3.59 thetanullk(q,k). k-th derivative at $z=0$ of `theta`(q,z).

The library syntax is `GEN thetanullk(GEN q, long k, long prec)`.

`GEN vecthetanullk(GEN q, long k, long prec)` returns the vector of all $\frac{d^i\theta}{dz^i}(q,0)$ for all odd $i = 1,3,\ldots,2k-1$. `GEN vecthetanullk_tau(GEN tau, long k, long prec)` returns `vecthetanullk_tau` at $q = \exp(2i\pi$`tau`$)$.

3.3.60 weber$(x,\{flag=0\})$. One of Weber's three f functions. If $flag = 0$, returns

$$f(x) = \exp(-i\pi/24)\cdot\eta((x+1)/2)\,/\,\eta(x) \quad \text{such that} \quad j = (f^{24}-16)^3/f^{24},$$

where j is the elliptic j-invariant (see the function `ellj`). If $flag = 1$, returns

$$f_1(x) = \eta(x/2)\,/\,\eta(x) \quad \text{such that} \quad j = (f_1^{24}+16)^3/f_1^{24}.$$

Finally, if $flag = 2$, returns

$$f_2(x) = \sqrt{2}\eta(2x)\,/\,\eta(x) \quad \text{such that} \quad j = (f_2^{24}+16)^3/f_2^{24}.$$

Note the identities $f^8 = f_1^8 + f_2^8$ and $ff_1f_2 = \sqrt{2}$.

The library syntax is `GEN weber0(GEN x, long flag, long prec)`. Also available are `GEN weberf(GEN x, long prec)`, `GEN weberf1(GEN x, long prec)` and `GEN weberf2(GEN x, long prec)`.

3.3.61 zeta(s). For s a complex number, Riemann's zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$, computed using the Euler-Maclaurin summation formula, except when s is of type integer, in which case it is computed using Bernoulli numbers for $s \leq 0$ or $s > 0$ and even, and using modular forms for $s > 0$ and odd.

For s a p-adic number, Kubota-Leopoldt zeta function at s, that is the unique continuous p-adic function on the p-adic integers that interpolates the values of $(1-p^{-k})\zeta(k)$ at negative integers k such that $k \equiv 1 \pmod{p-1}$ (resp. k is odd) if p is odd (resp. $p = 2$).

The library syntax is `GEN gzeta(GEN s, long prec)`.

3.3.62 zetamult(s). For s a vector of positive integers such that $s[1] \geq 2$, returns the multiple zeta value (MZV)

$$\zeta(s_1,\ldots,s_k) = \sum_{n_1>\ldots>n_k>0} n_1^{-s_1}\ldots n_k^{-s_k}.$$

```
? zetamult([2,1]) - zeta(3) \\ Euler's identity
%1 = 0.E-38
```

The library syntax is `GEN zetamult(GEN s, long prec)`.

3.4 Arithmetic functions.

These functions are by definition functions whose natural domain of definition is either $\mathbf{Z}$ (or $\mathbf{Z}_{>0}$). The way these functions are used is completely different from transcendental functions in that there are no automatic type conversions: in general only integers are accepted as arguments. An integer argument N can be given in the following alternate formats:

- `t_MAT`: its factorization `fa = factor(`N`)`,
- `t_VEC`: a pair `[`N`, fa]` giving both the integer and its factorization.

This allows to compute different arithmetic functions at a given N while factoring the latter only once.

```
? N = 10!; faN = factor(N);
? eulerphi(N)
%2 = 829440
? eulerphi(faN)
%3 = 829440
? eulerphi(S = [N, faN])
%4 = 829440
? sigma(S)
%5 = 15334088
```

3.4.1 Arithmetic functions and the factoring engine. All arithmetic functions in the narrow sense of the word — Euler's totient function, the Moebius function, the sums over divisors or powers of divisors etc.— call, after trial division by small primes, the same versatile factoring machinery described under `factorint`. It includes Shanks SQUFOF, Pollard Rho, ECM and MPQS stages, and has an early exit option for the functions **moebius** and (the integer function underlying) **issquarefree**. This machinery relies on a fairly strong probabilistic primality test, see `ispseudoprime`, but you may also set

```
default(factor_proven, 1)
```

to ensure that all tentative factorizations are fully proven. This should not slow down PARI too much, unless prime numbers with hundreds of decimal digits occur frequently in your application.

3.4.2 Orders in finite groups and Discrete Logarithm functions.

The following functions compute the order of an element in a finite group: `ellorder` (the rational points on an elliptic curve defined over a finite field), `fforder` (the multiplicative group of a finite field), `znorder` (the invertible elements in $\mathbf{Z}/n\mathbf{Z}$). The following functions compute discrete logarithms in the same groups (whenever this is meaningful) `elllog`, `fflog`, `znlog`.

All such functions allow an optional argument specifying an integer N, representing the order of the group. (The *order* functions also allows any non-zero multiple of the order, with a minor loss of efficiency.) That optional argument follows the same format as given above:

- `t_INT`: the integer N,
- `t_MAT`: the factorization `fa = factor(`N`)`,
- `t_VEC`: this is the preferred format and provides both the integer N and its factorization in a two-component vector `[`N`, fa]`.

When the group is fixed and many orders or discrete logarithms will be computed, it is much more efficient to initialize this data once and for all and pass it to the relevant functions, as in

```
? p = nextprime(10^40);
? v = [p-1, factor(p-1)]; \\ data for discrete log & order computations
? znorder(Mod(2,p), v)
%3 = 500000000000000000000000000028
? g = znprimroot(p);
? znlog(2, g, v)
%5 = 54303807090401490880187861137 4
```

3.4.3 Dirichlet characters.

The finite abelian group $G = (\mathbf{Z}/N\mathbf{Z})^*$ can be written $G = \oplus_{i \leq n}(\mathbf{Z}/d_i\mathbf{Z})g_i$, with $d_n \mid \ldots \mid d_2 \mid d_1$ (SNF condition), all $d_i > 0$, and $\prod_i d_i = \phi(N)$.

The SNF condition makes the d_i unique, but the generators g_i, of respective order d_i, are definitely not unique. The $\oplus$ notation means that all elements of G can be written uniquely as $\prod_i g_i^{n_i}$ where $n_i \in \mathbf{Z}/d_i\mathbf{Z}$. The g_i are the so-called *SNF generators* of G.

• a *character* on the abelian group $\oplus(\mathbf{Z}/d_j\mathbf{Z})g_j$ is given by a row vector $\chi = [a_1, \ldots, a_n]$ of integers $0 \leq a_i < d_i$ such that $\chi(g_j) = e(a_j/d_j)$ for all j, with the standard notation $e(x) := \exp(2i\pi x)$. In other words, $\chi(\prod g_j^{n_j}) = e(\sum a_j n_j/d_j)$.

This will be generalized to more general abelian groups in later sections (Hecke characters), but in the present case of $(\mathbf{Z}/N\mathbf{Z})^*$, there is a useful alternate convention : namely, it is not necessary to impose the SNF condition and we can use Chinese reminders instead. If $N = \prod p^{e_p}$ is the factorization of N into primes, the so-called *Conrey generators* of G are the generators of the $(\mathbf{Z}/p^{e_p}\mathbf{Z})^*$ lifted to $(\mathbf{Z}/N\mathbf{Z})^*$ by requesting that they be congruent to 1 modulo N/p^{e_p} (for p odd we take the smallest positive primitive root, and for $p = 2$ we take -1 if $e_2 > 1$ and additionally 5 if $e_2 > 2$). We can again write $G = \oplus_{i \leq n}(\mathbf{Z}/D_i\mathbf{Z})G_i$, where again $\prod_i D_i = \phi(N)$. These generators don't satisfy the SNF condition in general since their orders are now $(p-1)p^{e_p-1}$ for p odd; for $p = 2$, the generator -1 has order 2 and 5 has order 2^{e_2-2} ($e_2 > 2$). Nevertheless, any $m \in (\mathbf{Z}/N\mathbf{Z})^*$ can be uniquely decomposed as $\prod G_i^{m_i}$ for some m_i modulo D_i and we can define a character by $\chi(G_j) = e(m_j/D_j)$ for all j.

• The *column vector* of the m_j, $0 \leq m_j < D_j$ is called the *Conrey logarithm* of m (discrete logarithm in terms of the Conrey generators). Note that discrete logarithms in PARI/GP are always expressed as `t_COL`s.

• The attached character is called the *Conrey character* attached to m.

To sum up a Dirichlet character can be defined by a `t_INT` (the Conrey label m), a `t_COL` (the Conrey logarithm of m, in terms of the Conrey generators) or a `t_VEC` (in terms of the SNF generators). The `t_COL` format, i.e. Conrey logarithms, is the preferred (fastest) representation.

Concretely, this works as follows:

`G = idealstar(,N)` initializes $(\mathbf{Z}/N\mathbf{Z})^*$, which must be given as first arguments to all functions handling Dirichlet characters.

`znconreychar` transforms `t_INT` and `t_COL` to a SNF character.

`znconreylog` transforms `t_INT` and `t_VEC` to a Conrey logarithm.

`znconreyexp` transforms `t_VEC` and `t_COL` to a Conrey label.

Also available are `charconj`, `chardiv`, `charmul`, `charker`, `chareval`, `charorder`, `zncharinduce`, `znconreyconductor` (also computes the primitive character attached to the input character). The prefix `char` indicates that the function applies to all characters, the prefix `znchar` that it is specific to Dirichlet characters (on $(\mathbf{Z}/N\mathbf{Z})^*$) and the prefix `znconrey` that it is specific to Conrey representation.

3.4.4 addprimes($\{x = []\}$). Adds the integers contained in the vector x (or the single integer x) to a special table of "user-defined primes", and returns that table. Whenever `factor` is subsequently called, it will trial divide by the elements in this table. If x is empty or omitted, just returns the current list of extra primes.

The entries in x must be primes: there is no internal check, even if the `factor_proven` default is set. To remove primes from the list use `removeprimes`.

The library syntax is `GEN addprimes(GEN x = NULL)`.

3.4.5 bestappr($x, \{B\}$). Using variants of the extended Euclidean algorithm, returns a rational approximation a/b to x, whose denominator is limited by B, if present. If B is omitted, return the best approximation affordable given the input accuracy; if you are looking for true rational numbers, presumably approximated to sufficient accuracy, you should first try that option. Otherwise, B must be a positive real scalar (impose $0 < b \leq B$).

• If x is a `t_REAL` or a `t_FRAC`, this function uses continued fractions.

```
? bestappr(Pi, 100)
%1 = 22/7
? bestappr(0.142857142857142857142857142857)
%2 = 1/7
? bestappr([Pi, sqrt(2) + 'x], 10^3)
%3 = [355/113, x + 1393/985]
```

By definition, a/b is the best rational approximation to x if $|bx - a| < |vx - u|$ for all integers (u, v) with $0 < v \leq B$. (Which implies that n/d is a convergent of the continued fraction of x.)

• If x is a `t_INTMOD` modulo N or a `t_PADIC` of precision $N = p^k$, this function performs rational modular reconstruction modulo N. The routine then returns the unique rational number a/b in coprime integers $|a| < N/2B$ and $b \leq B$ which is congruent to x modulo N. Omitting B amounts to choosing it of the order of $\sqrt{N/2}$. If rational reconstruction is not possible (no suitable a/b exists), returns [].

```
? bestappr(Mod(18526731858, 11^10))
%1 = 1/7
? bestappr(Mod(18526731858, 11^20))
%2 = []
? bestappr(3 + 5 + 3*5^2 + 5^3 + 3*5^4 + 5^5 + 3*5^6 + O(5^7))
%2 = -1/3
```

In most concrete uses, B is a prime power and we performed Hensel lifting to obtain x.

The function applies recursively to components of complex objects (polynomials, vectors, ...). If rational reconstruction fails for even a single entry, return [].

The library syntax is `GEN bestappr(GEN x, GEN B = NULL)`.

3.4.6 bestapprPade$(x, \{B\})$. Using variants of the extended Euclidean algorithm, returns a rational function approximation a/b to x, whose denominator is limited by B, if present. If B is omitted, return the best approximation affordable given the input accuracy; if you are looking for true rational functions, presumably approximated to sufficient accuracy, you should first try that option. Otherwise, B must be a non-negative real (impose $0 \le \text{degree}(b) \le B$).

• If x is a `t_RFRAC` or `t_SER`, this function uses continued fractions.

```
? bestapprPade((1-x^11)/(1-x)+O(x^11))
%1 = 1/(-x + 1)
? bestapprPade([1/(1+x+O(x^10)), (x^3-2)/(x^3+1)], 1)
%2 =  [1/(x + 1), -2]
```

• If x is a `t_POLMOD` modulo N or a `t_SER` of precision $N = t^k$, this function performs rational modular reconstruction modulo N. The routine then returns the unique rational function a/b in coprime polynomials, with $\text{degree}(b) \le B$ which is congruent to x modulo N. Omitting B amounts to choosing it of the order of $N/2$. If rational reconstruction is not possible (no suitable a/b exists), returns [].

```
? bestapprPade(Mod(1+x+x^2+x^3+x^4, x^4-2))
%1 = (2*x - 1)/(x - 1)
? % * Mod(1,x^4-2)
%2 = Mod(x^3 + x^2 + x + 3, x^4 - 2)
? bestapprPade(Mod(1+x+x^2+x^3+x^5, x^9))
%2 = []
? bestapprPade(Mod(1+x+x^2+x^3+x^5, x^10))
%3 = (2*x^4 + x^3 - x - 1)/(-x^5 + x^3 + x^2 - 1)
```

The function applies recursively to components of complex objects (polynomials, vectors, ...). If rational reconstruction fails for even a single entry, return [].

The library syntax is `GEN bestapprPade(GEN x, long B)`.

3.4.7 bezout(x, y). Deprecated alias for `gcdext`

The library syntax is `GEN gcdext0(GEN x, GEN y)`.

3.4.8 bigomega(x). Number of prime divisors of the integer $|x|$ counted with multiplicity:

```
? factor(392)
%1 =
[2 3]

[7 2]

? bigomega(392)
%2 = 5;  \\ = 3+2
? omega(392)
%3 = 2;  \\ without multiplicity
```

The library syntax is `long bigomega(GEN x)`.

3.4.9 binomial(x, y). binomial coefficient $\binom{x}{y}$. Here y must be an integer, but x can be any PARI object.

The library syntax is `GEN binomial(GEN x, long y)`. The function `GEN binomialuu(ulong n, ulong k)` is also available, and so is `GEN vecbinome(long n)`, which returns a vector v with $n+1$ components such that $v[k+1] =$ `binomial`(n, k) for k from 0 up to n.

3.4.10 charconj(cyc, chi). Let cyc represent a finite abelian group by its elementary divisors, i.e. (d_j) represents $\sum_{j \le k} \mathbf{Z}/d_j\mathbf{Z}$ with $d_k \mid \ldots \mid d_1$; any object which has a `.cyc` method is also allowed, e.g. the output of `znstar` or `bnrinit`. A character on this group is given by a row vector $\chi = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = \exp(2\pi i \sum a_j n_j/d_j)$, where g_j denotes the generator (of order d_j) of the j-th cyclic component.

This function returns the conjugate character.

```
? cyc = [15,5]; chi = [1,1];
? charconj(cyc, chi)
%2 = [14, 4]
? bnf = bnfinit(x^2+23);
? bnf.cyc
%4 = [3]
? charconj(bnf, [1])
%5 = [2]
```

For Dirichlet characters (when `cyc` is `idealstar(,q)`), characters in Conrey representation are available, see Section 3.4.3 or `??character`:

```
? G = idealstar(,8);  \\ (Z/8Z)^*
? charorder(G, 3)  \\ Conrey label
%2 = 2
? chi = znconreylog(G, 3);
? charorder(G, chi)  \\ Conrey logarithm
%4 = 2
```

The library syntax is `GEN charconj0(GEN cyc, GEN chi)`. Also available is `GEN charconj(GEN cyc, GEN chi)`, when `cyc` is known to be a vector of elementary divisors and `chi` a compatible character (no checks).

3.4.11 chardiv(cyc, a, b). Let cyc represent a finite abelian group by its elementary divisors, i.e. (d_j) represents $\sum_{j \le k} \mathbf{Z}/d_j\mathbf{Z}$ with $d_k \mid \ldots \mid d_1$; any object which has a `.cyc` method is also allowed, e.g. the output of `znstar` or `bnrinit`. A character on this group is given by a row vector $a = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = \exp(2\pi i \sum a_j n_j/d_j)$, where g_j denotes the generator (of order d_j) of the j-th cyclic component.

Given two characters a and b, return the character $a/b = a\bar{b}$.

```
? cyc = [15,5]; a = [1,1]; b =  [2,4];
? chardiv(cyc, a,b)
%2 = [14, 2]
? bnf = bnfinit(x^2+23);
? bnf.cyc
%4 = [3]
```

```
? chardiv(bnf, [1], [2])
%5 = [2]
```

For Dirichlet characters on $(\mathbf{Z}/N\mathbf{Z})^*$, additional representations are available (Conrey labels, Conrey logarithm), see Section 3.4.3 or `??character`. If the two characters are in the same format, the result is given in the same format, otherwise a Conrey logarithm is used.

```
? G = idealstar(,100);
? G.cyc
%2 = [20, 2]
? a = [10, 1]; \\ usual representation for characters
? b = 7; \\ Conrey label;
? c = znconreylog(G, 11); \\ Conrey log
? chardiv(G, b,b)
%6 = 1   \\ Conrey label
? chardiv(G, a,b)
%7 = [0, 5]~  \\ Conrey log
? chardiv(G, a,c)
%7 = [0, 14]~   \\ Conrey log
```

The library syntax is `GEN chardiv0(GEN cyc, GEN a, GEN b)`. Also available is `GEN chardiv(GEN cyc, GEN a, GEN b)`, when `cyc` is known to be a vector of elementary divisors and a, b are compatible characters (no checks).

3.4.12 chareval$(G, chi, x, \{z\}))$. Let G be an abelian group structure affording a discrete logarithm method, e.g $G =$ `idealstar`$(, N)$ for $(\mathbf{Z}/N\mathbf{Z})^*$ or a `bnr` structure, let x be an element of G and let *chi* be a character of G (see the note below for details). This function returns the value of *chi* at x.

Note on characters. Let K be some field. If G is an abelian group, let $\chi : G \to K^*$ be a character of finite order and let o be a multiple of the character order such that $\chi(n) = \zeta^{c(n)}$ for some fixed $\zeta \in K^*$ of multiplicative order o and a unique morphism $c : G \to (\mathbf{Z}/o\mathbf{Z}, +)$. Our usual convention is to write

$$G = (\mathbf{Z}/o_1\mathbf{Z})g_1 \oplus \cdots \oplus (\mathbf{Z}/o_d\mathbf{Z})g_d$$

for some generators (g_i) of respective order d_i, where the group has exponent $o := \mathrm{lcm}_i o_i$. Since $\zeta^o = 1$, the vector (c_i) in $\prod(\mathbf{Z}/o_i\mathbf{Z})$ defines a character χ on G via $\chi(g_i) = \zeta^{c_i(o/o_i)}$ for all i. Classical Dirichlet characters have values in $K = \mathbf{C}$ and we can take $\zeta = \exp(2i\pi/o)$.

Note on Dirichlet characters. In the special case where *bid* is attached to $G = (\mathbf{Z}/q\mathbf{Z})^*$ (as per `bid = idealstar(,q)`), the Dirichlet character *chi* can be written in one of the usual 3 formats: a `t_VEC` in terms of `bid.gen` as above, a `t_COL` in terms of the Conrey generators, or a `t_INT` (Conrey label); see Section 3.4.3 or `??character`.

The character value is encoded as follows, depending on the optional argument z:

• If z is omitted: return the rational number $c(x)/o$ for x coprime to q, where we normalize $0 \leq c(x) < o$. If x can not be mapped to the group (e.g. x is not coprime to the conductor of a Dirichlet or Hecke character) we return the sentinel value -1.

• If z is an integer o, then we assume that o is a multiple of the character order and we return the integer $c(x)$ when x belongs to the group, and the sentinel value -1 otherwise.

• z can be of the form $[zeta, o]$, where *zeta* is an o-th root of 1 and o is a multiple of the character order. We return $\zeta^{c(x)}$ if x belongs to the group, and the sentinel value 0 otherwise. (Note that this coincides with the usual extension of Dirichlet characters to $\mathbf{Z}$, or of Hecke characters to general ideals.)

• Finally, z can be of the form $[vzeta, o]$, where *vzeta* is a vector of powers $\zeta^0, \ldots, \zeta^{o-1}$ of some o-th root of 1 and o is a multiple of the character order. As above, we return $\zeta^{c(x)}$ after a table lookup. Or the sentinel value 0.

The library syntax is `GEN chareval(GEN G, GEN chi, GEN x, GEN z) = NULL)`.

3.4.13 charker(cyc, chi). Let *cyc* represent a finite abelian group by its elementary divisors, i.e. (d_j) represents $\sum_{j \leq k} \mathbf{Z}/d_j\mathbf{Z}$ with $d_k \mid \ldots \mid d_1$; any object which has a `.cyc` method is also allowed, e.g. the output of `znstar` or `bnrinit`. A character on this group is given by a row vector $\chi = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = \exp(2\pi i \sum a_j n_j/d_j)$, where g_j denotes the generator (of order d_j) of the j-th cyclic component.

This function returns the kernel of χ, as a matrix K in HNF which is a left-divisor of `matdiagonal(d)`. Its columns express in terms of the g_j the generators of the subgroup. The determinant of K is the kernel index.

```
? cyc = [15,5]; chi = [1,1];
? charker(cyc, chi)
%2 =
[15 12]

[ 0  1]

? bnf = bnfinit(x^2+23);
? bnf.cyc
%4 = [3]
? charker(bnf, [1])
%5 =
[3]
```

Note that for Dirichlet characters (when `cyc` is `idealstar(,q)`), characters in Conrey representation are available, see Section 3.4.3 or `??character`.

```
? G = idealstar(,8);  \\ (Z/8Z)^*
? charker(G, 1) \\ Conrey label for trivial character
%2 =
```

```
[1 0]

[0 1]
```

The library syntax is `GEN charker0(GEN cyc, GEN chi)`. Also available is `GEN charker(GEN cyc, GEN chi)`, when `cyc` is known to be a vector of elementary divisors and `chi` a compatible character (no checks).

3.4.14 charmul(cyc, a, b). Let *cyc* represent a finite abelian group by its elementary divisors, i.e. (d_j) represents $\sum_{j\le k} \mathbf{Z}/d_j\mathbf{Z}$ with $d_k \mid \ldots \mid d_1$; any object which has a `.cyc` method is also allowed, e.g. the output of `znstar` or `bnrinit`. A character on this group is given by a row vector $a = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = \exp(2\pi i \sum a_j n_j/d_j)$, where g_j denotes the generator (of order d_j) of the j-th cyclic component.

Given two characters a and b, return the product character ab.

```
? cyc = [15,5]; a = [1,1]; b =  [2,4];
? charmul(cyc, a,b)
%2 = [3, 0]
? bnf = bnfinit(x^2+23);
? bnf.cyc
%4 = [3]
? charmul(bnf, [1], [2])
%5 = [0]
```

For Dirichlet characters on $(\mathbf{Z}/N\mathbf{Z})^*$, additional representations are available (Conrey labels, Conrey logarithm), see Section 3.4.3 or `??character`. If the two characters are in the same format, their product is given in the same format, otherwise a Conrey logarithm is used.

```
? G = idealstar(,100);
? G.cyc
%2 = [20, 2]
? a = [10, 1]; \\ usual representation for characters
? b = 7; \\ Conrey label;
? c = znconreylog(G, 11); \\ Conrey log
? charmul(G, b,b)
%6 = 49   \\ Conrey label
? charmul(G, a,b)
%7 = [0, 15]~  \\ Conrey log
? charmul(G, a,c)
%7 = [0, 6]~   \\ Conrey log
```

The library syntax is `GEN charmul0(GEN cyc, GEN a, GEN b)`. Also available is `GEN charmul(GEN cyc, GEN a, GEN b)`, when `cyc` is known to be a vector of elementary divisors and a, b are compatible characters (no checks).

3.4.15 charorder(cyc, chi). Let *cyc* represent a finite abelian group by its elementary divisors, i.e. (d_j) represents $\sum_{j\le k} \mathbf{Z}/d_j\mathbf{Z}$ with $d_k \mid \ldots \mid d_1$; any object which has a `.cyc` method is also allowed, e.g. the output of `znstar` or `bnrinit`. A character on this group is given by a row vector $\chi = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = \exp(2\pi i \sum a_j n_j/d_j)$, where g_j denotes the generator (of order d_j) of the j-th cyclic component.

This function returns the order of the character `chi`.

```
? cyc = [15,5]; chi = [1,1];
? charorder(cyc, chi)
%2 = 15
? bnf = bnfinit(x^2+23);
? bnf.cyc
%4 = [3]
? charorder(bnf, [1])
%5 = 3
```

For Dirichlet characters (when `cyc` is `idealstar(,q)`), characters in Conrey representation are available, see Section 3.4.3 or `??character`:

```
? G = idealstar(,100); \\ (Z/100Z)^*
? charorder(G, 7)   \\ Conrey label
%2 = 4
```

The library syntax is `GEN charorder0(GEN cyc, GEN chi)`. Also available is `GEN charorder(GEN cyc, GEN chi)`, when `cyc` is known to be a vector of elementary divisors and `chi` a compatible character (no checks).

3.4.16 chinese$(x, \{y\})$. If x and y are both intmods or both polmods, creates (with the same type) a z in the same residue class as x and in the same residue class as y, if it is possible.

```
? chinese(Mod(1,2), Mod(2,3))
%1 = Mod(5, 6)
? chinese(Mod(x,x^2-1), Mod(x+1,x^2+1))
%2 = Mod(-1/2*x^2 + x + 1/2, x^4 - 1)
```

This function also allows vector and matrix arguments, in which case the operation is recursively applied to each component of the vector or matrix.

```
? chinese([Mod(1,2),Mod(1,3)], [Mod(1,5),Mod(2,7)])
%3 = [Mod(1, 10), Mod(16, 21)]
```

For polynomial arguments in the same variable, the function is applied to each coefficient; if the polynomials have different degrees, the high degree terms are copied verbatim in the result, as if the missing high degree terms in the polynomial of lowest degree had been `Mod(0,1)`. Since the latter behavior is usually *not* the desired one, we propose to convert the polynomials to vectors of the same length first:

```
? P = x+1; Q = x^2+2*x+1;
? chinese(P*Mod(1,2), Q*Mod(1,3))
%4 = Mod(1, 3)*x^2 + Mod(5, 6)*x + Mod(3, 6)
? chinese(Vec(P,3)*Mod(1,2), Vec(Q,3)*Mod(1,3))
%5 = [Mod(1, 6), Mod(5, 6), Mod(4, 6)]
? Pol(%)
```

```
%6 = Mod(1, 6)*x^2 + Mod(5, 6)*x + Mod(4, 6)
```

If y is omitted, and x is a vector, `chinese` is applied recursively to the components of x, yielding a residue belonging to the same class as all components of x.

Finally `chinese`$(x, x) = x$ regardless of the type of x; this allows vector arguments to contain other data, so long as they are identical in both vectors.

The library syntax is `GEN chinese(GEN x, GEN y = NULL)`. `GEN chinese1(GEN x)` is also available.

3.4.17 content(x). Computes the gcd of all the coefficients of x, when this gcd makes sense. This is the natural definition if x is a polynomial (and by extension a power series) or a vector/matrix. This is in general a weaker notion than the *ideal* generated by the coefficients:

```
? content(2*x+y)
%1 = 1              \\ = gcd(2,y) over Q[y]
```

If x is a scalar, this simply returns the absolute value of x if x is rational (`t_INT` or `t_FRAC`), and either 1 (inexact input) or x (exact input) otherwise; the result should be identical to `gcd(x, 0)`.

The content of a rational function is the ratio of the contents of the numerator and the denominator. In recursive structures, if a matrix or vector *coefficient* x appears, the gcd is taken not with x, but with its content:

```
? content([ [2], 4*matid(3) ])
%1 = 2
```

The content of a `t_VECSMALL` is computed assuming the entries are signed integers.

The library syntax is `GEN content(GEN x)`.

3.4.18 contfrac$(x, \{b\}, \{nmax\})$. Returns the row vector whose components are the partial quotients of the continued fraction expansion of x. In other words, a result $[a_0, \ldots, a_n]$ means that $x \approx a_0 + 1/(a_1 + \ldots + 1/a_n)$. The output is normalized so that $a_n \neq 1$ (unless we also have $n = 0$).

The number of partial quotients $n + 1$ is limited by `nmax`. If `nmax` is omitted, the expansion stops at the last significant partial quotient.

```
? \p19
  realprecision = 19 significant digits
? contfrac(Pi)
%1 = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2]
? contfrac(Pi,, 3)  \\ n = 2
%2 = [3, 7, 15]
```

x can also be a rational function or a power series.

If a vector b is supplied, the numerators are equal to the coefficients of b, instead of all equal to 1 as above; more precisely, $x \approx (1/b_0)(a_0 + b_1/(a_1 + \ldots + b_n/a_n))$; for a numerical continued fraction (x real), the a_i are integers, as large as possible; if x is a rational function, they are polynomials with $\deg a_i = \deg b_i + 1$. The length of the result is then equal to the length of b, unless the next partial quotient cannot be reliably computed, in which case the expansion stops. This happens when a partial remainder is equal to zero (or too small compared to the available significant digits for x a `t_REAL`).

A direct implementation of the numerical continued fraction `contfrac(x,b)` described above would be

```
\\ "greedy" generalized continued fraction
cf(x, b) =
{ my( a= vector(#b), t );

  x *= b[1];
  for (i = 1, #b,
    a[i] = floor(x);
    t = x - a[i]; if (!t || i == #b, break);
    x = b[i+1] / t;
  ); a;
}
```

There is some degree of freedom when choosing the a_i; the program above can easily be modified to derive variants of the standard algorithm. In the same vein, although no builtin function implements the related Engel expansion (a special kind of Egyptian fraction decomposition: $x = 1/a_1 + 1/(a_1a_2) + \dots$), it can be obtained as follows:

```
\\ n terms of the Engel expansion of x
engel(x, n = 10) =
{ my( u = x, a = vector(n) );
  for (k = 1, n,
    a[k] = ceil(1/u);
    u = u*a[k] - 1;
    if (!u, break);
  ); a
}
```

Obsolete hack. (don't use this): if b is an integer, $nmax$ is ignored and the command is understood as `contfrac(`x`,,`b`)`.

The library syntax is `GEN contfrac0(GEN x, GEN b = NULL, long nmax)`. Also available are `GEN gboundcf(GEN x, long nmax)`, `GEN gcf(GEN x)` and `GEN gcf2(GEN b, GEN x)`.

3.4.19 contfracpnqn$(x, \{n = -1\})$. When x is a vector or a one-row matrix, x is considered as the list of partial quotients $[a_0, a_1, \dots, a_n]$ of a rational number, and the result is the 2 by 2 matrix $[p_n, p_{n-1}; q_n, q_{n-1}]$ in the standard notation of continued fractions, so $p_n/q_n = a_0 + 1/(a_1 + \dots + 1/a_n)$. If x is a matrix with two rows $[b_0, b_1, \dots, b_n]$ and $[a_0, a_1, \dots, a_n]$, this is then considered as a generalized continued fraction and we have similarly $p_n/q_n = (1/b_0)(a_0 + b_1/(a_1 + \dots + b_n/a_n))$. Note that in this case one usually has $b_0 = 1$.

If $n \geq 0$ is present, returns all convergents from p_0/q_0 up to p_n/q_n. (All convergents if x is too small to compute the $n + 1$ requested convergents.)

```
? a=contfrac(Pi,20)
%1 = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2]
? contfracpnqn(a,3)
%2 =
[3 22 333 355]

[1  7 106 113]
```

```
? contfracpnqn(a,7)
%3 =
[3 22 333 355 103993 104348 208341 312689]

[1  7 106 113  33102  33215  66317  99532]
```

The library syntax is `GEN contfracpnqn(GEN x, long n)`. also available is `GEN pnqn(GEN x)` for $n = -1$.

3.4.20 core$(n, \{flag = 0\})$. If n is an integer written as $n = df^2$ with d squarefree, returns d. If *flag* is non-zero, returns the two-element row vector $[d, f]$. By convention, we write $0 = 0 \times 1^2$, so `core(0, 1)` returns $[0, 1]$.

The library syntax is `GEN core0(GEN n, long flag)`. Also available are `GEN core(GEN n)` (*flag* $= 0$) and `GEN core2(GEN n)` (*flag* $= 1$)

3.4.21 coredisc$(n, \{flag = 0\})$. A *fundamental discriminant* is an integer of the form $t \equiv 1 \bmod 4$ or $4t \equiv 8, 12 \bmod 16$, with t squarefree (i.e. 1 or the discriminant of a quadratic number field). Given a non-zero integer n, this routine returns the (unique) fundamental discriminant d such that $n = df^2$, f a positive rational number. If *flag* is non-zero, returns the two-element row vector $[d, f]$. If n is congruent to 0 or 1 modulo 4, f is an integer, and a half-integer otherwise.

By convention, `coredisc(0, 1))` returns $[0, 1]$.

Note that `quaddisc`(n) returns the same value as `coredisc`(n), and also works with rational inputs $n \in \mathbf{Q}^*$.

The library syntax is `GEN coredisc0(GEN n, long flag)`. Also available are `GEN coredisc(GEN n)` (*flag* $= 0$) and `GEN coredisc2(GEN n)` (*flag* $= 1$)

3.4.22 dirdiv(x, y). x and y being vectors of perhaps different lengths but with $y[1] \neq 0$ considered as Dirichlet series, computes the quotient of x by y, again as a vector.

The library syntax is `GEN dirdiv(GEN x, GEN y)`.

3.4.23 direuler$(p = a, b, expr, \{c\})$. Computes the Dirichlet series attached to the Euler product of expression *expr* as p ranges through the primes from a to b. *expr* must be a polynomial or rational function in another variable than p (say X) and $expr(X)$ is understood as the local factor $expr(p^{-s})$.

The series is output as a vector of coefficients. If c is omitted, output the first b coefficients of the series; otherwise, output the first c coefficients. The following command computes the **sigma** function, attached to $\zeta(s)\zeta(s-1)$:

```
? direuler(p=2, 10, 1/((1-X)*(1-p*X)))
%1 = [1, 3, 4, 7, 6, 12, 8, 15, 13, 18]

? direuler(p=2, 10, 1/((1-X)*(1-p*X)), 5) \\ fewer terms
%2 = [1, 3, 4, 7, 6]
```

Setting $c < b$ is useless (the same effect would be achieved by setting $b = c$). If $c > b$, the computed coefficients are "missing" Euler factors:

```
? direuler(p=2, 10, 1/((1-X)*(1-p*X)), 15) \\ more terms, no longer = sigma !
%3 = [1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 0, 28, 0, 24, 24]
```

The library syntax is **direuler**`(void *E, GEN (*eval)(void*,GEN), GEN a, GEN b)`

3.4.24 dirmul(x, y). x and y being vectors of perhaps different lengths representing the Dirichlet series $\sum_n x_n n^{-s}$ and $\sum_n y_n n^{-s}$, computes the product of x by y, again as a vector.

```
? dirmul(vector(10,n,1), vector(10,n,moebius(n)))
%1 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```

The product length is the minimum of $\#x*v(y)$ and $\#y*v(x)$, where $v(x)$ is the index of the first non-zero coefficient.

```
? dirmul([0,1], [0,1]);
%2 = [0, 0, 0, 1]
```

The library syntax is `GEN dirmul(GEN x, GEN y)`.

3.4.25 divisors(x). Creates a row vector whose components are the divisors of x. The factorization of x (as output by `factor`) can be used instead.

By definition, these divisors are the products of the irreducible factors of n, as produced by `factor(n)`, raised to appropriate powers (no negative exponent may occur in the factorization). If n is an integer, they are the positive divisors, in increasing order.

The library syntax is `GEN divisors(GEN x)`.

3.4.26 eulerphi(x). Euler's ϕ (totient) function of the integer $|x|$, in other words $|(\mathbf{Z}/x\mathbf{Z})^*|$.

```
? eulerphi(40)
%1 = 16
```

According to this definition we let $\phi(0) := 2$, since $\mathbf{Z}^* = \{-1, 1\}$; this is consistent with `znstar(0)`: we have `znstar`(n)`.no = eulerphi(n)` for all $n \in \mathbf{Z}$.

The library syntax is `GEN eulerphi(GEN x)`.

3.4.27 factor$(x, \{lim\})$. General factorization function, where x is a rational (including integers), a complex number with rational real and imaginary parts, or a rational function (including polynomials). The result is a two-column matrix: the first contains the irreducibles dividing x (rational or Gaussian primes, irreducible polynomials), and the second the exponents. By convention, 0 is factored as 0^1.

Q and Q(i)**.** See `factorint` for more information about the algorithms used. The rational or Gaussian primes are in fact *pseudoprimes* (see `ispseudoprime`), a priori not rigorously proven primes. In fact, any factor which is $\leq 2^{64}$ (whose norm is $\leq 2^{64}$ for an irrational Gaussian prime) is a genuine prime. Use `isprime` to prove primality of other factors, as in

```
? fa = factor(2^2^7 + 1)
%1 =
[59649589127497217 1]

[5704689200685129054721 1]

? isprime( fa[,1] )
%2 = [1, 1]~   \\ both entries are proven primes
```

Another possibility is to set the global default `factor_proven`, which will perform a rigorous primality proof for each pseudoprime factor.

A `t_INT` argument *lim* can be added, meaning that we look only for prime factors $p < lim$. The limit *lim* must be non-negative. In this case, all but the last factor are proven primes, but the remaining factor may actually be a proven composite! If the remaining factor is less than lim^2, then it is prime.

```
? factor(2^2^7 +1, 10^5)
%3 =
[340282366920938463463374607431768211457 1]
```

Deprecated feature. Setting $lim = 0$ is the same as setting it to `primelimit` $+ 1$. Don't use this: it is unwise to rely on global variables when you can specify an explicit argument.

This routine uses trial division and perfect power tests, and should not be used for huge values of *lim* (at most 10^9, say): `factorint(, 1 + 8)` will in general be faster. The latter does not guarantee that all small prime factors are found, but it also finds larger factors, and in a much more efficient way.

```
? F = (2^2^7 + 1) * 1009 * 100003; factor(F, 10^5)  \\ fast, incomplete
time = 0 ms.
%4 =
[1009 1]

[340292575391946091617278508669991164503343 71 1]

? factor(F, 10^9)    \\ very slow
time = 6,892 ms.
%6 =
[1009 1]

[100003 1]

[340282366920938463463374607431768211457 1]

? factorint(F, 1+8)  \\ much faster, all small primes were found
time = 12 ms.
%7 =
[1009 1]

[100003 1]

[340282366920938463463374607431768211457 1]

? factor(F)   \\ complete factorisation
time = 112 ms.
%8 =
[1009 1]

[100003 1]

[59649589127497217 1]

[5704689200685129054721 1]
```

Over $\mathbf{Q}$, the prime factors are sorted in increasing order.

Rational functions. The polynomials or rational functions to be factored must have scalar coefficients. In particular PARI does not know how to factor *multivariate* polynomials. The following domains are currently supported: $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Q}_p$, finite fields and number fields. See `factormod` and `factorff` for the algorithms used over finite fields, `factornf` for the algorithms over number fields. Over $\mathbf{Q}$, van Hoeij's method is used, which is able to cope with hundreds of modular factors.

The routine guesses a sensible ring over which to factor: the smallest ring containing all coefficients, taking into account quotient structures induced by `t_INTMOD`s and `t_POLMOD`s (e.g. if a coefficient in $\mathbf{Z}/n\mathbf{Z}$ is known, all rational numbers encountered are first mapped to $\mathbf{Z}/n\mathbf{Z}$; different moduli will produce an error). Factoring modulo a non-prime number is not supported; to factor in $\mathbf{Q}_p$, use `t_PADIC` coefficients not `t_INTMOD` modulo p^n.

```
? T = x^2+1;
? factor(T);                          \\ over Q
? factor(T*Mod(1,3))                  \\ over F_3
? factor(T*ffgen(ffinit(3,2,'t))^0)   \\ over F_{3^2}
? factor(T*Mod(Mod(1,3), t^2+t+2))    \\ over F_{3^2}, again
? factor(T*(1 + O(3^6))               \\ over Q_3, precision 6
? factor(T*1.)                        \\ over R, current precision
? factor(T*(1.+0.*I))                 \\ over C
? factor(T*Mod(1, y^3-2))             \\ over Q(2^{1/3})
```

In most cases, it is clearer and simpler to call an explicit variant than to rely on the generic `factor` function and the above detection mechanism:

```
? factormod(T, 3)           \\ over F_3
? factorff(T, 3, t^2+t+2))  \\ over F_{3^2}
? factorpadic(T, 3,6)       \\ over Q_3, precision 6
? nffactor(y^3-2, T)        \\ over Q(2^{1/3})
? polroots(T)               \\ over C
```

Note that factorization of polynomials is done up to multiplication by a constant. In particular, the factors of rational polynomials will have integer coefficients, and the content of a polynomial or rational function is discarded and not included in the factorization. If needed, you can always ask for the content explicitly:

```
? factor(t^2 + 5/2*t + 1)
%1 =
[2*t + 1 1]

[t + 2 1]

? content(t^2 + 5/2*t + 1)
%2 = 1/2
```

The irreducible factors are sorted by increasing degree. See also `nffactor`.

The library syntax is `GEN gp_factor0(GEN x, GEN lim = NULL)`. This function should only be used by the gp interface. Use directly `GEN factor(GEN x)` or `GEN boundfact(GEN x, ulong lim)` . The obsolete function `GEN factor0(GEN x, long lim)` is kept for backward compatibility.

3.4.28 factorback$(f, \{e\})$. Gives back the factored object corresponding to a factorization. The integer 1 corresponds to the empty factorization.

If e is present, e and f must be vectors of the same length (e being integral), and the corresponding factorization is the product of the $f[i]^{e[i]}$.

If not, and f is vector, it is understood as in the preceding case with e a vector of 1s: we return the product of the $f[i]$. Finally, f can be a regular factorization, as produced with any `factor` command. A few examples:

```
? factor(12)
%1 =
[2 2]

[3 1]

? factorback(%)
%2 = 12
? factorback([2,3], [2,1])   \\ 2^3 * 3^1
%3 = 12
? factorback([5,2,3])
%4 = 30
```

The library syntax is `GEN factorback2(GEN f, GEN e = NULL)`. Also available is `GEN factorback(GEN f)` (case $e =$ `NULL`).

3.4.29 factorcantor(x, p). Factors the polynomial x modulo the prime p, using distinct degree plus Cantor-Zassenhaus. The coefficients of x must be operation-compatible with $\mathbf{Z}/p\mathbf{Z}$. The result is a two-column matrix, the first column being the irreducible polynomials dividing x, and the second the exponents. If you want only the *degrees* of the irreducible polynomials (for example for computing an L-function), use `factormod`$(x, p, 1)$. Note that the `factormod` algorithm is usually faster than `factorcantor`.

The library syntax is `GEN factcantor(GEN x, GEN p)`.

3.4.30 factorff$(x, \{p\}, \{a\})$. Factors the polynomial x in the field $\mathbf{F}_q$ defined by the irreducible polynomial a over $\mathbf{F}_p$. The coefficients of x must be operation-compatible with $\mathbf{Z}/p\mathbf{Z}$. The result is a two-column matrix: the first column contains the irreducible factors of x, and the second their exponents. If all the coefficients of x are in $\mathbf{F}_p$, a much faster algorithm is applied, using the computation of isomorphisms between finite fields.

Either a or p can omitted (in which case both are ignored) if x has `t_FFELT` coefficients; the function then becomes identical to `factor`:

```
? factorff(x^2 + 1, 5, y^2+3)  \\ over F_5[y]/(y^2+3) ~ F_25
%1 =
[Mod(Mod(1, 5), Mod(1, 5)*y^2 + Mod(3, 5))*x
 + Mod(Mod(2, 5), Mod(1, 5)*y^2 + Mod(3, 5)) 1]

[Mod(Mod(1, 5), Mod(1, 5)*y^2 + Mod(3, 5))*x
 + Mod(Mod(3, 5), Mod(1, 5)*y^2 + Mod(3, 5)) 1]
? t = ffgen(y^2 + Mod(3,5), 't); \\ a generator for F_25 as a t_FFELT
? factorff(x^2 + 1)   \\ not enough information to determine the base field
 ***   at top-level: factorff(x^2+1)
```

```
 ***                  ^---------------
 *** factorff: incorrect type in factorff.
? factorff(x^2 + t^0) \\ make sure a coeff. is a t_FFELT
%3 =
[x + 2 1]

[x + 3 1]
? factorff(x^2 + t + 1)
%11 =
[x + (2*t + 1) 1]

[x + (3*t + 4) 1]
```

Notice that the second syntax is easier to use and much more readable.

The library syntax is `GEN factorff(GEN x, GEN p = NULL, GEN a = NULL)`.

3.4.31 factorial(x). Factorial of x. The expression $x!$ gives a result which is an integer, while `factorial`(x) gives a real number.

The library syntax is `GEN mpfactr(long x, long prec)`. `GEN mpfact(long x)` returns $x!$ as a `t_INT`.

3.4.32 factorint$(x, \{flag = 0\})$. Factors the integer n into a product of pseudoprimes (see `ispseudoprime`), using a combination of the Shanks SQUFOF and Pollard Rho method (with modifications due to Brent), Lenstra's ECM (with modifications by Montgomery), and MPQS (the latter adapted from the LiDIA code with the kind permission of the LiDIA maintainers), as well as a search for pure powers. The output is a two-column matrix as for `factor`: the first column contains the "prime" divisors of n, the second one contains the (positive) exponents.

By convention 0 is factored as 0^1, and 1 as the empty factorization; also the divisors are by default not proven primes is they are larger than 2^{64}, they only failed the BPSW compositeness test (see `ispseudoprime`). Use `isprime` on the result if you want to guarantee primality or set the `factor_proven` default to 1. Entries of the private prime tables (see `addprimes`) are also included as is.

This gives direct access to the integer factoring engine called by most arithmetical functions. *flag* is optional; its binary digits mean 1: avoid MPQS, 2: skip first stage ECM (we may still fall back to it later), 4: avoid Rho and SQUFOF, 8: don't run final ECM (as a result, a huge composite may be declared to be prime). Note that a (strong) probabilistic primality test is used; thus composites might not be detected, although no example is known.

You are invited to play with the flag settings and watch the internals at work by using gp's `debug` default parameter (level 3 shows just the outline, 4 turns on time keeping, 5 and above show an increasing amount of internal details).

The library syntax is `GEN factorint(GEN x, long flag)`.

3.4.33 factormod$(x, p, \{flag = 0\})$. Factors the polynomial x modulo the prime integer p, using Berlekamp. The coefficients of x must be operation-compatible with $\mathbf{Z}/p\mathbf{Z}$. The result is a two-column matrix, the first column being the irreducible polynomials dividing x, and the second the exponents. If *flag* is non-zero, outputs only the *degrees* of the irreducible polynomials (for example, for computing an L-function). A different algorithm for computing the mod p factorization is `factorcantor` which is sometimes faster.

The library syntax is `GEN factormod0(GEN x, GEN p, long flag)`.

3.4.34 ffgen$(q, \{v\})$. Return a `t_FFELT` generator for the finite field with q elements; $q = p^f$ must be a prime power. This functions computes an irreducible monic polynomial $P \in \mathbf{F}_p[X]$ of degree f (via `ffinit`) and returns $g = X \pmod{P(X)}$. If `v` is given, the variable name is used to display g, else the variable x is used.

```
? g = ffgen(8, 't);
? g.mod
%2 = t^3 + t^2 + 1
? g.p
%3 = 2
? g.f
%4 = 3
? ffgen(6)
 ***   at top-level: ffgen(6)
 ***                 ^--------
 *** ffgen: not a prime number in ffgen: 6.
```

Alternative syntax: instead of a prime power $q = p^f$, one may input the pair $[p, f]$:

```
? g = ffgen([2,4], 't);
? g.p
%2 = 2
? g.mod
%3 = t^4 + t^3 + t^2 + t + 1
```

Finally, one may input directly the polynomial P (monic, irreducible, with `t_INTMOD` coefficients), and the function returns the generator $g = X \pmod{P(X)}$, inferring p from the coefficients of P. If `v` is given, the variable name is used to display g, else the variable of the polynomial P is used. If P is not irreducible, we create an invalid object and behaviour of functions dealing with the resulting `t_FFELT` is undefined; in fact, it is much more costly to test P for irreducibility than it would be to produce it via `ffinit`.

The library syntax is `GEN ffgen(GEN q, long v = -1)` where `v` is a variable number.

To create a generator for a prime finite field, the function `GEN p_to_GEN(GEN p, long v)` returns `1+ffgen(x*Mod(1,p),v)`.

3.4.35 ffinit$(p, n, \{v =' x\})$. Computes a monic polynomial of degree n which is irreducible over $\mathbf{F}_p$, where p is assumed to be prime. This function uses a fast variant of Adleman and Lenstra's algorithm.

It is useful in conjunction with `ffgen`; for instance if `P = ffinit(3,2)`, you can represent elements in $\mathbf{F}_{3^2}$ in term of `g = ffgen(P,'t)`. This can be abbreviated as `g = ffgen(3^2, 't)`, where the defining polynomial P can be later recovered as `g.mod`.

The library syntax is `GEN ffinit(GEN p, long n, long v = -1)` where `v` is a variable number.

3.4.36 fflog$(x, g, \{o\})$. Discrete logarithm of the finite field element x in base g, i.e. an e in $\mathbf{Z}$ such that $g^e = o$. If present, o represents the multiplicative order of g, see Section 3.4.2; the preferred format for this parameter is `[ord, factor(ord)]`, where `ord` is the order of g. It may be set as a side effect of calling `ffprimroot`.

If no o is given, assume that g is a primitive root. The result is undefined if e does not exist. This function uses

- a combination of generic discrete log algorithms (see `znlog`)
- a cubic sieve index calculus algorithm for large fields of degree at least 5.
- Coppersmith's algorithm for fields of characteristic at most 5.

```
? t = ffgen(ffinit(7,5));
? o = fforder(t)
%2 = 5602   \\ not a primitive root.
? fflog(t^10,t)
%3 = 10
? fflog(t^10,t, o)
%4 = 10
? g = ffprimroot(t, &o);
? o   \\ order is 16806, bundled with its factorization matrix
%6 = [16806, [2, 1; 3, 1; 2801, 1]]
? fforder(g, o)
%7 = 16806
? fflog(g^10000, g, o)
%8 = 10000
```

The library syntax is `GEN fflog(GEN x, GEN g, GEN o = NULL)`.

3.4.37 ffnbirred$(q, n\{, fl = 0\})$. Computes the number of monic irreducible polynomials over $\mathbf{F}_q$ of degree exactly n, ($flag = 0$ or omitted) or at most n ($flag = 1$).

The library syntax is `GEN ffnbirred0(GEN q, long n, long fl)`. Also available are `GEN ffnbirred(GEN q, long n)` (for $flag = 0$) and `GEN ffsumnbirred(GEN q, long n)` (for $flag = 1$).

3.4.38 fforder$(x, \{o\})$. Multiplicative order of the finite field element x. If o is present, it represents a multiple of the order of the element, see Section 3.4.2; the preferred format for this parameter is `[N, factor(N)]`, where `N` is the cardinality of the multiplicative group of the underlying finite field.

```
? t = ffgen(ffinit(nextprime(10^8), 5));
? g = ffprimroot(t, &o);  \\ o will be useful!
? fforder(g^1000000, o)
time = 0 ms.
%5 = 50000017500002450000171500006002500084 03
? fforder(g^1000000)
time = 16 ms. \\ noticeably slower, same result of course
%6 = 50000017500002450000171500006002500084 03
```

The library syntax is `GEN fforder(GEN x, GEN o = NULL)`.

3.4.39 ffprimroot$(x, \{\&o\})$. Return a primitive root of the multiplicative group of the definition field of the finite field element x (not necessarily the same as the field generated by x). If present, o is set to a vector `[ord, fa]`, where `ord` is the order of the group and `fa` its factorisation `factor(ord)`. This last parameter is useful in `fflog` and `fforder`, see Section 3.4.2.

```
? t = ffgen(ffinit(nextprime(10^7), 5));
? g = ffprimroot(t, &o);
? o[1]
%3 = 10000095000361000685900651605247609 8
? o[2]
%4 =
[2 1]

[7 2]

[31 1]

[41 1]

[67 1]

[1523 1]

[10498781 1]

[15992881 1]

[46858913131 1]

? fflog(g^1000000, g, o)
time = 1,312 ms.
%5 = 1000000
```

The library syntax is `GEN ffprimroot(GEN x, GEN *o = NULL)`.

3.4.40 fibonacci(x). x^{th} Fibonacci number.

The library syntax is `GEN fibo(long x)`.

3.4.41 gcd$(x, \{y\})$. Creates the greatest common divisor of x and y. If you also need the u and v such that $x * u + y * v = \gcd(x, y)$, use the `bezout` function. x and y can have rather quite general types, for instance both rational numbers. If y is omitted and x is a vector, returns the gcd of all components of x, i.e. this is equivalent to `content(x)`.

When x and y are both given and one of them is a vector/matrix type, the GCD is again taken recursively on each component, but in a different way. If y is a vector, resp. matrix, then the result has the same type as y, and components equal to `gcd(x, y[i])`, resp. `gcd(x, y[,i])`. Else if x is a vector/matrix the result has the same type as x and an analogous definition. Note that for these types, `gcd` is not commutative.

The algorithm used is a naive Euclid except for the following inputs:

• integers: use modified right-shift binary ("plus-minus" variant).

• univariate polynomials with coefficients in the same number field (in particular rational): use modular gcd algorithm.

• general polynomials: use the subresultant algorithm if coefficient explosion is likely (non modular coefficients).

If u and v are polynomials in the same variable with *inexact* coefficients, their gcd is defined to be scalar, so that

```
? a = x + 0.0; gcd(a,a)
%1 = 1
? b = y*x + O(y); gcd(b,b)
%2 = y
? c = 4*x + O(2^3); gcd(c,c)
%3 = 4
```

A good quantitative check to decide whether such a gcd "should be" non-trivial, is to use `polresultant`: a value close to 0 means that a small deformation of the inputs has non-trivial gcd. You may also use `gcdext`, which does try to compute an approximate gcd d and provides u, v to check whether $ux + vy$ is close to d.

The library syntax is `GEN ggcd0(GEN x, GEN y = NULL)`. Also available are `GEN ggcd(GEN x, GEN y)`, if `y` is not `NULL`, and `GEN content(GEN x)`, if `y` = `NULL`.

3.4.42 gcdext(x, y). Returns $[u, v, d]$ such that d is the gcd of x, y, $x * u + y * v = \gcd(x, y)$, and u and v minimal in a natural sense. The arguments must be integers or polynomials.

```
? [u, v, d] = gcdext(32,102)
%1 = [16, -5, 2]
? d
%2 = 2
? gcdext(x^2-x, x^2+x-2)
%3 = [-1/2, 1/2, x - 1]
```

If x, y are polynomials in the same variable and *inexact* coefficients, then compute u, v, d such that $x * u + y * v = d$, where d approximately divides both and x and y; in particular, we do not obtain `gcd(x,y)` which is *defined* to be a scalar in this case:

```
? a = x + 0.0; gcd(a,a)
%1 = 1

? gcdext(a,a)
%2 = [0, 1, x + 0.E-28]

? gcdext(x-Pi, 6*x^2-zeta(2))
%3 = [-6*x - 18.8495559, 1, 57.5726923]
```

For inexact inputs, the output is thus not well defined mathematically, but you obtain explicit polynomials to check whether the approximation is close enough for your needs.

The library syntax is `GEN gcdext0(GEN x, GEN y)`.

3.4.43 hilbert$(x, y, \{p\})$. Hilbert symbol of x and y modulo the prime p, $p = 0$ meaning the place at infinity (the result is undefined if $p \neq 0$ is not prime).

It is possible to omit p, in which case we take $p = 0$ if both x and y are rational, or one of them is a real number. And take $p = q$ if one of x, y is a `t_INTMOD` modulo q or a q-adic. (Incompatible types will raise an error.)

The library syntax is `long hilbert(GEN x, GEN y, GEN p = NULL)`.

3.4.44 isfundamental(x). True (1) if x is equal to 1 or to the discriminant of a quadratic field, false (0) otherwise.

The library syntax is `long isfundamental(GEN x)`.

3.4.45 ispolygonal($x, s, \{\&N\}$). True (1) if the integer x is an s-gonal number, false (0) if not. The parameter $s > 2$ must be a `t_INT`. If N is given, set it to n if x is the n-th s-gonal number.

```
? ispolygonal(36, 3, &N)
%1 = 1
? N
```

The library syntax is `long ispolygonal(GEN x, GEN s, GEN *N = NULL)`.

3.4.46 ispower($x, \{k\}, \{\&n\}$). If k is given, returns true (1) if x is a k-th power, false (0) if not. What it means to be a k-th power depends on the type of x; see `issquare` for details.

If k is omitted, only integers and fractions are allowed for x and the function returns the maximal $k \geq 2$ such that $x = n^k$ is a perfect power, or 0 if no such k exist; in particular `ispower(-1)`, `ispower(0)`, and `ispower(1)` all return 0.

If a third argument $\&n$ is given and x is indeed a k-th power, sets n to a k-th root of x.

For a `t_FFELT x`, instead of omitting `k` (which is not allowed for this type), it may be natural to set

```
k = (x.p ^ x.f - 1) / fforder(x)
```

The library syntax is `long ispower(GEN x, GEN k = NULL, GEN *n = NULL)`. Also available is `long gisanypower(GEN x, GEN *pty)` (k omitted).

3.4.47 ispowerful(x). True (1) if x is a powerful integer, false (0) if not; an integer is powerful if and only if its valuation at all primes dividing x is greater than 1.

```
? ispowerful(50)
%1 = 0
? ispowerful(100)
%2 = 1
? ispowerful(5^3*(10^1000+1)^2)
%3 = 1
```

The library syntax is `long ispowerful(GEN x)`.

3.4.48 isprime$(x, \{flag = 0\})$. True (1) if x is a prime number, false (0) otherwise. A prime number is a positive integer having exactly two distinct divisors among the natural numbers, namely 1 and itself.

This routine proves or disproves rigorously that a number is prime, which can be very slow when x is indeed prime and has more than 1000 digits, say. Use `ispseudoprime` to quickly check for compositeness. See also `factor`. It accepts vector/matrices arguments, and is then applied componentwise.

If *flag* $= 0$, use a combination of Baillie-PSW pseudo primality test (see `ispseudoprime`), Selfridge "$p-1$" test if $x-1$ is smooth enough, and Adleman-Pomerance-Rumely-Cohen-Lenstra (APRCL) for general x.

If *flag* $= 1$, use Selfridge-Pocklington-Lehmer "$p-1$" test and output a primality certificate as follows: return

• 0 if x is composite,

• 1 if x is small enough that passing Baillie-PSW test guarantees its primality (currently $x < 2^{64}$, as checked by Jan Feitsma),

• 2 if x is a large prime whose primality could only sensibly be proven (given the algorithms implemented in PARI) using the APRCL test.

• Otherwise (x is large and $x-1$ is smooth) output a three column matrix as a primality certificate. The first column contains prime divisors p of $x-1$ (such that $\prod p^{v_p(x-1)} > x^{1/3}$), the second the corresponding elements a_p as in Proposition 8.3.1 in GTM 138 , and the third the output of isprime(p,1).

The algorithm fails if one of the pseudo-prime factors is not prime, which is exceedingly unlikely and well worth a bug report. Note that if you monitor `isprime` at a high enough debug level, you may see warnings about untested integers being declared primes. This is normal: we ask for partial factorisations (sufficient to prove primality if the unfactored part is not too large), and `factor` warns us that the cofactor hasn't been tested. It may or may not be tested later, and may or may not be prime. This does not affect the validity of the whole `isprime` procedure.

If *flag* $= 2$, use APRCL.

The library syntax is `GEN gisprime(GEN x, long flag)`.

3.4.49 isprimepower$(x, \{\&n\})$. If $x = p^k$ is a prime power (p prime, $k > 0$), return k, else return 0. If a second argument $\&n$ is given and x is indeed the k-th power of a prime p, sets n to p.

The library syntax is `long isprimepower(GEN x, GEN *n = NULL)`.

3.4.50 ispseudoprime$(x, \{flag\})$. True (1) if x is a strong pseudo prime (see below), false (0) otherwise. If this function returns false, x is not prime; if, on the other hand it returns true, it is only highly likely that x is a prime number. Use `isprime` (which is of course much slower) to prove that x is indeed prime. The function accepts vector/matrices arguments, and is then applied componentwise.

If *flag* $= 0$, checks whether x has no small prime divisors (up to 101 included) and is a Baillie-Pomerance-Selfridge-Wagstaff pseudo prime. Such a pseudo prime passes a Rabin-Miller test for base 2, followed by a Lucas test for the sequence $(P, -1)$, P smallest positive integer such that $P^2 - 4$ is not a square mod x).

There are no known composite numbers passing the above test, although it is expected that infinitely many such numbers exist. In particular, all composites $\leq 2^{64}$ are correctly detected (checked using `http://www.cecm.sfu.ca/Pseudoprimes/index-2-to-64.html`).

If *flag* > 0, checks whether x is a strong Miller-Rabin pseudo prime for *flag* randomly chosen bases (with end-matching to catch square roots of -1).

The library syntax is `GEN gispseudoprime(GEN x, long flag)`.

3.4.51 ispseudoprimepower$(x, \{\&n\})$. If $x = p^k$ is a pseudo-prime power (p pseudo-prime as per `ispseudoprime`, $k > 0$), return k, else return 0. If a second argument $\&n$ is given and x is indeed the k-th power of a prime p, sets n to p.

More precisely, k is always the largest integer such that $x = n^k$ for some integer n and, when $n \leq 2^{64}$ the function returns $k > 0$ if and only if n is indeed prime. When $n > 2^{64}$ is larger than the threshold, the function may return 1 even though n is composite: it only passed an `ispseudoprime(n)` test.

The library syntax is `long ispseudoprimepower(GEN x, GEN *n = NULL)`.

3.4.52 issquare$(x, \{\&n\})$. True (1) if x is a square, false (0) if not. What "being a square" means depends on the type of x: all `t_COMPLEX` are squares, as well as all non-negative `t_REAL`; for exact types such as `t_INT`, `t_FRAC` and `t_INTMOD`, squares are numbers of the form s^2 with s in $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{Z}/N\mathbf{Z}$ respectively.

```
? issquare(3)          \\ as an integer
%1 = 0
? issquare(3.)         \\ as a real number
%2 = 1
? issquare(Mod(7, 8))  \\ in Z/8Z
%3 = 0
? issquare( 5 + O(13^4) )  \\ in Q_13
%4 = 0
```

If n is given, a square root of x is put into n.

```
? issquare(4, &n)
%1 = 1
? n
%2 = 2
```

For polynomials, either we detect that the characteristic is 2 (and check directly odd and even-power monomials) or we assume that 2 is invertible and check whether squaring the truncated power series for the square root yields the original input.

For `t_POLMOD` x, we only support `t_POLMOD`s of `t_INTMOD`s encoding finite fields, assuming without checking that the intmod modulus p is prime and that the polmod modulus is irreducible modulo p.

```
? issquare(Mod(Mod(2,3), x^2+1), &n)
%1 = 1
? n
%2 = Mod(Mod(2, 3)*x, Mod(1, 3)*x^2 + Mod(1, 3))
```

The library syntax is `long issquareall(GEN x, GEN *n = NULL)`. Also available is `long issquare(GEN x)`. Deprecated GP-specific functions `GEN gissquare(GEN x)` and `GEN gissquareall(GEN x, GEN *pt)` return `gen_0` and `gen_1` instead of a boolean value.

3.4.53 issquarefree(x). True (1) if x is squarefree, false (0) if not. Here x can be an integer or a polynomial.

The library syntax is `long issquarefree(GEN x)`.

3.4.54 istotient$(x, \{\&N\})$. True (1) if $x = \phi(n)$ for some integer n, false (0) if not.

```
? istotient(14)
%1 = 0
? istotient(100)
%2 = 0
```

If N is given, set $N = n$ as well.

```
? istotient(4, &n)
%1 = 1
? n
%2 = 10
```

The library syntax is `long istotient(GEN x, GEN *N = NULL)`.

3.4.55 kronecker(x, y). Kronecker symbol $(x|y)$, where x and y must be of type integer. By definition, this is the extension of Legendre symbol to $\mathbf{Z} \times \mathbf{Z}$ by total multiplicativity in both arguments with the following special rules for $y = 0, -1$ or 2:

- $(x|0) = 1$ if $|x| = 1$ and 0 otherwise.
- $(x|-1) = 1$ if $x \geq 0$ and -1 otherwise.
- $(x|2) = 0$ if x is even and 1 if $x = 1, -1 \bmod 8$ and -1 if $x = 3, -3 \bmod 8$.

The library syntax is `long kronecker(GEN x, GEN y)`.

3.4.56 lcm$(x, \{y\})$. Least common multiple of x and y, i.e. such that $\mathrm{lcm}(x, y) * \gcd(x, y) = x * y$, up to units. If y is omitted and x is a vector, returns the lcm of all components of x. For integer arguments, return the non-negative lcm.

When x and y are both given and one of them is a vector/matrix type, the LCM is again taken recursively on each component, but in a different way. If y is a vector, resp. matrix, then the result has the same type as y, and components equal to `lcm(x, y[i])`, resp. `lcm(x, y[,i])`. Else if x is a vector/matrix the result has the same type as x and an analogous definition. Note that for these types, `lcm` is not commutative.

Note that `lcm(v)` is quite different from

```
l = v[1]; for (i = 1, #v, l = lcm(l, v[i]))
```

Indeed, `lcm(v)` is a scalar, but `l` may not be (if one of the `v[i]` is a vector/matrix). The computation uses a divide-conquer tree and should be much more efficient, especially when using the GMP multiprecision kernel (and more subquadratic algorithms become available):

```
? v = vector(10^5, i, random);
```

```
? lcm(v);
time = 546 ms.
? l = v[1]; for (i = 1, #v, l = lcm(l, v[i]))
time = 4,561 ms.
```

The library syntax is `GEN glcm0(GEN x, GEN y = NULL)`.

3.4.57 logint$(x, b, \{\&z\})$. Return the largest integer e so that $b^e \leq x$, where the parameters $b > 1$ and $x > 0$ are both integers. If the parameter z is present, set it to b^e.

```
? logint(1000, 2)
%1 = 9
? 2^9
%2 = 512
? logint(1000, 2, &z)
%3 = 9
? z
%4 = 512
```

The number of digits used to write b in base x is `1 + logint(x,b)`:

```
? #digits(1000!, 10)
%5 = 2568
? logint(1000!, 10)
%6 = 2567
```

This function may conveniently replace

```
floor( log(x) / log(b) )
```

which may not give the correct answer since PARI does not guarantee exact rounding.

The library syntax is `long logint0(GEN x, GEN b, GEN *z = NULL)`.

3.4.58 moebius(x). Moebius μ-function of $|x|$. x must be of type integer.

The library syntax is `long moebius(GEN x)`.

3.4.59 nextprime(x). Finds the smallest pseudoprime (see `ispseudoprime`) greater than or equal to x. x can be of any real type. Note that if x is a pseudoprime, this function returns x and not the smallest pseudoprime strictly larger than x. To rigorously prove that the result is prime, use `isprime`.

The library syntax is `GEN nextprime(GEN x)`.

3.4.60 numbpart(n). Gives the number of unrestricted partitions of n, usually called $p(n)$ in the literature; in other words the number of nonnegative integer solutions to $a + 2b + 3c + \cdots = n$. n must be of type integer and $n < 10^{15}$ (with trivial values $p(n) = 0$ for $n < 0$ and $p(0) = 1$). The algorithm uses the Hardy-Ramanujan-Rademacher formula. To explicitly enumerate them, see `partitions`.

The library syntax is `GEN numbpart(GEN n)`.

3.4.61 numdiv(x). Number of divisors of $|x|$. x must be of type integer.

The library syntax is `GEN numdiv(GEN x)`.

3.4.62 omega(x). Number of distinct prime divisors of $|x|$. x must be of type integer.

```
? factor(392)
%1 =
[2 3]

[7 2]

? omega(392)
%2 = 2;  \\ without multiplicity
? bigomega(392)
%3 = 5;  \\ = 3+2, with multiplicity
```

The library syntax is `long omega(GEN x)`.

3.4.63 partitions$(k, \{a = k\}, \{n = k\}))$. Returns the vector of partitions of the integer k as a sum of positive integers (parts); for $k < 0$, it returns the empty set `[]`, and for $k = 0$ the trivial partition (no parts). A partition is given by a `t_VECSMALL`, where parts are sorted in nondecreasing order:

```
? partitions(3)
%1 = [Vecsmall([3]), Vecsmall([1, 2]), Vecsmall([1, 1, 1])]
```

correspond to 3, $1 + 2$ and $1 + 1 + 1$. The number of (unrestricted) partitions of k is given by `numbpart`:

```
? #partitions(50)
%1 = 204226
? numbpart(50)
%2 = 204226
```

Optional parameters n and a are as follows:

• $n = nmax$ (resp. $n = [nmin, nmax]$) restricts partitions to length less than $nmax$ (resp. length between $nmin$ and $nmax$), where the *length* is the number of nonzero entries.

• $a = amax$ (resp. $a = [amin, amax]$) restricts the parts to integers less than $amax$ (resp. between $amin$ and $amax$).

```
? partitions(4, 2)  \\ parts bounded by 2
%1 = [Vecsmall([2, 2]), Vecsmall([1, 1, 2]), Vecsmall([1, 1, 1, 1])]
? partitions(4,, 2) \\ at most 2 parts
%2 = [Vecsmall([4]), Vecsmall([1, 3]), Vecsmall([2, 2])]
? partitions(4,[0,3], 2) \\ at most 2 parts
%3 = [Vecsmall([4]), Vecsmall([1, 3]), Vecsmall([2, 2])]
```

By default, parts are positive and we remove zero entries unless $amin \leq 0$, in which case $nmin$ is ignored and X is of constant length $nmax$:

```
? partitions(4, [0,3])  \\ parts between 0 and 3
%1 = [Vecsmall([0, 0, 1, 3]), Vecsmall([0, 0, 2, 2]),\
      Vecsmall([0, 1, 1, 2]), Vecsmall([1, 1, 1, 1])]
```

The library syntax is `GEN partitions(long k, GEN a = NULL, GEN n) = NULL)`.

3.4.64 polrootsff$(x, \{p\}, \{a\})$. Returns the vector of distinct roots of the polynomial x in the field $\mathbf{F}_q$ defined by the irreducible polynomial a over $\mathbf{F}_p$. The coefficients of x must be operation-compatible with $\mathbf{Z}/p\mathbf{Z}$. Either a or p can omitted (in which case both are ignored) if x has `t_FFELT` coefficients:

```
? polrootsff(x^2 + 1, 5, y^2+3)  \\ over F_5[y]/(y^2+3) ~ F_25
%1 = [Mod(Mod(3, 5), Mod(1, 5)*y^2 + Mod(3, 5)),
      Mod(Mod(2, 5), Mod(1, 5)*y^2 + Mod(3, 5))]
? t = ffgen(y^2 + Mod(3,5), 't); \\ a generator for F_25 as a t_FFELT
? polrootsff(x^2 + 1)   \\ not enough information to determine the base field
 ***   at top-level: polrootsff(x^2+1)
 ***                 ^-----------------
 *** polrootsff: incorrect type in factorff.
? polrootsff(x^2 + t^0) \\ make sure one coeff. is a t_FFELT
%3 = [3, 2]
? polrootsff(x^2 + t + 1)
%4 = [2*t + 1, 3*t + 4]
```

Notice that the second syntax is easier to use and much more readable.

The library syntax is `GEN polrootsff(GEN x, GEN p = NULL, GEN a = NULL)`.

3.4.65 precprime(x). Finds the largest pseudoprime (see `ispseudoprime`) less than or equal to x. x can be of any real type. Returns 0 if $x \leq 1$. Note that if x is a prime, this function returns x and not the largest prime strictly smaller than x. To rigorously prove that the result is prime, use `isprime`.

The library syntax is `GEN precprime(GEN x)`.

3.4.66 prime(n). The n^{th} prime number

```
? prime(10^9)
%1 = 22801763489
```

Uses checkpointing and a naive $O(n)$ algorithm.

The library syntax is `GEN prime(long n)`.

3.4.67 primepi(x). The prime counting function. Returns the number of primes p, $p \leq x$.

```
? primepi(10)
%1 = 4;
? primes(5)
%2 = [2, 3, 5, 7, 11]
? primepi(10^11)
%3 = 4118054813
```

Uses checkpointing and a naive $O(x)$ algorithm.

The library syntax is `GEN primepi(GEN x)`.

3.4.68 primes(n). Creates a row vector whose components are the first n prime numbers. (Returns the empty vector for $n \leq 0$.) A `t_VEC` $n = [a, b]$ is also allowed, in which case the primes in $[a, b]$ are returned

```
? primes(10)     \\ the first 10 primes
%1 = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29]
? primes([0,29])  \\ the primes up to 29
%2 = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29]
? primes([15,30])
%3 = [17, 19, 23, 29]
```

The library syntax is `GEN primes0(GEN n)`.

3.4.69 qfbclassno$(D, \{flag = 0\})$. Ordinary class number of the quadratic order of discriminant D, for "small" values of D.

• if $D > 0$ or $flag = 1$, use a $O(|D|^{1/2})$ algorithm (compute $L(1, \chi_D)$ with the approximate functional equation). This is slower than `quadclassunit` as soon as $|D| \approx 10^2$ or so and is not meant to be used for large D.

• if $D < 0$ and $flag = 0$ (or omitted), use a $O(|D|^{1/4})$ algorithm (Shanks's baby-step/giant-step method). It should be faster than `quadclassunit` for small values of D, say $|D| < 10^{18}$.

Important warning. In the latter case, this function only implements part of Shanks's method (which allows to speed it up considerably). It gives unconditionnally correct results for $|D| < 2 \cdot 10^{10}$, but may give incorrect results for larger values if the class group has many cyclic factors. We thus recommend to double-check results using the function `quadclassunit`, which is about 2 to 3 times slower in the above range, assuming GRH. We currently have no counter-examples but they should exist: we'd appreciate a bug report if you find one.

Warning. Contrary to what its name implies, this routine does not compute the number of classes of binary primitive forms of discriminant D, which is equal to the *narrow* class number. The two notions are the same when $D < 0$ or the fundamental unit ε has negative norm; when $D > 0$ and $N\varepsilon > 0$, the number of classes of forms is twice the ordinary class number. This is a problem which we cannot fix for backward compatibility reasons. Use the following routine if you are only interested in the number of classes of forms:

```
QFBclassno(D) =
qfbclassno(D) * if (D < 0 || norm(quadunit(D)) < 0, 1, 2)
```

Here are a few examples:

```
? qfbclassno(400000028)
time = 3,140 ms.
%1 = 1
? quadclassunit(400000028).no
time = 20 ms. \\ much faster
%2 = 1
? qfbclassno(-400000028)
time = 0 ms.
%3 = 7253 \\ correct, and fast enough
? quadclassunit(-400000028).no
time = 0 ms.
```

```
%4 = 7253
```

See also `qfbhclassno`.

The library syntax is `GEN qfbclassno0(GEN D, long flag)`. The following functions are also available:

`GEN classno(GEN D)` ($flag = 0$)

`GEN classno2(GEN D)` ($flag = 1$).

Finally

`GEN hclassno(GEN D)` computes the class number of an imaginary quadratic field by counting reduced forms, an $O(|D|)$ algorithm.

3.4.70 qfbcompraw(x, y). composition of the binary quadratic forms x and y, without reduction of the result. This is useful e.g. to compute a generating element of an ideal. The result is undefined if x and y do not have the same discriminant.

The library syntax is `GEN qfbcompraw(GEN x, GEN y)`.

3.4.71 qfbhclassno(x). Hurwitz class number of x, where x is non-negative and congruent to 0 or 3 modulo 4. For $x > 5 \cdot 10^5$, we assume the GRH, and use `quadclassunit` with default parameters.

The library syntax is `GEN hclassno(GEN x)`.

3.4.72 qfbnucomp(x, y, L). composition of the primitive positive definite binary quadratic forms x and y (type `t_QFI`) using the NUCOMP and NUDUPL algorithms of Shanks, à la Atkin. L is any positive constant, but for optimal speed, one should take $L = |D/4|^{1/4}$, i.e. `sqrtnint(abs(D)>>2,4)`, where D is the common discriminant of x and y. When x and y do not have the same discriminant, the result is undefined.

The current implementation is slower than the generic routine for small D, and becomes faster when D has about 45 bits.

The library syntax is `GEN nucomp(GEN x, GEN y, GEN L)`. Also available is `GEN nudupl(GEN x, GEN L)` when $x = y$.

3.4.73 qfbnupow$(x, n, \{L\})$. n-th power of the primitive positive definite binary quadratic form x using Shanks's NUCOMP and NUDUPL algorithms; if set, L should be equal to `sqrtnint(abs(D)>>2,4)`, where $D < 0$ is the discriminant of x.

The current implementation is slower than the generic routine for small discriminant D, and becomes faster for $D \approx 2^{45}$.

The library syntax is `GEN nupow(GEN x, GEN n, GEN L = NULL)`.

3.4.74 qfbpowraw(x, n). n-th power of the binary quadratic form x, computed without doing any reduction (i.e. using `qfbcompraw`). Here n must be non-negative and $n < 2^{31}$.

The library syntax is `GEN qfbpowraw(GEN x, long n)`.

3.4.75 qfbprimeform(x, p). Prime binary quadratic form of discriminant x whose first coefficient is p, where $|p|$ is a prime number. By abuse of notation, $p = \pm 1$ is also valid and returns the unit form. Returns an error if x is not a quadratic residue mod p, or if $x < 0$ and $p < 0$. (Negative definite `t_QFI` are not implemented.) In the case where $x > 0$, the "distance" component of the form is set equal to zero according to the current precision.

The library syntax is `GEN primeform(GEN x, GEN p, long prec)`.

3.4.76 qfbred$(x, \{flag = 0\}, \{d\}, \{isd\}, \{sd\})$. Reduces the binary quadratic form x (updating Shanks's distance function if x is indefinite). The binary digits of *flag* are toggles meaning

1: perform a single reduction step

2: don't update Shanks's distance

The arguments d, isd, sd, if present, supply the values of the discriminant, $\lfloor \sqrt{d} \rfloor$, and $\sqrt{d}$ respectively (no checking is done of these facts). If $d < 0$ these values are useless, and all references to Shanks's distance are irrelevant.

The library syntax is `GEN qfbred0(GEN x, long flag, GEN d = NULL, GEN isd = NULL, GEN sd = NULL)`. Also available are

`GEN redimag(GEN x)` (for definite x),

and for indefinite forms:

`GEN redreal(GEN x)`

`GEN rhoreal(GEN x)` (= `qfbred(x,1)`),

`GEN redrealnod(GEN x, GEN isd)` (= `qfbred(x,2,,isd)`),

`GEN rhorealnod(GEN x, GEN isd)` (= `qfbred(x,3,,isd)`).

3.4.77 qfbredsl2$(x, \{data\})$. Reduction of the (real or imaginary) binary quadratic form x, return $[y, g]$ where y is reduced and g in $\mathrm{SL}(2, \mathbf{Z})$ is such that $g \cdot x = y$; *data*, if present, must be equal to $[D,$ `sqrtint`$(D)]$, where $D > 0$ is the discriminant of x. In case x is `t_QFR`, the distance component is unaffected.

The library syntax is `GEN qfbredsl2(GEN x, GEN data = NULL)`.

3.4.78 qfbsolve(Q, p). Solve the equation $Q(x, y) = p$ over the integers, where Q is a binary quadratic form and p a prime number.

Return $[x, y]$ as a two-components vector, or zero if there is no solution. Note that this function returns only one solution and not all the solutions.

Let $D = \mathrm{disc}Q$. The algorithm used runs in probabilistic polynomial time in p (through the computation of a square root of D modulo p); it is polynomial time in D if Q is imaginary, but exponential time if Q is real (through the computation of a full cycle of reduced forms). In the latter case, note that `bnfisprincipal` provides a solution in heuristic subexponential time in D assuming the GRH.

The library syntax is `GEN qfbsolve(GEN Q, GEN p)`.

3.4.79 quadclassunit$(D, \{flag = 0\}, \{tech = []\})$. Buchmann-McCurley's sub-exponential algorithm for computing the class group of a quadratic order of discriminant D.

This function should be used instead of `qfbclassno` or `quadregula` when $D < -10^{25}$, $D > 10^{10}$, or when the *structure* is wanted. It is a special case of `bnfinit`, which is slower, but more robust.

The result is a vector v whose components should be accessed using member functions:

- v`.no`: the class number
- v`.cyc`: a vector giving the structure of the class group as a product of cyclic groups;
- v`.gen`: a vector giving generators of those cyclic groups (as binary quadratic forms).
- v`.reg`: the regulator, computed to an accuracy which is the maximum of an internal accuracy determined by the program and the current default (note that once the regulator is known to a small accuracy it is trivial to compute it to very high accuracy, see the tutorial).

The *flag* is obsolete and should be left alone. In older versions, it supposedly computed the narrow class group when $D > 0$, but this did not work at all; use the general function `bnfnarrow`.

Optional parameter *tech* is a row vector of the form $[c_1, c_2]$, where $c_1 \leq c_2$ are non-negative real numbers which control the execution time and the stack size, see 3.8.7. The parameter is used as a threshold to balance the relation finding phase against the final linear algebra. Increasing the default c_1 means that relations are easier to find, but more relations are needed and the linear algebra will be harder. The default value for c_1 is 0 and means that it is taken equal to c_2. The parameter c_2 is mostly obsolete and should not be changed, but we still document it for completeness: we compute a tentative class group by generators and relations using a factorbase of prime ideals $\leq c_1(\log|D|)^2$, then prove that ideals of norm $\leq c_2(\log|D|)^2$ do not generate a larger group. By default an optimal c_2 is chosen, so that the result is provably correct under the GRH — a famous result of Bach states that $c_2 = 6$ is fine, but it is possible to improve on this algorithmically. You may provide a smaller c_2, it will be ignored (we use the provably correct one); you may provide a larger c_2 than the default value, which results in longer computing times for equally correct outputs (under GRH).

The library syntax is `GEN quadclassunit0(GEN D, long flag, GEN tech = NULL, long prec)`. If you really need to experiment with the *tech* parameter, it is usually more convenient to use `GEN Buchquad(GEN D, double c1, double c2, long prec)`

3.4.80 quaddisc(x). Discriminant of the étale algebra $\mathbf{Q}(\sqrt{x})$, where $x \in \mathbf{Q}^*$. This is the same as `coredisc`(d) where d is the integer square-free part of x, so x=df^2 with $f \in \mathbf{Q}^*$ and $d \in \mathbf{Z}$. This returns 0 for $x = 0$, 1 for x square and the discriminant of the quadratic field $\mathbf{Q}(\sqrt{x})$ otherwise.

```
? quaddisc(7)
%1 = 28
? quaddisc(-7)
%2 = -7
```

The library syntax is `GEN quaddisc(GEN x)`.

3.4.81 quadgen(D). Creates the quadratic number $\omega = (a + \sqrt{D})/2$ where $a = 0$ if $D \equiv 0 \bmod 4$, $a = 1$ if $D \equiv 1 \bmod 4$, so that $(1, \omega)$ is an integral basis for the quadratic order of discriminant D. D must be an integer congruent to 0 or 1 modulo 4, which is not a square.

The library syntax is `GEN quadgen(GEN D)`.

3.4.82 quadhilbert(D). Relative equation defining the Hilbert class field of the quadratic field of discriminant D.

If $D < 0$, uses complex multiplication (Schertz's variant).

If $D > 0$ Stark units are used and (in rare cases) a vector of extensions may be returned whose compositum is the requested class field. See `bnrstark` for details.

The library syntax is `GEN quadhilbert(GEN D, long prec)`.

3.4.83 quadpoly$(D, \{v =' x\})$. Creates the "canonical" quadratic polynomial (in the variable v) corresponding to the discriminant D, i.e. the minimal polynomial of `quadgen`(D). D must be an integer congruent to 0 or 1 modulo 4, which is not a square.

The library syntax is `GEN quadpoly0(GEN D, long v = -1)` where `v` is a variable number.

3.4.84 quadray(D, f). Relative equation for the ray class field of conductor f for the quadratic field of discriminant D using analytic methods. A `bnf` for $x^2 - D$ is also accepted in place of D.

For $D < 0$, uses the σ function and Schertz's method.

For $D > 0$, uses Stark's conjecture, and a vector of relative equations may be returned. See `bnrstark` for more details.

The library syntax is `GEN quadray(GEN D, GEN f, long prec)`.

3.4.85 quadregulator(x). Regulator of the quadratic field of positive discriminant x. Returns an error if x is not a discriminant (fundamental or not) or if x is a square. See also `quadclassunit` if x is large.

The library syntax is `GEN quadregulator(GEN x, long prec)`.

3.4.86 quadunit(D). Fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ where D is the positive discriminant of the field. If D is not a fundamental discriminant, this probably gives the fundamental unit of the corresponding order. D must be an integer congruent to 0 or 1 modulo 4, which is not a square; the result is a quadratic number (see Section 3.4.81).

The library syntax is `GEN quadunit(GEN D)`.

3.4.87 ramanujantau(n). Compute the value of Ramanujan's tau function at an individual n, assuming the truth of the GRH (to compute quickly class numbers of imaginary quadratic fields using `quadclassunit`). Algorithm in $\tilde{O}(n^{1/2})$ using $O(\log n)$ space. If all values up to N are required, then

$$\sum \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24}$$

will produce them in time $\tilde{O}(N)$, against $\tilde{O}(N^{3/2})$ for individual calls to `ramanujantau`; of course the space complexity then becomes $\tilde{O}(N)$.

```
? tauvec(N) = Vec(q*eta(q + O(q^N))^24);
? N = 10^4; v = tauvec(N);
time = 26 ms.
? ramanujantau(N)
%3 = -482606811957501440000
```

```
? w = vector(N, n, ramanujantau(n)); \\ much slower !
time = 13,190 ms.
? v == w
%4 = 1
```

The library syntax is `GEN ramanujantau(GEN n)`.

3.4.88 randomprime($\{N = 2^{31}\}$). Returns a strong pseudo prime (see `ispseudoprime`) in $[2, N-1]$. A `t_VEC` $N = [a, b]$ is also allowed, with $a \leq b$ in which case a pseudo prime $a \leq p \leq b$ is returned; if no prime exists in the interval, the function will run into an infinite loop. If the upper bound is less than 2^{64} the pseudo prime returned is a proven prime.

The library syntax is `GEN randomprime(GEN N = NULL)`.

3.4.89 removeprimes($\{x = []\}$). Removes the primes listed in x from the prime number table. In particular `removeprimes(addprimes())` empties the extra prime table. x can also be a single integer. List the current extra primes if x is omitted.

The library syntax is `GEN removeprimes(GEN x = NULL)`.

3.4.90 sigma($x, \{k = 1\}$). Sum of the k^{th} powers of the positive divisors of $|x|$. x and k must be of type integer.

The library syntax is `GEN sumdivk(GEN x, long k)`. Also available is `GEN sumdiv(GEN n)`, for $k = 1$.

3.4.91 sqrtint(x). Returns the integer square root of x, i.e. the largest integer y such that $y^2 \leq x$, where x a non-negative integer.

```
? N = 120938191237; sqrtint(N)
%1 = 347761
? sqrt(N)
%2 = 347761.687419704127476021309644140952 16
```

The library syntax is `GEN sqrtint(GEN x)`.

3.4.92 sqrtnint(x, n). Returns the integer n-th root of x, i.e. the largest integer y such that $y^n \leq x$, where x is a non-negative integer.

```
? N = 120938191237; sqrtnint(N, 5)
%1 = 164
? N^(1/5)
%2 = 164.631408498296608429586146769396773 91
```

The special case $n = 2$ is `sqrtint`

The library syntax is `GEN sqrtnint(GEN x, long n)`.

3.4.93 stirling$(n, k, \{flag = 1\})$. Stirling number of the first kind $s(n, k)$ ($flag = 1$, default) or of the second kind $S(n, k)$ ($flag$=2), where n, k are non-negative integers. The former is $(-1)^{n-k}$ times the number of permutations of n symbols with exactly k cycles; the latter is the number of ways of partitioning a set of n elements into k non-empty subsets. Note that if all $s(n, k)$ are needed, it is much faster to compute

$$\sum_k s(n, k)x^k = x(x-1)\dots(x-n+1).$$

Similarly, if a large number of $S(n, k)$ are needed for the same k, one should use

$$\sum_n S(n, k)x^n = \frac{x^k}{(1-x)\dots(1-kx)}.$$

(Should be implemented using a divide and conquer product.) Here are simple variants for n fixed:

```
/* list of s(n,k), k = 1..n */
vecstirling(n) = Vec( factorback(vector(n-1,i,1-i*'x)) )

/* list of S(n,k), k = 1..n */
vecstirling2(n) =
{ my(Q = x^(n-1), t);
  vector(n, i, t = divrem(Q, x-i); Q=t[1]; simplify(t[2]));
}
```

The library syntax is `GEN stirling(long n, long k, long flag)`. Also available are `GEN stirling1(ulong n, ulong k)` ($flag = 1$) and `GEN stirling2(ulong n, ulong k)` ($flag = 2$).

3.4.94 sumdedekind(h, k). Returns the Dedekind sum attached to the integers h and k, corresponding to a fast implementation of

```
s(h,k) = sum(n = 1, k-1, (n/k)*(frac(h*n/k) - 1/2))
```

The library syntax is `GEN sumdedekind(GEN h, GEN k)`.

3.4.95 sumdigits$(n, \{B = 10\})$. Sum of digits in the integer n, when written in base $B > 1$.

```
? sumdigits(123456789)
%1 = 45
? sumdigits(123456789, 2)
%1 = 16
```

Note that the sum of bits in n is also returned by `hammingweight`. This function is much faster than `vecsum(digits(n,B))` when B is 10 or a power of 2, and only slightly faster in other cases.

The library syntax is `GEN sumdigits0(GEN n, GEN B = NULL)`. Also available is `GEN sumdigits(GEN n)`, for $B = 10$.

3.4.96 zncharinduce(G, chi, N). Let G be attached to $(\mathbf{Z}/q\mathbf{Z})^*$ (as per `G = idealstar(,q)`) and let `chi` be a Dirichlet character on $(\mathbf{Z}/q\mathbf{Z})^*$, given by

- a `t_VEC`: a standard character on `bid.gen`,

- a `t_INT` or a `t_COL`: a Conrey index in $(\mathbf{Z}/q\mathbf{Z})^*$ or its Conrey logarithm; see Section 3.4.3 or `??character`.

Let N be a multiple of q, return the character modulo N induced by `chi`. As usual for arithmetic functions, the new modulus N can be given as a `t_INT`, via a factorization matrix or a pair `[N, factor(N)]`, or by `idealstar(,N)`.

```
? G = idealstar(,4);
? chi = znconreylog(G,1); \\ trivial character mod 4
? zncharinduce(G, chi, 80)  \\ now mod 80
%3 = [0, 0, 0]~
? zncharinduce(G, 1, 80)  \\ same using directly Conrey label
%4 = [0, 0, 0]~
? G2 = idealstar(,80);
? zncharinduce(G, 1, G2)  \\ same
%4 = [0, 0, 0]~
? chi = zncharinduce(G, 3, G2)  \\ induce the non-trivial character mod 4
%5 = [1, 0, 0]~
? znconreyconductor(G2, chi, &chi0)
%6 = [4, Mat([2, 2])]
? chi0
%7 = [1]~
```

Here is a larger example:

```
? G = idealstar(,126000);
? label = 1009;
? chi = znconreylog(G, label)
%3 = [0, 0, 0, 14, 0]~
? N0 = znconreyconductor(G, label, &chi0)
%4 = [125, Mat([5, 3])]
? chi0 \\ primitive character mod 5^3 attached to chi
%5 = [14]~
? G0 = idealstar(,N0);
? zncharinduce(G0, chi0, G) \\ induce back
%7 = [0, 0, 0, 14, 0]~
? znconreyexp(G, %)
%8 = 1009
```

The library syntax is `GEN zncharinduce(GEN G, GEN chi, GEN N)`.

3.4.97 zncharisodd(G, chi). Let G be attached to $(\mathbf{Z}/N\mathbf{Z})^*$ (as per `G = idealstar(,N)`) and let `chi` be a Dirichlet character on $(\mathbf{Z}/N\mathbf{Z})^*$, given by

- a `t_VEC`: a standard character on `bid.gen`,

- a `t_INT` or a `t_COL`: a Conrey index in $(\mathbf{Z}/q\mathbf{Z})^*$ or its Conrey logarithm; see Section 3.4.3 or `??character`.

Return 1 if and only if `chi`$(-1) = -1$ and 0 otherwise.

```
? G = idealstar(,8);
? zncharisodd(G, 1)  \\ trivial character
%2 = 0
? zncharisodd(G, 3)
%3 = 1
? chareval(G, 3, -1)
%4 = 1/2
```

The library syntax is `long zncharisodd(GEN G, GEN chi)`.

3.4.98 znchartokronecker$(G, chi, \{flag = 0\})$. Let G be attached to $(\mathbf{Z}/N\mathbf{Z})^*$ (as per `G = idealstar(,N)`) and let `chi` be a Dirichlet character on $(\mathbf{Z}/N\mathbf{Z})^*$, given by

- a `t_VEC`: a standard character on `bid.gen`,

- a `t_INT` or a `t_COL`: a Conrey index in $(\mathbf{Z}/q\mathbf{Z})^*$ or its Conrey logarithm; see Section 3.4.3 or `??character`.

If $flag = 0$, return the discriminant D if `chi` is real equal to the Kronecker symbol $(D/.)$ and 0 otherwise. The discriminant D is fundamental if and only if `chi` is primitive.

If $flag = 1$, return the fundamental discriminant attached to the corresponding primitive character.

```
? G = idealstar(,8); CHARS = [1,3,5,7]; \\ Conrey labels
? apply(t->znchartokronecker(G,t), CHARS)
%2 = [4, -8, 8, -4]
? apply(t->znchartokronecker(G,t,1), CHARS)
%3 = [1, -8, 8, -4]
```

The library syntax is `GEN znchartokronecker(GEN G, GEN chi, long flag)`.

3.4.99 znconreychar(bid, m). Given a bid attached to $(\mathbf{Z}/q\mathbf{Z})^*$ (as per `bid = idealstar(,q)`), this function returns the Dirichlet character attached to $m \in (\mathbf{Z}/q\mathbf{Z})^*$ via Conrey's logarithm, which establishes a "canonical" bijection between $(\mathbf{Z}/q\mathbf{Z})^*$ and its dual.

Let $q = \prod_p p^{e_p}$ be the factorization of q into distinct primes. For all odd p with $e_p > 0$, let g_p be the element in $(\mathbf{Z}/q\mathbf{Z})^*$ which is

- congruent to 1 mod q/p^{e_p},

- congruent mod p^{e_p} to the smallest integer whose order is $\phi(p^{e_p})$.

For $p = 2$, we let g_4 (if $2^{e_2} \geq 4$) and g_8 (if furthermore $(2^{e_2} \geq 8)$ be the elements in $(\mathbf{Z}/q\mathbf{Z})^*$ which are

- congruent to 1 mod $q/2^{e_2}$,

- $g_4 = -1 \bmod 2^{e_2}$,
- $g_8 = 5 \bmod 2^{e_2}$.

Then the g_p (and the extra g_4 and g_8 if $2^{e_2} \geq 2$) are independent generators of $(\mathbf{Z}/q\mathbf{Z})^*$, i.e. every m in $(\mathbf{Z}/q\mathbf{Z})^*$ can be written uniquely as $\prod_p g_p^{m_p}$, where m_p is defined modulo the order o_p of g_p and $p \in S_q$, the set of prime divisors of q together with 4 if $4 \mid q$ and 8 if $8 \mid q$. Note that the g_p are in general *not* SNF generators as produced by `znstar` or `idealstar` whenever $\omega(q) \geq 2$, although their number is the same. They however allow to handle the finite abelian group $(\mathbf{Z}/q\mathbf{Z})^*$ in a fast and elegant way. (Which unfortunately does not generalize to ray class groups or Hecke characters.)

The Conrey logarithm of m is the vector $(m_p)_{p \in S_q}$, obtained via `znconreylog`. The Conrey character $\chi_q(m, \cdot)$ attached to m mod q maps each g_p, $p \in S_q$ to $e(m_p/o_p)$, where $e(x) = \exp(2i\pi x)$. This function returns the Conrey character expressed in the standard PARI way in terms of the SNF generators `bid.gen`.

Note. It is useless to include the generators in the *bid*, except for debugging purposes: they are well defined from elementary matrix operations and Chinese remaindering, their explicit value as elements in $(\mathbf{Z}/q\mathbf{Z})^*$ is never used.

```
? G = idealstar(,8,2); /*add generators for debugging:*/
? G.cyc
%2 = [2, 2]  \\ Z/2 x Z/2
? G.gen
%3 = [7, 3]
? znconreychar(G,1)  \\ 1 is always the trivial character
%4 = [0, 0]
? znconreychar(G,2)  \\ 2 is not coprime to 8 !!!
  ***   at top-level: znconreychar(G,2)
  ***                 ^-----------------
  *** znconreychar: elements not coprime in Zideallog:
    2
    8
  ***   Break loop: type 'break' to go back to GP prompt
break>
? znconreychar(G,3)
%5 = [0, 1]
? znconreychar(G,5)
%6 = [1, 1]
? znconreychar(G,7)
%7 = [1, 0]
```

We indeed get all 4 characters of $(\mathbf{Z}/8\mathbf{Z})^*$.

For convenience, we allow to input the *Conrey logarithm* of m instead of m:

```
? G = idealstar(,55);
? znconreychar(G,7)
%2 = [7, 0]
? znconreychar(G, znconreylog(G,7))
%3 = [7, 0]
```

The library syntax is `GEN znconreychar(GEN bid, GEN m)`.

3.4.100 znconreyconductor(*bid*, *chi*, {&*chi0*}). Let *bid* be attached to $(\mathbf{Z}/q\mathbf{Z})^*$ (as per `bid = idealstar(,q)`) and `chi` be a Dirichlet character on $(\mathbf{Z}/q\mathbf{Z})^*$, given by

- a `t_VEC`: a standard character on `bid.gen`,

- a `t_INT` or a `t_COL`: a Conrey index in $(\mathbf{Z}/q\mathbf{Z})^*$ or its Conrey logarithm; see Section 3.4.3 or `??character`.

Return the conductor of `chi`, as the `t_INT` `bid.mod` if `chi` is primitive, and as a pair `[N, faN]` (with `faN` the factorization of N) otherwise.

If `chi0` is present, set it to the Conrey logarithm of the attached primitive character.

```
? G = idealstar(,126000);
? znconreyconductor(G,11)   \\ primitive
%2 = 126000
? znconreyconductor(G,1)    \\ trivial character, not primitive!
%3 = [1, matrix(0,2)]
? N0 = znconreyconductor(G,1009, &chi0) \\ character mod 5^3
%4 = [125, Mat([5, 3])]
? chi0
%5 = [14]~
? G0 = idealstar(,N0);      \\ format [N,factor(N)] accepted
? znconreyexp(G0, chi0)
%7 = 9
? znconreyconductor(G0, chi0) \\ now primitive, as expected
%8 = 125
```

The group `G0` is not computed as part of `znconreyconductor` because it needs to be computed only once per conductor, not once per character.

The library syntax is `GEN znconreyconductor(GEN bid, GEN chi, GEN *chi0 = NULL)`.

3.4.101 znconreyexp(*bid*, *chi*). Given a *bid* attached to $(\mathbf{Z}/q\mathbf{Z})^*$ (as per `bid = idealstar(,q)`), this function returns the Conrey exponential of the character *chi*: it returns the integer $m \in (\mathbf{Z}/q\mathbf{Z})^*$ such that `znconreylog(`*bid*`, `m`)` is *chi*.

The character *chi* is given either as a

- `t_VEC`: in terms of the generators *bid*`.gen`;

- `t_COL`: a Conrey logarithm.

```
? G = idealstar(,126000)
? znconreylog(G,1)
%2 = [0, 0, 0, 0, 0]~
? znconreyexp(G,%)
%3 = 1
? G.cyc \\ SNF generators
%4 = [300, 12, 2, 2, 2]
? chi = [100, 1, 0, 1, 0]; \\ some random character on SNF generators
? znconreylog(G, chi)  \\ in terms of Conrey generators
%6 = [0, 3, 3, 0, 2]~
```

```
? znconreyexp(G, %)  \\ apply to a Conrey log
%7 = 18251
? znconreyexp(G, chi) \\ ... or a char on SNF generators
%8 = 18251
? znconreychar(G,%)
%9 = [100, 1, 0, 1, 0]
```

The library syntax is `GEN znconreyexp(GEN bid, GEN chi)`.

3.4.102 znconreylog(bid, m). Given a *bid* attached to $(\mathbf{Z}/q\mathbf{Z})^*$ (as per `bid = idealstar(,q)`), this function returns the Conrey logarithm of $m \in (\mathbf{Z}/q\mathbf{Z})^*$.

Let $q = \prod_p p^{e_p}$ be the factorization of q into distinct primes, where we assume $e_2 = 0$ or $e_2 \geq 2$. (If $e_2 = 1$, we can ignore 2 from the factorization, as if we replaced q by $q/2$, since $(\mathbf{Z}/q\mathbf{Z})^* \sim (\mathbf{Z}/(q/2)\mathbf{Z})^*$.)

For all odd p with $e_p > 0$, let g_p be the element in $(\mathbf{Z}/q\mathbf{Z})^*$ which is

- congruent to 1 mod q/p^{e_p},
- congruent mod p^{e_p} to the smallest integer whose order is $\phi(p^{e_p})$ for p odd,

For $p = 2$, we let g_4 (if $2^{e_2} \geq 4$) and g_8 (if furthermore $(2^{e_2} \geq 8)$ be the elements in $(\mathbf{Z}/q\mathbf{Z})^*$ which are

- congruent to 1 mod $q/2^{e_2}$,
- $g_4 = -1 \bmod 2^{e_2}$,
- $g_8 = 5 \bmod 2^{e_2}$.

Then the g_p (and the extra g_4 and g_8 if $2^{e_2} \geq 2$) are independent generators of $\mathbf{Z}/q\mathbf{Z}^*$, i.e. every m in $(\mathbf{Z}/q\mathbf{Z})^*$ can be written uniquely as $\prod_p g_p^{m_p}$, where m_p is defined modulo the order o_p of g_p and $p \in S_q$, the set of prime divisors of q together with 4 if $4 \mid q$ and 8 if $8 \mid q$. Note that the g_p are in general *not* SNF generators as produced by `znstar` or `idealstar` whenever $\omega(q) \geq 2$, although their number is the same. They however allow to handle the finite abelian group $(\mathbf{Z}/q\mathbf{Z})^*$ in a fast and elegant way. (Which unfortunately does not generalize to ray class groups or Hecke characters.)

The Conrey logarithm of m is the vector $(m_p)_{p \in S_q}$. The inverse function `znconreyexp` recovers the Conrey label m from a character.

```
? G = idealstar(,126000);
? znconreylog(G,1)
%2 = [0, 0, 0, 0, 0]~
? znconreyexp(G, %)
%3 = 1
? znconreylog(G,2)  \\ 2 is not coprime to modulus !!!
  ***   at top-level: znconreylog(G,2)
  ***                 ^-----------------
  *** znconreylog: elements not coprime in Zideallog:
    2
    126000
  ***   Break loop: type 'break' to go back to GP prompt
break>
```

```
? znconreylog(G,11) \\ wrt. Conrey generators
%4 = [0, 3, 1, 76, 4]~
? log11 = ideallog(,11,G)   \\ wrt. SNF generators
%5 = [178, 3, -75, 1, 0]~
```

For convenience, we allow to input the ordinary discrete log of m, `ideallog(,m,bid)`, which allows to convert discrete logs from `bid.gen` generators to Conrey generators.

```
? znconreylog(G, log11)
%7 = [0, 3, 1, 76, 4]~
```

We also allow a character (`t_VEC`) on `bid.gen` and return its representation on the Conrey generators.

```
? G.cyc
%8 = [300, 12, 2, 2, 2]
? chi = [10,1,0,1,1];
? znconreylog(G, chi)
%10 = [1, 3, 3, 10, 2]~
? n = znconreyexp(G, chi)
%11 = 84149
? znconreychar(G, n)
%12 = [10, 1, 0, 1, 1]
```

The library syntax is `GEN znconreylog(GEN bid, GEN m)`.

3.4.103 zncoppersmith$(P, N, X, \{B = N\})$. N being an integer and $P \in \mathbf{Z}[X]$, finds all integers x with $|x| \leq X$ such that

$$\gcd(N, P(x)) \geq B,$$

using Coppersmith's algorithm (a famous application of the LLL algorithm). X must be smaller than $\exp(\log^2 B/(\deg(P)\log N))$: for $B = N$, this means $X < N^{1/\deg(P)}$. Some x larger than X may be returned if you are very lucky. The smaller B (or the larger X), the slower the routine will be. The strength of Coppersmith method is the ability to find roots modulo a general *composite* N: if N is a prime or a prime power, `polrootsmod` or `polrootspadic` will be much faster.

We shall now present two simple applications. The first one is finding non-trivial factors of N, given some partial information on the factors; in that case B must obviously be smaller than the largest non-trivial divisor of N.

```
setrand(1); \\ to make the example reproducible
interval = [10^30, 10^31];
p = randomprime(interval);
q = randomprime(interval); N = p*q;
p0 = p % 10^20; \\ assume we know 1) p > 10^29, 2) the last 19 digits of p
L = zncoppersmith(10^19*x + p0, N, 10^12, 10^29)

\\ result in 10ms.
%6 = [738281386540]
? gcd(L[1] * 10^19 + p0, N) == p
%7 = 1
```

and we recovered p, faster than by trying all possibilities $< 10^{12}$.

The second application is an attack on RSA with low exponent, when the message x is short and the padding P is known to the attacker. We use the same RSA modulus N as in the first example:

```
setrand(1);
P = random(N);     \\ known padding
e = 3;             \\ small public encryption exponent
X = floor(N^0.3); \\ N^(1/e - epsilon)
x0 = random(X);    \\ unknown short message
C = lift( (Mod(x0,N) + P)^e ); \\ known ciphertext, with padding P
zncoppersmith((P + x)^3 - C, N, X)

\\ result in 244ms.
%14 = [2679982004001230401]

? %[1] == x0
%15 = 1
```

We guessed an integer of the order of 10^{18}, almost instantly.

The library syntax is `GEN zncoppersmith(GEN P, GEN N, GEN X, GEN B = NULL)`.

3.4.104 znlog$(x, g, \{o\})$. This functions allows two distinct modes of operation depending on g:

• if g is the output of `znstar` (with initialization), we compute the discrete logarithm of x with respect to the generators contained in the structure. See `ideallog` for details.

• else g is an explicit element in $(\mathbf{Z}/N\mathbf{Z})^*$, we compute the discrete logarithm of x in $(\mathbf{Z}/N\mathbf{Z})^*$ in base g. The rest of this entry describes the latter possibility.

The result is [] when x is not a power of g, though the function may also enter an infinite loop in this case.

If present, o represents the multiplicative order of g, see Section 3.4.2; the preferred format for this parameter is `[ord, factor(ord)]`, where `ord` is the order of g. This provides a definite speedup when the discrete log problem is simple:

```
? p = nextprime(10^4); g = znprimroot(p); o = [p-1, factor(p-1)];
? for(i=1,10^4, znlog(i, g, o))
time = 205 ms.
? for(i=1,10^4, znlog(i, g))
time = 244 ms. \\ a little slower
```

The result is undefined if g is not invertible mod N or if the supplied order is incorrect.

This function uses

• a combination of generic discrete log algorithms (see below).

• in $(\mathbf{Z}/N\mathbf{Z})^*$ when N is prime: a linear sieve index calculus method, suitable for $N < 10^{50}$, say, is used for large prime divisors of the order.

The generic discrete log algorithms are:

• Pohlig-Hellman algorithm, to reduce to groups of prime order q, where $q|p-1$ and p is an odd prime divisor of N,

• Shanks baby-step/giant-step ($q < 2^{32}$ is small),

• Pollard rho method ($q > 2^{32}$).

The latter two algorithms require $O(\sqrt{q})$ operations in the group on average, hence will not be able to treat cases where $q > 10^{30}$, say. In addition, Pollard rho is not able to handle the case where there are no solutions: it will enter an infinite loop.

```
? g = znprimroot(101)
%1 = Mod(2,101)
? znlog(5, g)
%2 = 24
? g^24
%3 = Mod(5, 101)

? G = znprimroot(2 * 101^10)
%4 = Mod(110462212541120451003, 220924425082240902002)
? znlog(5, G)
%5 = 76210072736547066624
? G^% == 5
%6 = 1
? N = 2^4*3^2*5^3*7^4*11; g = Mod(13, N); znlog(g^110, g)
%7 = 110
? znlog(6, Mod(2,3))  \\ no solution
%8 = []
```

For convenience, g is also allowed to be a p-adic number:

```
? g = 3+O(5^10); znlog(2, g)
%1 = 1015243
? g^%
%2 = 2 + O(5^10)
```

The library syntax is `GEN znlog0(GEN x, GEN g, GEN o = NULL)`. The function `GEN znlog(GEN x, GEN g, GEN o)` is also available

3.4.105 znorder$(x, \{o\})$. x must be an integer mod n, and the result is the order of x in the multiplicative group $(\mathbf{Z}/n\mathbf{Z})^*$. Returns an error if x is not invertible. The parameter o, if present, represents a non-zero multiple of the order of x, see Section 3.4.2; the preferred format for this parameter is `[ord, factor(ord)]`, where `ord = eulerphi(n)` is the cardinality of the group.

The library syntax is `GEN znorder(GEN x, GEN o = NULL)`. Also available is `GEN order(GEN x)`.

3.4.106 znprimroot(n). Returns a primitive root (generator) of $(\mathbf{Z}/n\mathbf{Z})^*$, whenever this latter group is cyclic ($n = 4$ or $n = 2p^k$ or $n = p^k$, where p is an odd prime and $k \geq 0$). If the group is not cyclic, the result is undefined. If n is a prime power, then the smallest positive primitive root is returned. This may not be true for $n = 2p^k$, p odd.

Note that this function requires factoring $p - 1$ for p as above, in order to determine the exact order of elements in $(\mathbf{Z}/n\mathbf{Z})^*$: this is likely to be costly if p is large.

The library syntax is `GEN znprimroot(GEN n)`.

3.4.107 znstar$(n, \{flag = 0\})$. Gives the structure of the multiplicative group $(\mathbf{Z}/n\mathbf{Z})^*$. The output G depends on the value of *flag*:

• *flag* $= 0$ (default), an abelian group structure $[h, d, g]$, where $h = \phi(n)$ is the order (`G.no`), d (`G.cyc`) is a k-component row-vector d of integers d_i such that $d_i > 1$, $d_i \mid d_{i-1}$ for $i \geq 2$ and

$$(\mathbf{Z}/n\mathbf{Z})^* \simeq \prod_{i=1}^{k} (\mathbf{Z}/d_i\mathbf{Z}),$$

and g (`G.gen`) is a k-component row vector giving generators of the image of the cyclic groups $\mathbf{Z}/d_i\mathbf{Z}$.

• *flag* $= 1$ the result is a `bid` structure without generators (which are well defined but not explicitly computed, which saves time); this allows computing discrite logarithms using `znlog` (also in the non-cyclic case!).

• *flag* $= 2$ same as *flag* $= 1$ with generators.

```
? G = znstar(40)
%1 = [16, [4, 2, 2], [Mod(17, 40), Mod(21, 40), Mod(11, 40)]]
? G.no   \\ eulerphi(40)
%2 = 16
? G.cyc  \\ cycle structure
%3 = [4, 2, 2]
? G.gen  \\ generators for the cyclic components
%4 = [Mod(17, 40), Mod(21, 40), Mod(11, 40)]
? apply(znorder, G.gen)
%5 = [4, 2, 2]
```

According to the above definitions, `znstar(0)` is `[2, [2], [-1]]`, corresponding to $\mathbf{Z}^*$.

The library syntax is `GEN znstar0(GEN n, long flag)`. Instead the above hardcoded numerical flags, one should rather use `GEN ZNstar(GEN N, long flag)`, where `flag` is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (return a full `bid`, not a group), possibly 0. This offers one more combination: no gen and no init.

3.5 Elliptic curves.

3.5.1 Elliptic curve structures. An elliptic curve is given by a Weierstrass model

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

whose discriminant is non-zero. Affine points on `E` are represented as two-component vectors `[x,y]`; the point at infinity, i.e. the identity element of the group law, is represented by the one-component vector `[0]`.

Given a vector of coefficients $[a_1, a_2, a_3, a_4, a_6]$, the function `ellinit` initializes and returns an *ell* structure. (An additional optional argument allows to specify the base field in case it cannot be inferred from the curve coefficients.) This structure contains data needed by elliptic curve related functions, and is generally passed as a first argument. Expensive data are skipped on initialization: they will be dynamically computed when (and if) needed, and then inserted in the structure. The precise layout of the *ell* structure is left undefined and should never be used directly. The following member functions are available, depending on the underlying domain.

3.5.1.1 All domains.

• `a1`, `a2`, `a3`, `a4`, `a6`: coefficients of the elliptic curve.

• `b2`, `b4`, `b6`, `b8`: b-invariants of the curve; in characteristic $\neq 2$, for $Y = 2y + a_1x + a3$, the curve equation becomes

$$Y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6 =: g(x).$$

• `c4`, `c6`: c-invariants of the curve; in characteristic $\neq 2, 3$, for $X = x + b_2/12$ and $Y = 2y + a_1x + a3$, the curve equation becomes

$$Y^2 = 4X^3 - (c_4/12)X - (c_6/216).$$

• `disc`: discriminant of the curve. This is only required to be non-zero, not necessarily a unit.

• `j`: j-invariant of the curve.

These are used as follows:

```
? E = ellinit([0,0,0, a4,a6]);
? E.b4
%2 = 2*a4
? E.disc
%3 = -64*a4^3 - 432*a6^2
```

3.5.1.2 Curves over R.

This in particular includes curves defined over $\mathbf{Q}$. All member functions in this section return data, as it is currently stored in the structure, if present; and otherwise compute it to the default accuracy, that was fixed *at the time of ellinit* (via a `t_REAL` D domain argument, or `realprecision` by default). The function `ellperiods` allows to recompute (and cache) the following data to *current* `realprecision`.

• `area`: volume of the complex lattice defining E.

• `roots` is a vector whose three components contain the complex roots of the right hand side $g(x)$ of the attached b-model $Y^2 = g(x)$. If the roots are all real, they are ordered by decreasing value. If only one is real, it is the first component.

• `omega`: $[\omega_1, \omega_2]$, periods forming a basis of the complex lattice defining E. The first component ω_1 is the (positive) real period, in other words the integral of the Néron differential $dx/(2y + a_1x + a_3)$ over the connected component of the identity component of $E(\mathbf{R})$. The second component ω_2 is a complex period, such that $\tau = \frac{\omega_1}{\omega_2}$ belongs to Poincaré's half-plane (positive imaginary part); not necessarily to the standard fundamental domain. It is normalized so that $\Im(\omega_2) < 0$ and either $\Re(\omega_2) = 0$, when `E.disc > 0` ($E(\mathbf{R})$ has two connected components), or $\Re(\omega_2) = \omega_1/2$

• `eta` is a row vector containing the quasi-periods η_1 and η_2 such that $\eta_i = 2\zeta(\omega_i/2)$, where ζ is the Weierstrass zeta function attached to the period lattice; see `ellzeta`. In particular, the Legendre relation holds: $\eta_2\omega_1 - \eta_1\omega_2 = 2\pi i$.

Warning. As for the orientation of the basis of the period lattice, beware that many sources use the inverse convention where ω_2/ω_1 has positive imaginary part and our ω_2 is the negative of theirs. Our convention $\tau = \omega_1/\omega_2$ ensures that the action of PSL_2 is the natural one:

$$[a, b; c, d] \cdot \tau = (a\tau + b)/(c\tau + d) = (a\omega_1 + b\omega_2)/(c\omega_1 + d\omega_2),$$

instead of a twisted one. (Our *tau* is $-1/\tau$ in the above inverse convention.)

3.5.1.3 Curves over $\mathbf{Q}_p$.

We advise to input a model defined over $\mathbf{Q}$ for such curves. In any case, if you input an approximate model with `t_PADIC` coefficients, it will be replaced by a lift to $\mathbf{Q}$ (an exact model "close" to the one that was input) and all quantities will then be computed in terms of this lifted model.

For the time being only curves with multiplicative reduction (split or non-split), i.e. $v_p(j) < 0$, are supported by non-trivial functions. In this case the curve is analytically isomorphic to $\bar{\mathbf{Q}}_p^*/q^{\mathbf{Z}} := E_q(\bar{\mathbf{Q}}_p)$, for some p-adic integer q (the Tate period). In particular, we have $j(q) = j(E)$.

- `p` is the residual characteristic

- `roots` is a vector with a single component, equal to the p-adic root e_1 of the right hand side $g(x)$ of the attached b-model $Y^2 = g(x)$. The point $(e_1, 0)$ corresponds to $-1 \in \bar{\mathbf{Q}}_p^*/q^{\mathbf{Z}}$ under the Tate parametrization.

- `tate` returns $[u^2, u, q, [a, b], L, Ei]$ in the notation of Henniart-Mestre (CRAS t. 308, p. 391–395, 1989): q is as above, $u \in \mathbf{Q}_p(\sqrt{-c_6})$ is such that $\phi^* dx/(2y + a_1x + a3) = udt/t$, where $\phi : E_q \to E$ is an isomorphism (well defined up to sign) and dt/t is the canonical invariant differential on the Tate curve; $u^2 \in \mathbf{Q}_p$ does not depend on ϕ. (Technicality: if $u \notin \mathbf{Q}_p$, it is stored as a quadratic `t_POLMOD`.) The parameters $[a, b]$ satisfy $4u^2b \cdot \mathrm{agm}(\sqrt{a/b}, 1)^2 = 1$ as in Theorem 2 (*loc. cit.*). `Ei` describes the sequence of 2-isogenous curves (with kernel generated by $[0, 0]$) $E_i : y^2 = x(x + A_i)(x + A_i - B_i)$ converging quadratically towards the singular curve E_∞. Finally, L is Mazur-Tate-Teitelbaum's $\mathcal{L}$-invariant, equal to $\log_p q/v_p(q)$.

3.5.1.4 Curves over $\mathbf{F}_q$.

- `p` is the characteristic of $\mathbf{F}_q$.

- `no` is $\#E(\mathbf{F}_q)$.

- `cyc` gives the cycle structure of $E(\mathbf{F}_q)$.

- `gen` returns the generators of $E(\mathbf{F}_q)$.

- `group` returns [`no`, `cyc`, `gen`], i.e. $E(\mathbf{F}_q)$ as an abelian group structure.

3.5.1.5 Curves over $\mathbf{Q}$.

All functions should return a correct result, whether the model is minimal or not, but it is a good idea to stick to minimal models whenever $\gcd(c_4, c_6)$ is easy to factor (minor speed-up). The construction

```
E = ellminimalmodel(E0, &v)
```

replaces the original model E_0 by a minimal model E, and the variable change v allows to go between the two models:

```
ellchangepoint(P0, v)
```

`ellchangepointinv(P, v)`

respectively map the point P_0 on E_0 to its image on E, and the point P on E to its pre-image on E_0.

A few routines — namely `ellgenerators`, `ellidentify`, `ellsearch`, `forell` — require the optional package `elldata` (John Cremona's database) to be installed. In that case, the function `ellinit` will allow alternative inputs, e.g. `ellinit("11a1")`. Functions using this package need to load chunks of a large database in memory and require at least 2MB stack to avoid stack overflows.

• `gen` returns the generators of $E(\mathbf{Q})$, if known (from John Cremona's database)

3.5.1.6 Curves over number fields.

• `nf` return the *nf* structure attached to the number field over which E is defined.

• `bnf` return the *bnf* structure attached to the number field over which E is defined or raise an error (if only an *nf* is available).

3.5.2 ellL1$(e, \{r = 0\})$. Returns the value at $s = 1$ of the derivative of order r of the L-function of the elliptic curve e.

```
? e = ellinit("11a1"); \\ order of vanishing is 0
? ellL1(e)
%2 = 0.25384186085591068433775892 33
? e = ellinit("389a1");  \\ order of vanishing is 2
? ellL1(e)
%4 = -5.384067311837218089235032414 E-29
? ellL1(e, 1)
%5 = 0
? ellL1(e, 2)
%6 = 1.518633000576853540460385214
```

The main use of this function, after computing at *low* accuracy the order of vanishing using `ellanalyticrank`, is to compute the leading term at *high* accuracy to check (or use) the Birch and Swinnerton-Dyer conjecture:

```
? \p18
  realprecision = 18 significant digits
? e = ellinit("5077a1"); ellanalyticrank(e)
time = 8 ms.
%1 = [3, 10.3910994007158041]
? \p200
  realprecision = 202 significant digits (200 digits displayed)
? ellL1(e, 3)
time = 104 ms.
%3 = 10.391099400715804138751850510360917069726356375657009279 7[...]
```

The library syntax is `GEN ellL1_bitprec(GEN e, long r, long bitprec)`.

3.5.3 elladd$(E, z1, z2)$. Sum of the points $z1$ and $z2$ on the elliptic curve corresponding to E.

The library syntax is `GEN elladd(GEN E, GEN z1, GEN z2)`.

3.5.4 ellak(E, n). Computes the coefficient a_n of the L-function of the elliptic curve $E/\mathbf{Q}$, i.e. coefficients of a newform of weight 2 by the modularity theorem (Taniyama-Shimura-Weil conjecture). E must be an `ell` structure over $\mathbf{Q}$ as output by `ellinit`. E must be given by an integral model, not necessarily minimal, although a minimal model will make the function faster.

```
? E = ellinit([0,1]);
? ellak(E, 10)
%2 = 0
? e = ellinit([5^4,5^6]); \\ not minimal at 5
? ellak(e, 5) \\ wasteful but works
%3 = -3
? E = ellminimalmodel(e); \\ now minimal
? ellak(E, 5)
%5 = -3
```

If the model is not minimal at a number of bad primes, then the function will be slower on those n divisible by the bad primes. The speed should be comparable for other n:

```
? for(i=1,10^6, ellak(E,5))
time = 820 ms.
? for(i=1,10^6, ellak(e,5)) \\ 5 is bad, markedly slower
time = 1,249 ms.
```

```
? for(i=1,10^5,ellak(E,5*i))
time = 977 ms.
? for(i=1,10^5,ellak(e,5*i)) \\ still slower but not so much on average
time = 1,008 ms.
```

The library syntax is `GEN akell(GEN E, GEN n)`.

3.5.5 ellan(E, n). Computes the vector of the first n Fourier coefficients a_k corresponding to the elliptic curve E defined over a number field. If E is defined over $\mathbf{Q}$, the curve may be given by an arbitrary model, not necessarily minimal, although a minimal model will make the function faster. Over a more general number field, the model must be locally minimal at all primes above 2 and 3.

The library syntax is `GEN ellan(GEN E, long n)`. Also available is `GEN ellanQ_zv(GEN e, long n)`, which returns a `t_VECSMALL` instead of a `t_VEC`, saving on memory.

3.5.6 ellanalyticrank$(e, \{eps\})$. Returns the order of vanishing at $s = 1$ of the L-function of the elliptic curve e and the value of the first non-zero derivative. To determine this order, it is assumed that any value less than `eps` is zero. If no value of `eps` is given, a value of half the current precision is used.

```
? e = ellinit("11a1"); \\ rank 0
? ellanalyticrank(e)
%2 = [0, 0.25384186085591068433775892335]
? e = ellinit("37a1"); \\ rank 1
? ellanalyticrank(e)
%4 = [1, 0.30599977383405230182048368350]
? e = ellinit("389a1"); \\ rank 2
? ellanalyticrank(e)
%6 = [2, 1.5186330005768535404603852140]
? e = ellinit("5077a1"); \\ rank 3
```

```
? ellanalyticrank(e)
%8 = [3, 10.391099400715804138751850 35]
```

The library syntax is `GEN ellanalyticrank_bitprec(GEN e, GEN eps = NULL, long bitprec)`.

3.5.7 ellap$(E, \{p\})$. Let E be an `ell` structure as output by `ellinit`, defined over a number field or a finite field $\mathbf{F}_q$. The argument p is best left omitted if the curve is defined over a finite field, and must be a prime number or a maximal ideal otherwise. This function computes the trace of Frobenius t for the elliptic curve E, defined by the equation $\#E(\mathbf{F}_q) = q+1-t$ (for primes of good reduction).

When the characteristic of the finite field is large, the availability of the `seadata` package will speed the computation.

If the curve is defined over $\mathbf{Q}$, p must be explicitly given and the function computes the trace of the reduction over $\mathbf{F}_p$. The trace of Frobenius is also the a_p coefficient in the curve L-series $L(E,s) = \sum_n a_n n^{-s}$, whence the function name. The equation must be integral at p but need not be minimal at p; of course, a minimal model will be more efficient.

```
? E = ellinit([0,1]);  \\ y^2 = x^3 + 0.x + 1, defined over Q
? ellap(E, 7) \\ 7 necessary here
%2 = -4       \\ #E(F_7) = 7+1-(-4) = 12
? ellcard(E, 7)
%3 = 12       \\ OK
? E = ellinit([0,1], 11);  \\ defined over F_11
? ellap(E)       \\ no need to repeat 11
%4 = 0
? ellap(E, 11)   \\ ... but it also works
%5 = 0
? ellgroup(E, 13) \\ ouch, inconsistent input!
   ***   at top-level: ellap(E,13)
   ***                 ^-----------
   *** ellap: inconsistent moduli in Rg_to_Fp:
     11
     13
? Fq = ffgen(ffinit(11,3), 'a); \\ defines F_q := F_{11^3}
? E = ellinit([a+1,a], Fq);  \\ y^2 = x^3 + (a+1)x + a, defined over F_q
? ellap(E)
%8 = -3
```

If the curve is defined over a more general number field than $\mathbf{Q}$, the maximal ideal p must be explicitly given in `idealprimedec` format. If p is above 2 or 3, the function currently assumes (without checking) that the given model is locally minimal at p. There is no restriction at other primes.

```
? K = nfinit(a^2+1); E = ellinit([1+a,0,1,0,0], K);
? fa = idealfactor(K, E.disc)
%2 =
[  [5, [-2, 1]~, 1, 1, [2, -1; 1, 2]] 1]

[[13, [5, 1]~, 1, 1, [-5, -1; 1, -5]] 2]
```

```
? ellap(E, fa[1,1])
%3 = -1 \\ non-split multiplicative reduction
? ellap(E, fa[2,1])
%4 = 1  \\ split multiplicative reduction
? P17 = idealprimedec(K,17)[1];
? ellap(E, P17)
%6 = 6  \\ good reduction
? E2 = ellchangecurve(E, [17,0,0,0]);
? ellap(E2, P17)
%8 = 6  \\ same, starting from a non-miminal model
? P3 = idealprimedec(K,3)[1];
? E3 = ellchangecurve(E, [3,0,0,0]);
? ellap(E, P3)  \\ OK: E is minimal at P3
%11 = -2
? ellap(E3, P3) \\ junk: E3 is not minimal at P3 | 3
%12 = 0
```

Algorithms used. If $E/\mathbf{F}_q$ has CM by a principal imaginary quadratic order we use a fast explicit formula (involving essentially Kronecker symbols and Cornacchia's algorithm), in $O(\log q)^2$. Otherwise, we use Shanks-Mestre's baby-step/giant-step method, which runs in time $\tilde{O}(q^{1/4})$ using $\tilde{O}(q^{1/4})$ storage, hence becomes unreasonable when q has about 30 digits. Above this range, the `SEA` algorithm becomes available, heuristically in $\tilde{O}(\log q)^4$, and primes of the order of 200 digits become feasible. In small characteristic we use Mestre's (p=2), Kohel's (p=3,5,7,13), Satoh-Harley (all in $\tilde{O}(p^2n^2)$) or Kedlaya's (in $\tilde{O}(pn^3)$) algorithms.

The library syntax is `GEN ellap(GEN E, GEN p = NULL)`.

3.5.8 ellbil$(E, z1, z2)$. Deprecated alias for `ellheight(E,P,Q)`.

The library syntax is `GEN bilhell(GEN E, GEN z1, GEN z2, long prec)`.

3.5.9 ellcard$(E, \{p\})$. Let E be an `ell` structure as output by `ellinit`, defined over $\mathbf{Q}$ or a finite field $\mathbf{F}_q$. The argument p is best left omitted if the curve is defined over a finite field, and must be a prime number otherwise. This function computes the order of the group $E(\mathbf{F}_q)$ (as would be computed by `ellgroup`).

When the characteristic of the finite field is large, the availability of the `seadata` package will speed the computation.

If the curve is defined over $\mathbf{Q}$, p must be explicitly given and the function computes the cardinality of the reduction over $\mathbf{F}_p$; the equation need not be minimal at p, but a minimal model will be more efficient. The reduction is allowed to be singular, and we return the order of the group of non-singular points in this case.

The library syntax is `GEN ellcard(GEN E,  GEN p = NULL)`. Also available is `GEN ellcard(GEN E, GEN p)` where p is not `NULL`.

3.5.10 ellchangecurve(E, v). Changes the data for the elliptic curve E by changing the coordinates using the vector `v=[u,r,s,t]`, i.e. if x' and y' are the new coordinates, then $x = u^2x' + r$, $y = u^3y' + su^2x' + t$. E must be an `ell` structure as output by `ellinit`. The special case $v = 1$ is also used instead of $[1, 0, 0, 0]$ to denote the trivial coordinate change.

The library syntax is `GEN ellchangecurve(GEN E, GEN v)`.

3.5.11 ellchangepoint(x, v). Changes the coordinates of the point or vector of points x using the vector `v=[u,r,s,t]`, i.e. if x' and y' are the new coordinates, then $x = u^2x' + r$, $y = u^3y' + su^2x' + t$ (see also `ellchangecurve`).

```
? E0 = ellinit([1,1]); P0 = [0,1]; v = [1,2,3,4];
? E = ellchangecurve(E0, v);
? P = ellchangepoint(P0,v)
%3 = [-2, 3]
? ellisoncurve(E, P)
%4 = 1
? ellchangepointinv(P,v)
%5 = [0, 1]
```

The library syntax is `GEN ellchangepoint(GEN x, GEN v)`. The reciprocal function `GEN ellchangepointinv(GEN x, GEN ch)` inverts the coordinate change.

3.5.12 ellchangepointinv(x, v). Changes the coordinates of the point or vector of points x using the inverse of the isomorphism attached to `v=[u,r,s,t]`, i.e. if x' and y' are the old coordinates, then $x = u^2x' + r$, $y = u^3y' + su^2x' + t$ (inverse of `ellchangepoint`).

```
? E0 = ellinit([1,1]); P0 = [0,1]; v = [1,2,3,4];
? E = ellchangecurve(E0, v);
? P = ellchangepoint(P0,v)
%3 = [-2, 3]
? ellisoncurve(E, P)
%4 = 1
? ellchangepointinv(P,v)
%5 = [0, 1]  \\ we get back P0
```

The library syntax is `GEN ellchangepointinv(GEN x, GEN v)`.

3.5.13 ellconvertname$(name)$. Converts an elliptic curve name, as found in the `elldata` database, from a string to a triplet $[conductor, isogeny\ class, index]$. It will also convert a triplet back to a curve name. Examples:

```
? ellconvertname("123b1")
%1 = [123, 1, 1]
? ellconvertname(%)
%2 = "123b1"
```

The library syntax is `GEN ellconvertname(GEN name)`.

3.5.14 elldivpol$(E, n, \{v = ' x\})$. n-division polynomial f_n for the curve E in the variable v. In standard notation, for any affine point $P = (X, Y)$ on the curve, we have

$$[n]P = (\phi_n(P)\psi_n(P) : \omega_n(P) : \psi_n(P)^3)$$

for some polynomials ϕ_n, ω_n, ψ_n in $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6][X, Y]$. We have $f_n(X) = \psi_n(X)$ for n odd, and $f_n(X) = \psi_n(X, Y)(2Y + a_1X + a_3)$ for n even. We have

$$f_1 = 1, \quad f_2 = 4X^3 + b_2X^2 + 2b_4X + b_6, \quad f_3 = 3X^4 + b_2X^3 + 3b_4X^2 + 3b_6X + b8,$$
$$f_4 = f_2(2X^6 + b_2X^5 + 5b_4X^4 + 10b_6X^3 + 10b_8X^2 + (b_2b_8 - b_4b_6)X + (b_8b_4 - b_6^2)), \ldots$$

For $n \geq 2$, the roots of f_n are the X-coordinates of points in $E[n]$.

The library syntax is `GEN elldivpol(GEN E, long n, long v = -1)` where `v` is a variable number.

3.5.15 elleisnum$(w, k, \{flag = 0\})$. k being an even positive integer, computes the numerical value of the Eisenstein series of weight k at the lattice w, as given by `ellperiods`, namely

$$(2i\pi/\omega_2)^k\Big(1 + 2/\zeta(1-k)\sum_{n\geq 1} n^{k-1}q^n/(1-q^n)\Big),$$

where $q = \exp(2i\pi\tau)$ and $\tau := \omega_1/\omega_2$ belongs to the complex upper half-plane. It is also possible to directly input $w = [\omega_1, \omega_2]$, or an elliptic curve E as given by `ellinit`.

```
? w = ellperiods([1,I]);
? elleisnum(w, 4)
%2 = 2268.8726415508062275167367584190557607
? elleisnum(w, 6)
%3 = -3.977978632282564763 E-33
? E = ellinit([1, 0]);
? elleisnum(E, 4, 1)
%5 = -47.999999999999999999999999999999999998
```

When *flag* is non-zero and $k = 4$ or 6, returns the elliptic invariants g_2 or g_3, such that

$$y^2 = 4x^3 - g_2x - g_3$$

is a Weierstrass equation for E.

The library syntax is `GEN elleisnum(GEN w, long k, long flag, long prec)`.

3.5.16 elleta(w). Returns the quasi-periods $[\eta_1, \eta_2]$ attached to the lattice basis $w = [\omega_1, \omega_2]$. Alternatively, w can be an elliptic curve E as output by `ellinit`, in which case, the quasi periods attached to the period lattice basis E`.omega` (namely, E`.eta`) are returned.

```
? elleta([1, I])
%1 = [3.1415926535897932384626433832, 9.424777960769379715387930149*I]
```

The library syntax is `GEN elleta(GEN w, long prec)`.

3.5.17 ellformaldifferential$(E, \{n = seriesprecision\}, \{t =' x\})$. Let $\omega := dx/(2y + a_1x + a_3)$ be the invariant differential form attached to the model E of some elliptic curve (`ellinit` form), and $\eta := x(t)\omega$. Return n terms (`seriesprecision` by default) of $f(t), g(t)$ two power series in the formal parameter $t = -x/y$ such that $\omega = f(t)dt$, $\eta = g(t)dt$:

$$f(t) = 1 + a_1t + (a_1^2 + a_2)t^2 + \ldots, \quad g(t) = t^{-2} + \ldots$$

```
? E = ellinit([-1,1/4]); [f,g] = ellformaldifferential(E,7,'t);
? f
%2 = 1 - 2*t^4 + 3/4*t^6 + O(t^7)
? g
%3 = t^-2 - t^2 + 1/2*t^4 + O(t^5)
```

The library syntax is `GEN ellformaldifferential(GEN E, long precdl, long n = -1)` where `n` is a variable number.

3.5.18 ellformalexp$(E, \{n = \textit{seriesprecision}\}, \{z =' x\})$. The elliptic formal exponential `Exp` attached to E is the isomorphism from the formal additive law to the formal group of E. It is normalized so as to be the inverse of the elliptic logarithm (see `ellformallog`): `Exp` $\circ L = \mathrm{Id}$. Return n terms of this power series:

```
? E=ellinit([-1,1/4]); Exp = ellformalexp(E,10,'z)
%1 = z + 2/5*z^5 - 3/28*z^7 + 2/15*z^9 + O(z^11)
? L = ellformallog(E,10,'t);
? subst(Exp,z,L)
%3 = t + O(t^11)
```

The library syntax is `GEN ellformalexp(GEN E, long precdl, long n = -1)` where `n` is a variable number.

3.5.19 ellformallog$(E, \{n = \textit{seriesprecision}\}, \{v =' x\})$. The formal elliptic logarithm is a series L in $tK[[t]]$ such that $dL = \omega = dx/(2y + a_1x + a_3)$, the canonical invariant differential attached to the model E. It gives an isomorphism from the formal group of E to the additive formal group.

```
? E = ellinit([-1,1/4]); L = ellformallog(E, 9, 't)
%1 = t - 2/5*t^5 + 3/28*t^7 + 2/3*t^9 + O(t^10)
? [f,g] = ellformaldifferential(E,8,'t);
? L' - f
%3 = O(t^8)
```

The library syntax is `GEN ellformallog(GEN E, long precdl, long n = -1)` where `n` is a variable number.

3.5.20 ellformalpoint$(E, \{n = \textit{seriesprecision}\}, \{v =' x\})$. If E is an elliptic curve, return the coordinates $x(t), y(t)$ in the formal group of the elliptic curve E in the formal parameter $t = -x/y$ at ∞:

$$x = t^{-2} - a_1t^{-1} - a_2 - a_3t + \dots$$

$$y = -t^{-3} - a_1t^{-2} - a_2t^{-1} - a_3 + \dots$$

Return n terms (`seriesprecision` by default) of these two power series, whose coefficients are in $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$.

```
? E = ellinit([0,0,1,-1,0]); [x,y] = ellformalpoint(E,8,'t);
? x
%2 = t^-2 - t + t^2 - t^4 + 2*t^5 + O(t^6)
? y
%3 = -t^-3 + 1 - t + t^3 - 2*t^4 + O(t^5)
? E = ellinit([0,1/2]); ellformalpoint(E,7)
%4 = [x^-2 - 1/2*x^4 + O(x^5), -x^-3 + 1/2*x^3 + O(x^4)]
```

The library syntax is `GEN ellformalpoint(GEN E, long precdl, long n = -1)` where `n` is a variable number.

3.5.21 ellformalw$(E, \{n = \textit{seriesprecision}\}, \{t =' x\})$. Return the formal power series w attached to the elliptic curve E, in the variable t:

$$w(t) = t^3 + a_1 t^4 + (a_2 + a_1^2)t^5 + \cdots + O(t^{n+3}),$$

which is the formal expansion of $-1/y$ in the formal parameter $t := -x/y$ at ∞ (take $n =$ `seriesprecision` if n is omitted). The coefficients of w belong to $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$.

```
? E=ellinit([3,2,-4,-2,5]); ellformalw(E, 5, 't)
%1 = t^3 + 3*t^4 + 11*t^5 + 35*t^6 + 101*t^7 + O(t^8)
```

The library syntax is `GEN ellformalw(GEN E, long precdl, long n = -1)` where `n` is a variable number.

3.5.22 ellfromeqn(P). Given a genus 1 plane curve, defined by the affine equation $f(x, y) = 0$, return the coefficients $[a_1, a_2, a_3, a_4, a_6]$ of a Weierstrass equation for its Jacobian. This allows to recover a Weierstrass model for an elliptic curve given by a general plane cubic or by a binary quartic or biquadratic model. The function implements the $f \mapsto f^*$ formulae of Artin, Tate and Villegas (Advances in Math. 198 (2005), pp. 366–382).

In the example below, the function is used to convert between twisted Edwards coordinates and Weierstrass coordinates.

```
? e = ellfromeqn(a*x^2+y^2 - (1+d*x^2*y^2))
%1 = [0, -a - d, 0, -4*d*a, 4*d*a^2 + 4*d^2*a]
? E = ellinit(ellfromeqn(y^2-x^2 - 1 +(121665/121666*x^2*y^2)),2^255-19);
? isprime(ellcard(E) / 8)
%3 = 1
```

The elliptic curve attached to the sum of two cubes is given by

```
? ellfromeqn(x^3+y^3 - a)
%1 = [0, 0, -9*a, 0, -27*a^2]
```

Congruent number problem:. Let n be an integer, if $a^2 + b^2 = c^2$ and $ab = 2n$, then by substituting b by $2n/a$ in the first equation, we get $((a^2 + (2n/a)^2) - c^2)a^2 = 0$. We set $x = a$, $y = ac$.

```
? En = ellfromeqn((x^2 + (2*n/x)^2 - (y/x)^2)*x^2)
%1 = [0, 0, 0, -16*n^2, 0]
```

For example 23 is congruent since the curve has a point of infinite order, namely:

```
? ellheegner( ellinit(subst(En, n, 23)) )
%2 = [168100/289, 68053440/4913]
```

The library syntax is `GEN ellfromeqn(GEN P)`.

3.5.23 ellfromj(j). Returns the coefficients $[a_1, a_2, a_3, a_4, a_6]$ of a fixed elliptic curve with j-invariant j.

The library syntax is `GEN ellfromj(GEN j)`.

3.5.24 ellgenerators(E). If E is an elliptic curve over the rationals, return a $\mathbf{Z}$-basis of the free part of the Mordell-Weil group attached to E. This relies on the `elldata` database being installed and referencing the curve, and so is only available for curves over $\mathbf{Z}$ of small conductors. If E is an elliptic curve over a finite field $\mathbf{F}_q$ as output by `ellinit`, return a minimal set of generators for the group $E(\mathbf{F}_q)$.

The library syntax is `GEN ellgenerators(GEN E)`.

3.5.25 ellglobalred(E). Let E be an `ell` structure as output by `ellinit` attached to an elliptic curve defined over a number field. This function calculates the arithmetic conductor and the global Tamagawa number c. The result $[N, v, c, F, L]$ is slightly different if E is defined over $\mathbf{Q}$ (domain $D = 1$ in `ellinit`) or over a number field (domain D is a number field structure, including `nfinit(x)` representing $\mathbf{Q}$!):

• N is the arithmetic conductor of the curve,

• v is an obsolete field, left in place for backward compatibility. If E is defined over $\mathbf{Q}$, v gives the coordinate change for E to the standard minimal integral model (`ellminimalmodel` provides it in a cheaper way); if E is defined over another number field, v gives a coordinate change to an integral model (`ellintegralmodel` provides it in a cheaper way).

• c is the product of the local Tamagawa numbers c_p, a quantity which enters in the Birch and Swinnerton-Dyer conjecture,

• F is the factorization of N,

• L is a vector, whose i-th entry contains the local data at the i-th prime ideal divisor of N, i.e. `L[i] = elllocalred(E,F[i,1])`. If E is defined over $\mathbf{Q}$, the local coordinate change has been deleted and replaced by a 0; if E is defined over another number field the local coordinate change to a local minimal model is given relative to the integral model afforded by v (so either start from an integral model so that v be trivial, or apply v first).

The library syntax is `GEN ellglobalred(GEN E)`.

3.5.26 ellgroup$(E, \{p\}, \{flag\})$. Let E be an `ell` structure as output by `ellinit`, defined over $\mathbf{Q}$ or a finite field $\mathbf{F}_q$. The argument p is best left omitted if the curve is defined over a finite field, and must be a prime number otherwise. This function computes the structure of the group $E(\mathbf{F}_q) \sim \mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$, with $d_2 \mid d_1$.

If the curve is defined over $\mathbf{Q}$, p must be explicitly given and the function computes the structure of the reduction over $\mathbf{F}_p$; the equation need not be minimal at p, but a minimal model will be more efficient. The reduction is allowed to be singular, and we return the structure of the (cyclic) group of non-singular points in this case.

If the flag is 0 (default), return $[d_1]$ or $[d_1, d_2]$, if $d_2 > 1$. If the flag is 1, return a triple $[h, cyc, gen]$, where h is the curve cardinality, cyc gives the group structure as a product of cyclic groups (as per $flag = 0$). More precisely, if $d_2 > 1$, the output is $[d_1 d_2, [d_1, d_2], [P, Q]]$ where P is of order d_1 and $[P, Q]$ generates the curve.

Caution. It is not guaranteed that Q has order d_2, which in the worst case requires an expensive discrete log computation. Only that `ellweilpairing(E, P, Q, d1)` has order d_2.

```
? E = ellinit([0,1]);  \\ y^2 = x^3 + 0.x + 1, defined over Q
? ellgroup(E, 7)
%2 = [6, 2] \\ Z/6 x Z/2, non-cyclic
? E = ellinit([0,1] * Mod(1,11));  \\ defined over F_11
? ellgroup(E)   \\ no need to repeat 11
%4 = [12]
? ellgroup(E, 11)   \\ ... but it also works
%5 = [12]
? ellgroup(E, 13) \\ ouch, inconsistent input!
   ***   at top-level: ellgroup(E,13)
   ***                 ^--------------
   *** ellgroup: inconsistent moduli in Rg_to_Fp:
     11
     13
? ellgroup(E, 7, 1)
%6 = [12, [6, 2], [[Mod(2, 7), Mod(4, 7)], [Mod(4, 7), Mod(4, 7)]]]
```

If E is defined over $\mathbf{Q}$, we allow singular reduction and in this case we return the structure of the group of non-singular points, satisfying $\#E_{ns}(\mathbf{F}_p) = p - a_p$.

```
? E = ellinit([0,5]);
? ellgroup(E, 5, 1)
%2 = [5, [5], [[Mod(4, 5), Mod(2, 5)]]]
? ellap(E, 5)
%3 = 0 \\ additive reduction at 5
? E = ellinit([0,-1,0,35,0]);
? ellgroup(E, 5, 1)
%5 = [4, [4], [[Mod(2, 5), Mod(2, 5)]]]
? ellap(E, 5)
%6 = 1 \\ split multiplicative reduction at 5
? ellgroup(E, 7, 1)
%7 = [8, [8], [[Mod(3, 7), Mod(5, 7)]]]
? ellap(E, 7)
%8 = -1 \\ non-split multiplicative reduction at 7
```

The library syntax is `GEN ellgroup0(GEN E, GEN p = NULL, long flag)`. Also available is `GEN ellgroup(GEN E, GEN p)`, corresponding to *flag*$= 0$.

3.5.27 ellheegner(E). Let E be an elliptic curve over the rationals, assumed to be of (analytic) rank 1. This returns a non-torsion rational point on the curve, whose canonical height is equal to the product of the elliptic regulator by the analytic Sha.

This uses the Heegner point method, described in Cohen GTM 239; the complexity is proportional to the product of the square root of the conductor and the height of the point (thus, it is preferable to apply it to strong Weil curves).

```
? E = ellinit([-157^2,0]);
? u = ellheegner(E); print(u[1], "\n", u[2])
6964897098259649425445822 5/16613623166818526754080 4
```

```
5389624350896046150780043072587852183335/6771681655607745599922849543574240 8
? ellheegner(ellinit([0,1]))         \\ E has rank 0 !
 ***   at top-level: ellheegner(E=ellinit
 ***                 ^--------------------
 *** ellheegner: The curve has even analytic rank.
```

The library syntax is `GEN ellheegner(GEN E)`.

3.5.28 ellheight$(E, P, \{Q\})$. Global Néron-Tate height $h(P)$ of the point P on the elliptic curve $E/\mathbf{Q}$, using the normalization in Cremona's *Algorithms for modular elliptic curves*. E must be an `ell` as output by `ellinit`; it needs not be given by a minimal model although the computation will be faster if it is.

If the argument Q is present, computes the value of the bilinear form $(h(P+Q)-h(P-Q))/4$.

The library syntax is `GEN ellheight0(GEN E, GEN P, GEN Q = NULL, long prec)`. Also available is `GEN ellheight(GEN E, GEN P, long prec)` (Q omitted).

3.5.29 ellheightmatrix(E, x). x being a vector of points, this function outputs the Gram matrix of x with respect to the Néron-Tate height, in other words, the (i, j) component of the matrix is equal to `ellbil(`E`,x[`i`],x[`j`])`. The rank of this matrix, at least in some approximate sense, gives the rank of the set of points, and if x is a basis of the Mordell-Weil group of E, its determinant is equal to the regulator of E. Note our height normalization follows Cremona's *Algorithms for modular elliptic curves*: this matrix should be divided by 2 to be in accordance with, e.g., Silverman's normalizations.

The library syntax is `GEN ellheightmatrix(GEN E, GEN x, long prec)`.

3.5.30 ellidentify(E). Look up the elliptic curve E, defined by an arbitrary model over $\mathbf{Q}$, in the `elldata` database. Return `[[N, M, G], C]` where N is the curve name in Cremona's elliptic curve database, M is the minimal model, G is a $\mathbf{Z}$-basis of the free part of the Mordell-Weil group $E(\mathbf{Q})$ and C is the change of coordinates change, suitable for `ellchangecurve`.

The library syntax is `GEN ellidentify(GEN E)`.

3.5.31 ellinit$(x, \{D = 1\})$. Initialize an `ell` structure, attached to the elliptic curve E. E is either

- a 5-component vector $[a_1, a_2, a_3, a_4, a_6]$ defining the elliptic curve with Weierstrass equation

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6,$$

- a 2-component vector $[a_4, a_6]$ defining the elliptic curve with short Weierstrass equation

$$Y^2 = X^3 + a_4X + a_6,$$

- a character string in Cremona's notation, e.g. `"11a1"`, in which case the curve is retrieved from the `elldata` database if available.

The optional argument D describes the domain over which the curve is defined:

- the `t_INT` 1 (default): the field of rational numbers $\mathbf{Q}$.

- a `t_INT` p, where p is a prime number: the prime finite field $\mathbf{F}_p$.

• an `t_INTMOD` `Mod(a, p)`, where p is a prime number: the prime finite field $\mathbf{F}_p$.

• a `t_FFELT`, as returned by `ffgen`: the corresponding finite field $\mathbf{F}_q$.

• a `t_PADIC`, $O(p^n)$: the field $\mathbf{Q}_p$, where p-adic quantities will be computed to a relative accuracy of n digits. We advise to input a model defined over $\mathbf{Q}$ for such curves. In any case, if you input an approximate model with `t_PADIC` coefficients, it will be replaced by a lift to $\mathbf{Q}$ (an exact model "close" to the one that was input) and all quantities will then be computed in terms of this lifted model, at the given accuracy.

• a `t_REAL` x: the field $\mathbf{C}$ of complex numbers, where floating point quantities are by default computed to a relative accuracy of `precision`(x). If no such argument is given, the value of `realprecision` at the time `ellinit` is called will be used.

• a number field K, given by a `nf` or `bnf` structure; a `bnf` is required for `ellminimalmodel`.

• a prime ideal `p`, given by a `prid` structure; valid if x is a curve defined over a number field K and the equation is integral and minimal at `p`.

This argument D is indicative: the curve coefficients are checked for compatibility, possibly changing D; for instance if $D = 1$ and an `t_INTMOD` is found. If inconsistencies are detected, an error is raised:

```
? ellinit([1 + O(5), 1], O(7));
 ***   at top-level: ellinit([1+O(5),1],O
 ***                 ^--------------------
 *** ellinit: inconsistent moduli in ellinit: 7 != 5
```

If the curve coefficients are too general to fit any of the above domain categories, only basic operations, such as point addition, will be supported later.

If the curve (seen over the domain D) is singular, fail and return an empty vector [].

```
? E = ellinit([0,0,0,0,1]); \\ y^2 = x^3 + 1, over Q
? E = ellinit([0,1]);       \\ the same curve, short form
? E = ellinit("36a1");      \\ sill the same curve, Cremona's notations
? E = ellinit([0,1], 2)     \\ over F2: singular curve
%4 = []
? E = ellinit(['a4,'a6] * Mod(1,5));  \\ over F_5[a4,a6], basic support !
```

The result of `ellinit` is an *ell* structure. It contains at least the following information in its components:

$$a_1, a_2, a_3, a_4, a_6, b_2, b_4, b_6, b_8, c_4, c_6, \Delta, j.$$

All are accessible via member functions. In particular, the discriminant is E`.disc`, and the j-invariant is E`.j`.

```
? E = ellinit([a4, a6]);
? E.disc
%2 = -64*a4^3 - 432*a6^2
? E.j
%3 = -6912*a4^3/(-4*a4^3 - 27*a6^2)
```

Further components contain domain-specific data, which are in general dynamic: only computed when needed, and then cached in the structure.

```
? E = ellinit([2,3], 10^60+7);  \\ E over F_p, p large
? ellap(E)
time = 4,440 ms.
%2 = -137626826951057988490454040406082
? ellcard(E);  \\ now instantaneous !
time = 0 ms.
? ellgenerators(E);
time = 5,965 ms.
? ellgenerators(E); \\ second time instantaneous
time = 0 ms.
```

See the description of member functions related to elliptic curves at the beginning of this section.

The library syntax is `GEN ellinit(GEN x, GEN D = NULL, long prec)`.

3.5.32 ellintegralmodel$(E, \{\&v\})$. Let E be an `ell` structure over a number field K. This function returns an integral model. If v is present, sets $v = [u, 0, 0, 0]$ to the corresponding change of variable: the return value is identical to that of `ellchangecurve(E, v)`.

The library syntax is `GEN ellintegralmodel(GEN E, GEN *v = NULL)`.

3.5.33 ellisdivisible$(E, P, n, \{\&Q\}))$. Given E/K a number field and P in $E(K)$ return 1 if $P = [n]R$ for some R in $E(K)$ and set Q to one such R; and return 0 otherwise. The integer $n \geq 0$ may be given as `ellxn(E,n)`, if many points need to be tested.

```
? K = nfinit(polcyclo(11,t));
? E = ellinit([0,-1,1,0,0], K);
? P = [0,0];
? ellorder(E,P)
%4 = 5
? ellisdivisible(E,P,5, &Q)
%5 = 1
? lift(Q)
%6 = [-t^7-t^6-t^5-t^4+1, -t^9-2*t^8-2*t^7-3*t^6-3*t^5-2*t^4-2*t^3-t^2-1]
? ellorder(E, Q)
%7 = 25
```

The algebraic complexity of the underlying algorithm is in $O(n^4)$, so it is advisable to first factor n, then use a chain of checks attached to the prime divisors of n: the function will do it itself unless n is given in `ellxn` form.

The library syntax is `long ellisdivisible(GEN E, GEN P, GEN n, GEN *Q) = NULL)`.

3.5.34 ellisogeny$(E, G, \{only\_image = 0\}, \{x =' x\}, \{y =' y\})$. Given an elliptic curve E, a finite subgroup G of E is given either as a generating point P (for a cyclic G) or as a polynomial whose roots vanish on the x-coordinates of the non-zero elements of G (general case and more efficient if available). This function returns the $[a_1, a_2, a_3, a_4, a_6]$ invariants of the quotient elliptic curve E/G and (if *only_image* is zero (the default)) a vector of rational functions $[f, g, h]$ such that the isogeny $E \to E/G$ is given by $(x, y) \mapsto (f(x)/h(x)^2, g(x, y)/h(x)^3)$.

```
? E = ellinit([0,1]);
? elltors(E)
%2 = [6, [6], [[2, 3]]]
? ellisogeny(E, [2,3], 1)  \\ Weierstrass model for E/<P>
%3 = [0, 0, 0, -135, -594]
? ellisogeny(E,[-1,0])
%4 = [[0,0,0,-15,22], [x^3+2*x^2+4*x+3, y*x^3+3*y*x^2-2*y, x+1]]
```

The library syntax is `GEN ellisogeny(GEN E, GEN G, long only_image, long x = -1, long y = -1)` where `x`, `y` are variable numbers.

3.5.35 ellisogenyapply(f, g). Given an isogeny of elliptic curves $f : E' \to E$ (being the result of a call to `ellisogeny`), apply f to g:

- if g is a point P in the domain of f, return the image $f(P)$;
- if $g : E'' \to E'$ is a compatible isogeny, return the composite isogeny $f \circ g : E'' \to E$.

```
? one = ffgen(101, 't)^0;
? E = ellinit([6, 53, 85, 32, 34] * one);
? P = [84, 71] * one;
? ellorder(E, P)
%4 = 5
? [F, f] = ellisogeny(E, P);  \\ f: E->F = E/<P>
? ellisogenyapply(f, P)
%6 = [0]
? F = ellinit(F);
? Q = [89, 44] * one;
? ellorder(F, Q)
%9 = 2
? [G, g] = ellisogeny(F, Q); \\  g: F->G = F/<Q>
? gof = ellisogenyapply(g, f); \\ gof: E -> G
```

The library syntax is `GEN ellisogenyapply(GEN f, GEN g)`.

3.5.36 ellisomat$(E, \{fl = 0\})$. Given an elliptic curve E defined over $\mathbf{Q}$, compute representatives of the isomorphism classes of elliptic curves $\mathbf{Q}$-isogenous to E. The function returns a vector $[L, M]$ where L is a list of triples $[E_i, f_i, g_i]$, where E_i is an elliptic curve in $[a_4, a_6]$ form, $f_i : E \to E_i$ is a rational isogeny, $g_i : E_i \to E$ is the dual isogeny of f_i, and M is the matrix such that $M_{i,j}$ is the degree of the isogeny between E_i and E_j. Furthermore the first curve E_1 is isomorphic to E by f_1. If the flag $fl = 1$, the f_i and g_i are not computed, which saves time, and L is the list of the curves E_i.

```
? E = ellinit("14a1");
? [L,M] = ellisomat(E);
```

```
? LE = apply(x->x[1], L)  \\ list of curves
%3 = [[215/48,-5291/864],[-675/16,6831/32],[-8185/48,-742643/864],
     [-1705/48,-57707/864],[-13635/16,306207/32],[-131065/48,-47449331/864]]
? L[2][2]  \\ isogeny f_2
%4 = [x^3+3/4*x^2+19/2*x-311/12,
      1/2*x^4+(y+1)*x^3+(y-4)*x^2+(-9*y+23)*x+(55*y+55/2),x+1/3]
? L[2][3]  \\ dual isogeny g_2
%5 = [1/9*x^3-1/4*x^2-141/16*x+5613/64,
      -1/18*x^4+(1/27*y-1/3)*x^3+(-1/12*y+87/16)*x^2+(49/16*y-48)*x
      +(-3601/64*y+16947/512),x-3/4]
? apply(E->ellidentify(ellinit(E))[1][1], LE)
%6 = ["14a1","14a4","14a3","14a2","14a6","14a5"]
? M
%7 =
[1  3  3 2  6  6]

[3  1  9 6  2 18]

[3  9  1 6 18  2]

[2  6  6 1  3  3]

[6  2 18 3  1  9]

[6 18  2 3  9  1]
```

The library syntax is `GEN ellisomat(GEN E, long fl)`.

3.5.37 ellisoncurve(E, z). Gives 1 (i.e. true) if the point z is on the elliptic curve E, 0 otherwise. If E or z have imprecise coefficients, an attempt is made to take this into account, i.e. an imprecise equality is checked, not a precise one. It is allowed for z to be a vector of points in which case a vector (of the same type) is returned.

The library syntax is `GEN ellisoncurve(GEN E, GEN z)`. Also available is `int oncurve(GEN E, GEN z)` which does not accept vectors of points.

3.5.38 ellissupersingular$(E, \{p\})$. Return 1 if the elliptic curve E defined over a number field or a finite field is supersingular at p, and 0 otherwise. If the curve is defined over a number field, p must be explicitly given, and must be a prime number, resp. a maximal ideal, if the curve is defined over $\mathbf{Q}$, resp. a general number field: we return 1 if and only if E has supersingular good reduction at p.

Alternatively, E can be given by its j-invariant in a finite field. In this case p must be omitted.

```
? g = ffprimroot(ffgen(7^5))
%1 = x^3 + 2*x^2 + 3*x + 1
? [g^n | n <- [1 .. 7^5 - 1], ellissupersingular(g^n)]
%2 = [6]

? K = nfinit(y^3-2); P = idealprimedec(K, 2)[1];
? E = ellinit([y,1], K);
? ellissupersingular(E, P)
%5 = 1
```

The library syntax is `GEN ellissupersingular(GEN E, GEN p = NULL)`. Also available is `int elljissupersingular(GEN j)` where j is a j-invariant of a curve over a finite field.

3.5.39 ellj(x). Elliptic j-invariant. x must be a complex number with positive imaginary part, or convertible into a power series or a p-adic number with positive valuation.

The library syntax is `GEN jell(GEN x, long prec)`.

3.5.40 elllocalred(E, p). Calculates the Kodaira type of the local fiber of the elliptic curve E at p. E must be an `ell` structure as output by `ellinit`, over $\mathbf{Q}$ (p a rational prime) or a number field K (p a maximal ideal given by a `prid` structure), and is assumed to have all its coefficients a_i integral. The result is a 4-component vector $[f, kod, v, c]$. Here f is the exponent of p in the arithmetic conductor of E, and kod is the Kodaira type which is coded as follows:

1 means good reduction (type I_0), 2, 3 and 4 mean types II, III and IV respectively, $4+\nu$ with $\nu > 0$ means type I_ν; finally the opposite values -1, -2, etc. refer to the starred types I_0^*, II^*, etc. The third component v is itself a vector $[u, r, s, t]$ giving the coordinate changes done during the local reduction; $u = 1$ if and only if the given equation was already minimal at p. Finally, the last component c is the local Tamagawa number c_p.

The library syntax is `GEN elllocalred(GEN E, GEN p)`.

3.5.41 elllog$(E, P, G, \{o\})$. Given two points P and G on the elliptic curve $E/\mathbf{F}_q$, returns the discrete logarithm of P in base G, i.e. the smallest non-negative integer n such that $P = [n]G$. See `znlog` for the limitations of the underlying discrete log algorithms. If present, o represents the order of G, see Section 3.4.2; the preferred format for this parameter is `[N, factor(N)]`, where N is the order of G.

If no o is given, assume that G generates the curve. The function also assumes that P is a multiple of G.

```
? a = ffgen(ffinit(2,8),'a);
? E = ellinit([a,1,0,0,1]);  \\ over F_{2^8}
? x = a^3; y = ellordinate(E,x)[1];
? P = [x,y]; G = ellmul(E, P, 113);
? ord = [242, factor(242)]; \\ P generates a group of order 242. Initialize.
? ellorder(E, G, ord)
%4 = 242
? e = elllog(E, P, G, ord)
%5 = 15
? ellmul(E,G,e) == P
%6 = 1
```

The library syntax is `GEN elllog(GEN E, GEN P, GEN G, GEN o = NULL)`.

3.5.42 elllseries$(E, s, \{A = 1\})$. This function is deprecated, use `lfun(E,s)` instead.

E being an elliptic curve, given by an arbitrary model over $\mathbf{Q}$ as output by `ellinit`, this function computes the value of the L-series of E at the (complex) point s. This function uses an $O(N^{1/2})$ algorithm, where N is the conductor.

The optional parameter A fixes a cutoff point for the integral and is best left omitted; the result must be independent of A, up to `realprecision`, so this allows to check the function's accuracy.

The library syntax is `GEN elllseries(GEN E, GEN s, GEN A = NULL, long prec)`.

3.5.43 ellminimalmodel$(E, \{\&v\})$. Let E be an `ell` structure over a number field K. This function determines whether E admits a global minimal integral model. If so, it returns it and sets $v = [u, r, s, t]$ to the corresponding change of variable: the return value is identical to that of `ellchangecurve(E, v)`.

Else return the (non-principal) Weierstrass class of E, i.e. the class of $\prod \mathfrak{p}^{(v_{\mathfrak{p}}\Delta - \delta_{\mathfrak{p}})/12}$ where $\Delta =$ `E.disc` is the model's discriminant and $\mathfrak{p}_{\mathfrak{p}}^{\delta}$ is the local minimal discriminant. This function requires either that E be defined over the rational field $\mathbf{Q}$ (with domain $D = 1$ in `ellinit`), in which case a global minimal model always exists, or over a number field given by a *bnf* structure. The Weierstrass class is given in `bnfisprincipal` format, i.e. in terms of the `K.gen` generators.

The resulting model has integral coefficients and is everywhere minimal, the coefficients a_1 and a_3 are reduced modulo 2 (in terms of the fixed integral basis `K.zk`) and a_2 is reduced modulo 3. Over $\mathbf{Q}$, we further require that a_1 and a_3 be 0 or 1, that a_2 be 0 or ± 1 and that $u > 0$ in the change of variable: both the model and the change of variable v are then unique.

```
? e = ellinit([6,6,12,55,233]);  \\ over Q
? E = ellminimalmodel(e, &v);
? E[1..5]
%3 = [0, 0, 0, 1, 1]
? v
%4 = [2, -5, -3, 9]

? K = bnfinit(a^2-65);  \\ over a non-principal number field
? K.cyc
%2 = [2]
? u = Mod(8+a, K.pol);
? E = ellinit([1,40*u+1,0,25*u^2,0], K);
? ellminimalmodel(E) \\ no global minimal model exists over Z_K
%6 = [1]~
```

The library syntax is `GEN ellminimalmodel(GEN E, GEN *v = NULL)`.

3.5.44 ellminimaltwist$(E, \{flag = 0\})$. Let E be an elliptic curve defined over $\mathbf{Q}$, return a discriminant D such that the twist of E by D is minimal among all possible quadratic twists, i.e. if $flag = 0$, its minimal model has minimal discriminant, or if $flag = 1$, it has minimal conductor.

In the example below, we find a curve with j-invariant 3 and minimal conductor.

```
? E=ellminimalmodel(ellinit(ellfromj(3)));
? ellglobalred(E)[1]
%2 = 357075
? D = ellminimaltwist(E,1)
%3 = -15
? E2=ellminimalmodel(ellinit(elltwist(E,D)));
? ellglobalred(E2)[1]
%5 = 14283
```

The library syntax is `GEN ellminimaltwist0(GEN E, long flag)`. Also available are `GEN ellminimaltwist(E)` for $flag = 0$, and `GEN ellminimaltwistcond(E)` for $flag = 1$.

3.5.45 ellmoddegree(e). e being an elliptic curve defined over $\mathbf{Q}$ output by `ellinit`, compute the modular degree of e divided by the square of the Manin constant. Return $[D, err]$, where D is a rational number and err is exponent of the truncation error.

The library syntax is `GEN ellmoddegree(GEN e, long bitprec)`.

3.5.46 ellmodulareqn$(N, \{x\}, \{y\})$. Given a prime $N < 500$, return a vector $[P, t]$ where $P(x, y)$ is a modular equation of level N, i.e. a bivariate polynomial with integer coefficients; t indicates the type of this equation: either *canonical* ($t = 0$) or *Atkin* ($t = 1$). This function requires the `seadata` package and its only use is to give access to the package contents. See `polmodular` for a more general and more flexible function.

Let j be the j-invariant function. The polynomial P satisfies the functional equation,

$$P(f, j) = P(f \mid W_N, j \mid W_N) = 0$$

for some modular function $f = f_N$ (hand-picked for each fixed N to minimize its size, see below), where $W_N(\tau) = -1/(N\tau)$ is the Atkin-Lehner involution. These two equations allow to compute the values of the classical modular polynomial Φ_N, such that $\Phi_N(j(\tau), j(N\tau)) = 0$, while being much smaller than the latter. More precisely, we have $j(W_N(\tau)) = j(N\tau)$; the function f is invariant under $\Gamma_0(N)$ and also satisfies

• for Atkin type: $f \mid W_N = f$;

• for canonical type: let $s = 12/\gcd(12, N-1)$, then $f \mid W_N = N^s/f$. In this case, f has a simple definition: $f(\tau) = N^s \left(\eta(N\tau)/\eta(\tau)\right)^{2s}$, where η is Dedekind's eta function.

The following GP function returns values of the classical modular polynomial by eliminating $f_N(\tau)$ in the above functional equation, for $N \leq 31$ or $N \in \{41, 47, 59, 71\}$.

```
classicaleqn(N, X='X, Y='Y)=
{
  my([P,t] = ellmodulareqn(N), Q, d);
  if (poldegree(P,'y) > 2, error("level unavailable in classicaleqn"));
  if (t == 0, \\ Canonical
    my(s = 12/gcd(12,N-1));
    Q = 'x^(N+1) * substvec(P,['x,'y],[N^s/'x,Y]);
    d = N^(s*(2*N+1)) * (-1)^(N+1);
  , \\ Atkin
    Q = subst(P,'y,Y);
    d = (X-Y)^(N+1));
  polresultant(subst(P,'y,X), Q) / d;
}
```

The library syntax is `GEN ellmodulareqn(long N, long x = -1, long y = -1)` where `x`, `y` are variable numbers.

3.5.47 ellmul(E, z, n). Computes $[n]z$, where z is a point on the elliptic curve E. The exponent n is in $\mathbf{Z}$, or may be a complex quadratic integer if the curve E has complex multiplication by n (if not, an error message is issued).

```
? Ei = ellinit([1,0]); z = [0,0];
? ellmul(Ei, z, 10)
%2 = [0]     \\ unsurprising: z has order 2
? ellmul(Ei, z, I)
%3 = [0, 0]  \\ Ei has complex multiplication by Z[i]
? ellmul(Ei, z, quadgen(-4))
%4 = [0, 0]  \\ an alternative syntax for the same query
? Ej  = ellinit([0,1]); z = [-1,0];
? ellmul(Ej, z, I)
  ***   at top-level: ellmul(Ej,z,I)
  ***                 ^--------------
  *** ellmul: not a complex multiplication in ellmul.
? ellmul(Ej, z, 1+quadgen(-3))
%6 = [1 - w, 0]
```

The simple-minded algorithm for the CM case assumes that we are in characteristic 0, and that the quadratic order to which n belongs has small discriminant.

The library syntax is `GEN ellmul(GEN E, GEN z, GEN n)`.

3.5.48 ellneg(E, z). Opposite of the point z on elliptic curve E.

The library syntax is `GEN ellneg(GEN E, GEN z)`.

3.5.49 ellnonsingularmultiple(E, P). Given an elliptic curve $E/\mathbf{Q}$ (more precisely, a model defined over $\mathbf{Q}$ of a curve) and a rational point $P \in E(\mathbf{Q})$, returns the pair $[R, n]$, where n is the least positive integer such that $R := [n]P$ has good reduction at every prime. More precisely, its image in a minimal model is everywhere non-singular.

```
? e = ellinit("57a1"); P = [2,-2];
? ellnonsingularmultiple(e, P)
%2 = [[1, -1], 2]
? e = ellinit("396b2"); P = [35, -198];
? [R,n] = ellnonsingularmultiple(e, P);
? n
%5 = 12
```

The library syntax is `GEN ellnonsingularmultiple(GEN E, GEN P)`.

3.5.50 ellorder$(E, z, \{o\})$. Gives the order of the point z on the elliptic curve E, defined over a finite field or a number field. Return (the impossible value) zero if the point has infinite order.

```
? E = ellinit([-157^2,0]);  \\ the "157-is-congruent" curve
? P = [2,2]; ellorder(E, P)
%2 = 2
? P = ellheegner(E); ellorder(E, P) \\ infinite order
%3 = 0
? K = nfinit(polcyclo(11,t)); E=ellinit("11a3", K); T = elltors(E);
? ellorder(E, T.gen[1])
%5 = 25
? E = ellinit(ellfromj(ffgen(5^10)));
? ellcard(E)
%7 = 9762580
? P = random(E); ellorder(E, P)
%8 = 4881290
? p = 2^160+7; E = ellinit([1,2], p);
? N = ellcard(E)
%9 = 1461501637330902918203686560289225285992592471152
? o = [N, factor(N)];
? for(i=1,100, ellorder(E,random(E)))
time = 260 ms.
```

The parameter o, is now mostly useless, and kept for backward compatibility. If present, it represents a non-zero multiple of the order of z, see Section 3.4.2; the preferred format for this parameter is `[ord, factor(ord)]`, where `ord` is the cardinality of the curve. It is no longer needed since PARI is now able to compute it over large finite fields (was restricted to small prime fields at the time this feature was introduced), *and* caches the result in E so that it is computed and factored only once. Modifying the last example, we see that including this extra parameter provides no improvement:

```
? o = [N, factor(N)];
? for(i=1,100, ellorder(E,random(E),o))
time = 260 ms.
```

The library syntax is `GEN ellorder(GEN E, GEN z, GEN o = NULL)`. The obsolete form `GEN orderell(GEN e, GEN z)` should no longer be used.

3.5.51 ellordinate(E, x). Gives a 0, 1 or 2-component vector containing the y-coordinates of the points of the curve E having x as x-coordinate.

The library syntax is `GEN ellordinate(GEN E, GEN x, long prec)`.

3.5.52 ellpadicL$(E, p, n, \{s = 0\}, \{r = 0\}, \{D = 1\})$. Returns the value (or r-th derivative) on a character χ^s of $\mathbf{Z}_p^*$ of the p-adic L-function of the elliptic curve $E/\mathbf{Q}$, twisted by D, given modulo p^n.

Characters. The set of continuous characters of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ is identified to $\mathbf{Z}_p^*$ via the cyclotomic character χ with values in $\overline{\mathbf{Q}_p}^*$. Denote by $\tau : \mathbf{Z}_p^* \to \mathbf{Z}_p^*$ the Teichmüller character, with values in the $(p-1)$-th roots of 1 for $p \neq 2$, and $\{-1, 1\}$ for $p = 2$; finally, let $\langle\chi\rangle = \chi\tau^{-1}$, with values in $1 + 2p\mathbf{Z}_p$. In GP, the continuous character of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ given by $\langle\chi\rangle^{s_1}\tau^{s_2}$ is represented by the pair of integers $s = (s_1, s_2)$, with $s_1 \in \mathbf{Z}_p$ and $s_2 \bmod p-1$ for $p > 2$, (resp. mod 2 for $p = 2$); s may be also an integer, representing (s, s) or χ^s.

The p-adic L function. The p-adic L function L_p is defined on the set of continuous characters of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$, as $\int_{\mathbf{Z}_p^*} \chi^s d\mu$ for a certain p-adic distribution μ on $\mathbf{Z}_p^*$. The derivative is given by

$$L_p^{(r)}(E, \chi^s) = \int_{\mathbf{Z}_p^*} \log_p^r(a)\chi^s(a)d\mu(a).$$

More precisely:

• When E has good supersingular reduction, L_p takes its values in $\mathbf{Q}_p \otimes H^1_{dR}(E/\mathbf{Q})$ and satisfies

$$(1 - p^{-1}F)^{-2}L_p(E, \chi^0) = (L(E, 1)/\Omega) \cdot \omega$$

where F is the Frobenius, $L(E, 1)$ is the value of the complex L function at 1, ω is the Néron differential and Ω the attached period on $E(\mathbf{R})$. Here, χ^0 represents the trivial character.

The function returns the components of $L_p^{(r)}(E, \chi^s)$ in the basis $(\omega, F(\omega))$.

• When E has ordinary good reduction, this method only defines the projection of $L_p(E, \chi^s)$ on the α-eigenspace, where α is the unit eigenvalue for F. This is what the function returns. We have

$$(1 - \alpha^{-1})^{-2}L_{p,\alpha}(E, \chi^0) = L(E, 1)/\Omega.$$

Two supersingular examples:

```
? cxL(e) = bestappr( ellL1(e) / e.omega[1] );
? e = ellinit("17a1"); p=3; \\ supersingular, a3 = 0
? L = ellpadicL(e,p,4);
? F = [0,-p;1,ellap(e,p)]; \\ Frobenius matrix in the basis (omega,F(omega))
? (1-p^(-1)*F)^-2 * L / cxL(e)
%5 = [1 + O(3^5), O(3^5)]~ \\ [1,0]~
? e = ellinit("116a1"); p=3; \\ supersingular, a3 != 0~
? L = ellpadicL(e,p,4);
? F = [0,-p; 1,ellap(e,p)];
? (1-p^(-1)*F)^-2*L~ / cxL(e)
%9 = [1 + O(3^4), O(3^5)]~
```

Good ordinary reduction:

```
? e = ellinit("17a1"); p=5; ap = ellap(e,p)
%1 = -2 \\ ordinary
? L = ellpadicL(e,p,4)
%2 = 4 + 3*5 + 4*5^2 + 2*5^3 + O(5^4)
? al = padicappr(x^2 - ap*x + p, ap + O(p^7))[1];
? (1-al^(-1))^(-2) * L / cxL(e)
%4 = 1 + O(5^4)
```

Twist and Teichmüller:

```
? e = ellinit("17a1"); p=5; \\ ordinary
\\ 2nd derivative at tau^1, twist by -7
? ellpadicL(e, p, 4, [0,1], 2, -7)
%2 = 2*5^2 + 5^3 + O(5^4)
```

This function is a special case of `mspadicL`, and it also appears as the first term of `mspadicseries`:

```
? e = ellinit("17a1"); p=5;
? L = ellpadicL(e,p,4)
%2 = 4 + 3*5 + 4*5^2 + 2*5^3 + O(5^4)
? [M,phi] = msfromell(e, 1);
? Mp = mspadicinit(M, p, 4);
? mu = mspadicmoments(Mp, phi);
? mspadicL(mu)
%6 = 4 + 3*5 + 4*5^2 + 2*5^3 + 2*5^4 + 5^5 + O(5^6)
? mspadicseries(mu)
%7 = (4 + 3*5 + 4*5^2 + 2*5^3 + 2*5^4 + 5^5 + O(5^6))
      + (3 + 3*5 + 5^2 + 5^3 + O(5^4))*x
      + (2 + 3*5 + 5^2 + O(5^3))*x^2
      + (3 + 4*5 + 4*5^2 + O(5^3))*x^3
      + (3 + 2*5 + O(5^2))*x^4 + O(x^5)
```

These are more cumbersome than `ellpadicL` but allow to compute at different characters, or successive derivatives, or to twist by a quadratic character essentially for the cost of a single call to `ellpadicL` due to precomputations.

The library syntax is `GEN ellpadicL(GEN E, GEN p, long n, GEN s = NULL, long r, GEN D = NULL)`.

3.5.53 ellpadicfrobenius(E, p, n). If $p > 2$ is a prime and E is a elliptic curve on $\mathbf{Q}$ with good reduction at p, return the matrix of the Frobenius endomorphism φ on the crystalline module $D_p(E) = \mathbf{Q}_p \otimes H^1_{dR}(E/\mathbf{Q})$ with respect to the basis of the given model $(\omega, \eta = x\omega)$, where $\omega = dx/(2y + a_1x + a_3)$ is the invariant differential. The characteristic polynomial of φ is $x^2 - a_p x + p$. The matrix is computed to absolute p-adic precision p^n.

```
? E = ellinit([1,-1,1,0,0]);
? F = ellpadicfrobenius(E,5,3);
? lift(F)
%3 =
[120 29]

[ 55  5]
? charpoly(F)
%4 = x^2 + O(5^3)*x + (5 + O(5^3))
? ellap(E, 5)
%5 = 0
```

The library syntax is `GEN ellpadicfrobenius(GEN E, long p, long n)`.

3.5.54 ellpadicheight$(E, p, n, P, \{Q\})$. Cyclotomic p-adic height of the rational point P on the elliptic curve E (defined over $\mathbf{Q}$), given to n p-adic digits. If the argument Q is present, computes the value of the bilinear form $(h(P+Q) - h(P-Q))/4$.

Let $D_{dR}(E) := H^1_{dR}(E) \otimes_{\mathbf{Q}} \mathbf{Q}_p$ be the $\mathbf{Q}_p$ vector space spanned by ω (invariant differential $dx/(2y + a_1x + a3)$ related to the given model) and $\eta = x\omega$. Then the cyclotomic p-adic height associates to $P \in E(\mathbf{Q})$ an element $f\omega + g\eta$ in D_{dR}. This routine returns the vector $[f, g]$ to n p-adic digits.

If $P \in E(\mathbf{Q})$ is in the kernel of reduction mod p and if its reduction at all finite places is non singular, then $g = -(\log_E P)^2$, where $\log_E$ is the logarithm for the formal group of E at p.

If furthermore the model is of the form $Y^2 = X^3 + aX + b$ and $P = (x, y)$, then

$$f = \log_p(\texttt{denominator}(x)) - 2\log_p(\sigma(P))$$

where $\sigma(P)$ is given by `ellsigma`(E, P).

Recall (*Advanced topics in the arithmetic of elliptic curves*, Theorem 3.2) that the local height function over the complex numbers is of the form

$$\lambda(z) = -\log(|\texttt{E.disc}|)/6 + \Re(z\eta(z)) - 2\log(\sigma(z).$$

(N.B. our normalization for local and global heights is twice that of Silverman's).

```
? E = ellinit([1,-1,1,0,0]); P = [0,0];
? ellpadicheight(E,5,4, P)
%2 = [3*5 + 5^2 + 2*5^3 + O(5^4), 5^2 + 4*5^4 + O(5^6)]
? E = ellinit("11a1"); P = [5,5]; \\ torsion point
? ellpadicheight(E,19,6, P)
%4 = O(19^6)
? E = ellinit([0,0,1,-4,2]); P = [-2,1];
? ellpadicheight(E,3,5, P)
%6 = [2*3^2 + 2*3^3 + 3^4 + O(3^5), 2*3^2 + 3^4 + 2*3^5 + 3^6 + O(3^7)]
? ellpadicheight(E,3,5, P, elladd(E,P,P))
```

One can replace the parameter p prime by a vector $[p, [a, b]]$, in which case the routine returns the p-adic number $af + bg$.

When E has good ordinary reduction at p, the "canonical" p-adic height is given by

```
s2 = ellpadics2(E,p,n);
ellpadicheight(E, [p,[1,-s2]], n, P)
```

Since s_2 does not depend on P, it is preferable to compute it only once:

```
? E = ellinit("5077a1"); p = 5; n = 7;
? s2 = ellpadics2(E,p,n);
? M = ellpadicheightmatrix(E,[p,[1,-s2]], n, E.gen);
? matdet(M)   \\ p-adic regulator
%4 = 5 + 5^2 + 4*5^3 + 2*5^4 + 2*5^5 + 5^6 + O(5^7)
```

The library syntax is `GEN ellpadicheight0(GEN E, GEN p, long n, GEN P, GEN Q = NULL)`

.

3.5.55 ellpadicheightmatrix(E, p, n, v). v being a vector of points, this function outputs the Gram matrix of v with respect to the cyclotomic p-adic height, given to n p-adic digits; in other words, the (i, j) component of the matrix is equal to `ellpadicheight`$(E, p, n, v[i], v[j]) = [f, g]$.

See `ellpadicheight`; in particular one can replace the parameter p prime by a vector $[p, [a, b]]$, in which case the routine returns the matrix containing the p-adic numbers $af + bg$.

The library syntax is `GEN ellpadicheightmatrix(GEN E, GEN p, long n, GEN v)`.

3.5.56 ellpadiclog(E, p, n, P). Given E defined over $K = \mathbf{Q}$ or $\mathbf{Q}_p$ and $P = [x, y]$ on $E(K)$ in the kernel of reduction mod p, let $t(P) = -x/y$ be the formal group parameter; this function returns $L(t)$, where L denotes the formal logarithm (mapping the formal group of E to the additive formal group) attached to the canonical invariant differential: $dL = dx/(2y + a_1x + a_3)$.

The library syntax is `GEN ellpadiclog(GEN E, GEN p, long n, GEN P)`.

3.5.57 ellpadics2(E, p, n). If $p > 2$ is a prime and $E/\mathbf{Q}$ is a elliptic curve with ordinary good reduction at p, returns the slope of the unit eigenvector of `ellpadicfrobenius(E,p,n)`, i.e. the action of Frobenius φ on the crystalline module $D_p(E) = \mathbf{Q}_p \otimes H^1_{dR}(E/\mathbf{Q})$ in the basis of the given model $(\omega, \eta = x\omega)$, where ω is the invariant differential $dx/(2y + a_1x + a_3)$. In other words, $\eta + s_2\omega$ is an eigenvector for the unit eigenvalue of φ.

This slope is the unique $c \in 3^{-1}\mathbf{Z}_p$ such that the odd solution $\sigma(t) = t + O(t^2)$ of

$$-d(\frac{1}{\sigma}\frac{d\sigma}{\omega}) = (x(t) + c)\omega$$

is in $t\mathbf{Z}_p[[t]]$.

It is equal to $b_2/12 - E_2/12$ where E_2 is the value of the Katz p-adic Eisenstein series of weight 2 on (E, ω). This is used to construct a canonical p-adic height when E has good ordinary reduction at p as follows

```
s2 = ellpadics2(E,p,n);
h(E,p,n, P, s2) = ellpadicheight(E, [p,[1,-s2]],n, P);
```

Since s_2 does not depend on the point P, we compute it only once.

The library syntax is `GEN ellpadics2(GEN E, GEN p, long n)`.

3.5.58 ellperiods$(w, \{flag = 0\})$. Let w describe a complex period lattice ($w = [w_1, w_2]$ or an `ellinit` structure). Returns normalized periods $[W_1, W_2]$ generating the same lattice such that $\tau := W_1/W_2$ has positive imaginary part and lies in the standard fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$.

If *flag* $= 1$, the function returns $[[W_1, W_2], [\eta_1, \eta_2]]$, where η_1 and η_2 are the quasi-periods attached to $[W_1, W_2]$, satisfying $\eta_1 W_2 - \eta_2 W_1 = 2i\pi$.

The output of this function is meant to be used as the first argument given to ellwp, ellzeta, ellsigma or elleisnum. Quasi-periods are needed by ellzeta and ellsigma only.

The library syntax is `GEN ellperiods(GEN w, long flag, long prec)`.

3.5.59 ellpointtoz(E, P). If $E/\mathbf{C} \simeq \mathbf{C}/\Lambda$ is a complex elliptic curve ($\Lambda =$ `E.omega`), computes a complex number z, well-defined modulo the lattice Λ, corresponding to the point P; i.e. such that $P = [\wp_\Lambda(z), \wp'_\Lambda(z)]$ satisfies the equation

$$y^2 = 4x^3 - g_2 x - g_3,$$

where g_2, g_3 are the elliptic invariants.

If E is defined over $\mathbf{R}$ and $P \in E(\mathbf{R})$, we have more precisely, $0 \leq \Re(t) < w1$ and $0 \leq \Im(t) < \Im(w2)$, where $(w1, w2)$ are the real and complex periods of E.

```
? E = ellinit([0,1]); P = [2,3];
? z = ellpointtoz(E, P)
%2 = 3.5054552633136356529375476976257353387
? ellwp(E, z)
%3 = 2.0000000000000000000000000000000000000
? ellztopoint(E, z) - P
%4 = [6.372367644529809109 E-58, 7.646841173435770930 E-57]
? ellpointtoz(E, [0]) \\ the point at infinity
%5 = 0
```

If $E/\mathbf{Q}_p$ has multiplicative reduction, then $E/\bar{\mathbf{Q}}_p$ is analytically isomorphic to $\bar{\mathbf{Q}}_p^*/q^{\mathbf{Z}}$ (Tate curve) for some p-adic integer q. The behaviour is then as follows:

• If the reduction is split (E.`tate`[2] is a `t_PADIC`), we have an isomorphism $\phi : E(\mathbf{Q}_p) \simeq \mathbf{Q}_p^*/q^{\mathbf{Z}}$ and the function returns $\phi(P) \in \mathbf{Q}_p$.

• If the reduction is *not* split (E.`tate`[2] is a `t_POLMOD`), we only have an isomorphism $\phi : E(K) \simeq K^*/q^{\mathbf{Z}}$ over the unramified quadratic extension $K/\mathbf{Q}_p$. In this case, the output $\phi(P) \in K$ is a `t_POLMOD`.

```
? E = ellinit([0,-1,1,0,0], O(11^5)); P = [0,0];
? [u2,u,q] = E.tate; type(u) \\ split multiplicative reduction
%2 = "t_PADIC"
? ellmul(E, P, 5)  \\ P has order 5
%3 = [0]
? z = ellpointtoz(E, [0,0])
%4 = 3 + 11^2 + 2*11^3 + 3*11^4 + O(11^5)
? z^5
%5 = 1 + O(11^5)
? E = ellinit(ellfromj(1/4), O(2^6)); x=1/2; y=ellordinate(E,x)[1];
? z = ellpointtoz(E,[x,y]); \\ t_POLMOD of t_POL with t_PADIC coeffs
? liftint(z) \\ lift all p-adics
%8 = Mod(8*u + 7, u^2 + 437)
```

The library syntax is `GEN zell(GEN E, GEN P, long prec)`.

3.5.60 ellpow(E, z, n). Deprecated alias for `ellmul`.

The library syntax is `GEN ellmul(GEN E, GEN z, GEN n)`.

3.5.61 ellrootno$(E, \{p\})$. E being an `ell` structure over $\mathbf{Q}$ as output by `ellinit`, this function computes the local root number of its L-series at the place p (at the infinite place if $p = 0$). If p is omitted, return the global root number. Note that the global root number is the sign of the functional equation and conjecturally is the parity of the rank of the Mordell-Weil group. The equation for E needs not be minimal at p, but if the model is already minimal the function will run faster.

The library syntax is `long ellrootno(GEN E, GEN p = NULL)`.

3.5.62 ellsea$(E, \{tors = 0\})$. Let E be an *ell* structure as output by `ellinit`, defined over a finite field $\mathbf{F}_q$. This low-level function computes the order of the group $E(\mathbf{F}_q)$ using the SEA algorithm; compared to the high-level function `ellcard`, which includes SEA among its choice of algorithms, the `tors` argument allows to speed up a search for curves having almost prime order. When `tors` is set to a non-zero value, the function returns 0 as soon as it detects that the order has a small prime factor not dividing `tors`; SEA considers modular polynomials of increasing prime degree ℓ and we return 0 as soon as we hit an ℓ (coprime to `tors`) dividing $\#E(\mathbf{F}_q)$:

```
? ellsea(ellinit([1,1], 2^56+3477), 1)
%1 = 72057594135613381
? forprime(p=2^128,oo, q = ellcard(ellinit([1,1],p)); if(isprime(q),break))
time = 6,571 ms.
? forprime(p=2^128,oo, q = ellsea(ellinit([1,1],p),1);if(isprime(q),break))
time = 522 ms.
```

In particular, set `tors` to 1 if you want a curve with prime order, to 2 if you want to allow a cofactor which is a power of two (e.g. for Edwards's curves), etc. The early exit on bad curves yields a massive speedup compared to running the cardinal algorithm to completion.

The following function returns a curve of prime order over $\mathbf{F}_p$.

```
cryptocurve(p) =
{
  while(1,
    my(E, N, j = Mod(random(p), p));
    E = ellinit(ellfromj(j));
    N = ellsea(E, 1); if(!N, continue);
    if (isprime(N), return(E));
    \\ try the quadratic twist for free
    if (isprime(2*p+2 - N), return(ellinit(elltwist(E))));
  );
}
? p = randomprime([2^255, 2^256]);
? E = cryptocurve(p); \\ insist on prime order
%2 = 47,447ms
```

The same example without early abort (using `ellsea(E,1)` instead of `ellsea(E)`) runs for about 5 minutes before finding a suitable curve.

The availability of the `seadata` package will speed up the computation, and is strongly recommended. The generic function `ellcard` should be preferred when you only want to compute the cardinal of a given curve without caring about it having almost prime order:

• If the characteristic is too small ($p \leq 7$) or the field cardinality is tiny ($q \leq 523$) the generic algorithm `ellcard` is used instead and the `tors` argument is ignored. (The reason for this is that SEA is not implemented for $p \leq 7$ and that if $q \leq 523$ it is likely to run into an infinite loop.)

• If the field cardinality is smaller than about 2^{50}, the generic algorithm will be faster.

• Contrary to `ellcard`, `ellsea` does not store the computed cardinality in E.

The library syntax is `GEN ellsea(GEN E, ulong tors)`.

3.5.63 ellsearch(N). This function finds all curves in the `elldata` database satisfying the constraint defined by the argument N:

• if N is a character string, it selects a given curve, e.g. `"11a1"`, or curves in the given isogeny class, e.g. `"11a"`, or curves with given conductor, e.g. `"11"`;

• if N is a vector of integers, it encodes the same constraints as the character string above, according to the `ellconvertname` correspondance, e.g. `[11,0,1]` for `"11a1"`, `[11,0]` for `"11a"` and `[11]` for `"11"`;

• if N is an integer, curves with conductor N are selected.

If N codes a full curve name, for instance `"11a1"` or `[11,0,1]`, the output format is $[N, [a_1, a_2, a_3, a_4, a_6], G]$ where $[a_1, a_2, a_3, a_4, a_6]$ are the coefficients of the Weierstrass equation of the curve and G is a $\mathbf{Z}$-basis of the free part of the Mordell-Weil group attached to the curve.

```
? ellsearch("11a3")
%1 = ["11a3", [0, -1, 1, 0, 0], []]
? ellsearch([11,0,3])
%2 = ["11a3", [0, -1, 1, 0, 0], []]
```

If N is not a full curve name, then the output is a vector of all matching curves in the above format:

```
? ellsearch("11a")
%1 = [["11a1", [0, -1, 1, -10, -20], []],
      ["11a2", [0, -1, 1, -7820, -263580], []],
      ["11a3", [0, -1, 1, 0, 0], []]]
? ellsearch("11b")
%2 = []
```

The library syntax is `GEN ellsearch(GEN N)`. Also available is `GEN ellsearchcurve(GEN N)` that only accepts complete curve names (as `t_STR`).

3.5.64 ellsigma$(L, \{z =' x\}, \{flag = 0\})$. Computes the value at z of the Weierstrass σ function attached to the lattice L as given by `ellperiods`$(,1)$: including quasi-periods is useful, otherwise there are recomputed from scratch for each new z.

$$\sigma(z, L) = z \prod_{\omega \in L^*} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}.$$

It is also possible to directly input $L = [\omega_1, \omega_2]$, or an elliptic curve E as given by `ellinit` ($L =$ `E.omega`).

```
? w = ellperiods([1,I], 1);
? ellsigma(w, 1/2)
%2 = 0.47494937998792065033250463632798296855
? E = ellinit([1,0]);
? ellsigma(E) \\ at 'x, implicitly at default seriesprecision
%4 = x + 1/60*x^5 - 1/10080*x^9 - 23/259459200*x^13 + O(x^17)
```

If *flag* $= 1$, computes an arbitrary determination of $\log(\sigma(z))$.

The library syntax is `GEN ellsigma(GEN L, GEN z = NULL, long flag, long prec)`.

3.5.65 ellsub$(E, z1, z2)$. Difference of the points $z1$ and $z2$ on the elliptic curve corresponding to E.

The library syntax is `GEN ellsub(GEN E, GEN z1, GEN z2)`.

3.5.66 elltaniyama$(E, \{d = seriesprecision\})$. Computes the modular parametrization of the elliptic curve $E/\mathbf{Q}$, where E is an `ell` structure as output by `ellinit`. This returns a two-component vector $[u, v]$ of power series, given to d significant terms (`seriesprecision` by default), characterized by the following two properties. First the point (u, v) satisfies the equation of the elliptic curve. Second, let N be the conductor of E and $\Phi : X_0(N) \to E$ be a modular parametrization; the pullback by Φ of the Néron differential $du/(2v + a_1u + a_3)$ is equal to $2i\pi f(z)dz$, a holomorphic differential form. The variable used in the power series for u and v is x, which is implicitly understood to be equal to $\exp(2i\pi z)$.

The algorithm assumes that E is a *strong* Weil curve and that the Manin constant is equal to 1: in fact, $f(x) = \sum_{n>0}$ `ellan`$(E, n)x^n$.

The library syntax is `GEN elltaniyama(GEN E, long precdl)`.

3.5.67 elltatepairing(E, P, Q, m). Computes the Tate pairing of the two points P and Q on the elliptic curve E. The point P must be of m-torsion.

The library syntax is `GEN elltatepairing(GEN E, GEN P, GEN Q, GEN m)`.

3.5.68 elltors(E). If E is an elliptic curve defined over a number field or a finite field, outputs the torsion subgroup of E as a 3-component vector `[t,v1,v2]`, where `t` is the order of the torsion group, `v1` gives the structure of the torsion group as a product of cyclic groups (sorted by decreasing order), and `v2` gives generators for these cyclic groups. E must be an `ell` structure as output by `ellinit`.

```
?  E = ellinit([-1,0]);
?  elltors(E)
%1 = [4, [2, 2], [[0, 0], [1, 0]]]
```

Here, the torsion subgroup is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, with generators $[0, 0]$ and $[1, 0]$.

The library syntax is `GEN elltors(GEN E)`.

3.5.69 elltwist$(E, \{P\})$. Returns the coefficients $[a_1, a_2, a_3, a_4, a_6]$ of the twist of the elliptic curve E by the quadratic extension of the coefficient ring defined by P (when P is a polynomial) or `quadpoly(P)` when P is an integer. If E is defined over a finite field, then P can be omitted, in which case a random model of the unique non-trivial twist is returned. If E is defined over a number field, the model should be replaced by a minimal model (if one exists).

Example: Twist by discriminant -3:

```
? elltwist(ellinit([0,a2,0,a4,a6]),-3)
%1 = [0,-3*a2,0,9*a4,-27*a6]
```

Twist by the Artin-Shreier extension given by $x^2 + x + T$ in characteristic 2:

```
? lift(elltwist(ellinit([a1,a2,a3,a4,a6]*Mod(1,2)),x^2+x+T))
%1 = [a1,a2+a1^2*T,a3,a4,a6+a3^2*T]
```

Twist of an elliptic curve defined over a finite field:

```
? E=ellinit([1,7]*Mod(1,19));lift(elltwist(E))
%1 = [0,0,0,11,12]
```

The library syntax is `GEN elltwist(GEN E, GEN P = NULL)`.

3.5.70 ellweilpairing(E, P, Q, m). Computes the Weil pairing of the two points of m-torsion P and Q on the elliptic curve E.

The library syntax is `GEN ellweilpairing(GEN E, GEN P, GEN Q, GEN m)`.

3.5.71 ellwp$(w, \{z =' x\}, \{flag = 0\})$. Computes the value at z of the Weierstrass $\wp$ function attached to the lattice w as given by `ellperiods`. It is also possible to directly input $w = [\omega_1, \omega_2]$, or an elliptic curve E as given by `ellinit` ($w =$ `E.omega`).

```
? w = ellperiods([1,I]);
? ellwp(w, 1/2)
%2 = 6.875185818020372827490095779810557197 8
? E = ellinit([1,1]);
? ellwp(E, 1/2)
%4 = 3.941311242701647464604828246270915138 9
```

One can also compute the series expansion around $z = 0$:

```
? E = ellinit([1,0]);
? ellwp(E)              \\ 'x implicitly at default seriesprecision
%5 = x^-2 - 1/5*x^2 + 1/75*x^6 - 2/4875*x^10 + O(x^14)
? ellwp(E, x + O(x^12)) \\ explicit precision
%6 = x^-2 - 1/5*x^2 + 1/75*x^6 + O(x^9)
```

Optional *flag* means 0 (default): compute only $\wp(z)$, 1: compute $[\wp(z), \wp'(z)]$.

The library syntax is `GEN ellwp0(GEN w, GEN z = NULL, long flag, long prec)`. For *flag* $= 0$, we also have `GEN ellwp(GEN w, GEN z, long prec)`, and `GEN ellwpseries(GEN E, long v, long precdl)` for the power series in variable v.

3.5.72 ellxn$(E, n, \{v =' x\})$. In standard notation, for any affine point $P = (v, w)$ on the curve E, we have

$$[n]P = (\phi_n(P)\psi_n(P) : \omega_n(P) : \psi_n(P)^3)$$

for some polynomials ϕ_n, ω_n, ψ_n in $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6][v, w]$. This function returns $[\phi_n(P), \psi_n(P)^2]$, which give the numerator and denominator of the abcissa of $[n]P$ and depend only on v.

The library syntax is `GEN ellxn(GEN E, long n, long v = -1)` where `v` is a variable number.

3.5.73 ellzeta$(w, \{z =' x\})$. Computes the value at z of the Weierstrass ζ function attached to the lattice w as given by `ellperiods`$(, 1)$: including quasi-periods is useful, otherwise there are recomputed from scratch for each new z.

$$\zeta(z, L) = \frac{1}{z} + z^2 \sum_{\omega \in L^*} \frac{1}{\omega^2(z - \omega)}.$$

It is also possible to directly input $w = [\omega_1, \omega_2]$, or an elliptic curve E as given by `ellinit` ($w =$ `E.omega`). The quasi-periods of ζ, such that

$$\zeta(z + a\omega_1 + b\omega_2) = \zeta(z) + a\eta_1 + b\eta_2$$

for integers a and b are obtained as $\eta_i = 2\zeta(\omega_i/2)$. Or using directly `elleta`.

```
? w = ellperiods([1,I],1);
? ellzeta(w, 1/2)
%2 = 1.5707963267948966192313216916397514421
? E = ellinit([1,0]);
? ellzeta(E, E.omega[1]/2)
%4 = 0.84721308479397908660649912348219163647
```

One can also compute the series expansion around $z = 0$ (the quasi-periods are useless in this case):

```
? E = ellinit([0,1]);
? ellzeta(E) \\ at 'x, implicitly at default seriesprecision
%4 = x^-1 + 1/35*x^5 - 1/7007*x^11 + O(x^15)
? ellzeta(E, x + O(x^20)) \\ explicit precision
%5 = x^-1 + 1/35*x^5 - 1/7007*x^11 + 1/1440257*x^17 + O(x^18)
```

The library syntax is `GEN ellzeta(GEN w, GEN z = NULL, long prec)`.

3.5.74 ellztopoint(E, z). E being an *ell* as output by `ellinit`, computes the coordinates $[x, y]$ on the curve E corresponding to the complex number z. Hence this is the inverse function of `ellpointtoz`. In other words, if the curve is put in Weierstrass form $y^2 = 4x^3 - g_2x - g_3$, $[x, y]$ represents the Weierstrass $\wp$-function and its derivative. More precisely, we have

$$x = \wp(z) - b_2/12, \quad y = \wp'(z) - (a_1x + a_3)/2.$$

If z is in the lattice defining E over $\mathbf{C}$, the result is the point at infinity [0].

The library syntax is `GEN pointell(GEN E, GEN z, long prec)`.

3.5.75 genus2red$(PQ, \{p\})$. Let PQ be a polynomial P, resp. a vector $[P, Q]$ of polynomials, with rational coefficients. Determines the reduction at $p > 2$ of the (proper, smooth) genus 2 curve $C/\mathbf{Q}$, defined by the hyperelliptic equation $y^2 = P(x)$, resp. $y^2 + Q(x) * y = P(x)$. (The special fiber X_p of the minimal regular model X of C over $\mathbf{Z}$.)

If p is omitted, determines the reduction type for all (odd) prime divisors of the discriminant.

This function was rewritten from an implementation of Liu's algorithm by Cohen and Liu (1994), `genus2reduction-0.3`, see `http://www.math.u-bordeaux.fr/~liu/G2R/`.

CAVEAT. The function interface may change: for the time being, it returns $[N, FaN, T, V]$ where N is either the local conductor at p or the global conductor, FaN is its factorization, $y^2 = T$ defines a minimal model over $\mathbf{Z}[1/2]$ and V describes the reduction type at the various considered p. Unfortunately, the program is not complete for $p = 2$, and we may return the odd part of the conductor only: this is the case if the factorization includes the (impossible) term 2^{-1}; if the factorization contains another power of 2, then this is the exact local conductor at 2 and N is the global conductor.

```
? default(debuglevel, 1);
? genus2red(x^6 + 3*x^3 + 63, 3)
(potential) stable reduction: [1, []]
reduction at p: [III{9}] page 184, [3, 3], f = 10
%1 = [59049, Mat([3, 10]), x^6 + 3*x^3 + 63, [3, [1, []],
       ["[III{9}] page 184", [3, 3]]]]
? [N, FaN, T, V] = genus2red(x^3-x^2-1, x^2-x);  \\ X_1(13), global reduction
p = 13
(potential) stable reduction: [5, [Mod(0, 13), Mod(0, 13)]]
reduction at p: [I{0}-II-0] page 159, [], f = 2
? N
%3 = 169
? FaN
%4 = Mat([13, 2])   \\ in particular, good reduction at 2 !
? T
%5 = x^6 + 58*x^5 + 1401*x^4 + 18038*x^3 + 130546*x^2 + 503516*x + 808561
? V
%6 = [[13, [5, [Mod(0, 13), Mod(0, 13)]], ["[I{0}-II-0] page 159", []]]]
```

We now first describe the format of the vector $V = V_p$ in the case where p was specified (local reduction at p): it is a triple $[p, stable, red]$. The component $stable = [type, vecj]$ contains information about the stable reduction after a field extension; depending on *type*s, the stable reduction is

• 1: smooth (i.e. the curve has potentially good reduction). The Jacobian $J(C)$ has potentially good reduction.

• 2: an elliptic curve E with an ordinary double point; *vecj* contains j mod p, the modular invariant of E. The (potential) semi-abelian reduction of $J(C)$ is the extension of an elliptic curve (with modular invariant j mod p) by a torus.

• 3: a projective line with two ordinary double points. The Jacobian $J(C)$ has potentially multiplicative reduction.

• 4: the union of two projective lines crossing transversally at three points. The Jacobian $J(C)$ has potentially multiplicative reduction.

• 5: the union of two elliptic curves E_1 and E_2 intersecting transversally at one point; *vecj* contains their modular invariants j_1 and j_2, which may live in a quadratic extension of $\mathbf{F}_p$ and need not be distinct. The Jacobian $J(C)$ has potentially good reduction, isomorphic to the product of the reductions of E_1 and E_2.

• 6: the union of an elliptic curve E and a projective line which has an ordinary double point, and these two components intersect transversally at one point; *vecj* contains j mod p, the modular invariant of E. The (potential) semi-abelian reduction of $J(C)$ is the extension of an elliptic curve (with modular invariant j mod p) by a torus.

• 7: as in type 6, but the two components are both singular. The Jacobian $J(C)$ has potentially multiplicative reduction.

The component *red* = [*NUtype*, *neron*] contains two data concerning the reduction at p without any ramified field extension.

The *NUtype* is a `t_STR` describing the reduction at p of C, following Namikawa-Ueno, *The complete classification of fibers in pencils of curves of genus two*, Manuscripta Math., vol. 9, (1973), pages 143-186. The reduction symbol is followed by the corresponding page number or page range in this article.

The second datum *neron* is the group of connected components (over an algebraic closure of $\mathbf{F}_p$) of the Néron model of $J(C)$, given as a finite abelian group (vector of elementary divisors).

If $p = 2$, the *red* component may be omitted altogether (and replaced by `[]`, in the case where the program could not compute it. When p was not specified, V is the vector of all V_p, for all considered p.

Notes about Namikawa-Ueno types.

• A lower index is denoted between braces: for instance, `[I{2}-II-5]` means `[I_2-II-5]`.

• If K and K' are Kodaira symbols for singular fibers of elliptic curves, then `[`K`-`K'`-m]` and `[`K'`-`K`-m]` are the same.

We define a total ordering on Kodaira symbol by fixing `I` $<$ `I*` $<$ `II` $<$ `II*`, ... If the reduction type is the same, we order by the number of components, e.g. $\mathtt{I}_2 < \mathtt{I}_4$, etc. Then we normalize our output so that $K \leq K'$.

• `[`K`-`K'`--1]` is `[`K`-`K'`-`α`]` in the notation of Namikawa-Ueno.

• The figure `[2I_0-m]` in Namikawa-Ueno, page 159, must be denoted by `[2I_0-(m+1)]`.

The library syntax is `GEN genus2red(GEN PQ, GEN p = NULL)`.

3.5.76 hyperellcharpoly(X). X being a non-singular hyperelliptic curve defined over a finite field, return the characteristic polynomial of the Frobenius automorphism. X can be given either by a squarefree polynomial P such that $X : y^2 = P(x)$ or by a vector $[P, Q]$ such that $X : y^2 + Q(x) \times y = P(x)$ and $Q^2 + 4P$ is squarefree.

The library syntax is `GEN hyperellcharpoly(GEN X)`.

3.5.77 hyperellpadicfrobenius(Q,p,n). Let X be the curve defined by $y^2 = Q(x)$, where Q is a polynomial of degree d over $\mathbf{Q}$ and $p \geq d$ a prime such that X has good reduction at p return the matrix of the Frobenius endomorphism φ on the crystalline module $D_p(X) = \mathbf{Q}_p \otimes H^1_{dR}(X/\mathbf{Q})$ with respect to the basis of the given model $(\omega, x\omega, \ldots, x^{g-1}\omega)$, where $\omega = dx/(2y)$ is the invariant differential, where g is the genus of X (either $d = 2g+1$ or $d = 2g+2$). The characteristic polynomial of φ is the numerator of the zeta-function of the reduction of the curve X modulo p. The matrix is computed to absolute p-adic precision p^n.

The library syntax is `GEN hyperellpadicfrobenius(GEN Q, ulong p, long n)`.

3.6 L-functions.

This section describes routines related to L-functions. We first introduce the basic concept and notations, then explain how to represent them in GP. Let $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, where Γ is Euler's gamma function. Given $d \geq 1$ and a d-tuple $A = [\alpha_1, \ldots, \alpha_d]$ of complex numbers, we let $\gamma_A(s) = \prod_{\alpha \in A} \Gamma_{\mathbf{R}}(s+\alpha)$.

Given a sequence $a = (a_n)_{n \geq 1}$ of complex numbers (such that $a_1 = 1$), a positive *conductor* $N \in \mathbf{Z}$, and a *gamma factor* γ_A as above, we consider the Dirichlet series

$$L(a,s) = \sum_{n \geq 1} a_n n^{-s}$$

and the attached completed function

$$\Lambda(a,s) = N^{s/2}\gamma_A(s) \cdot L(a,s).$$

Such a datum defines an *L-function* if it satisfies the three following assumptions:

• [Convergence] The $a_n = O_\epsilon(n^{k_1+\epsilon})$ have polynomial growth, equivalently $L(s)$ converges absolutely in some right half-plane $\Re(s) > k_1 + 1$.

• [Analytic continuation] $L(s)$ has a meromorphic continuation to the whole complex plane with finitely many poles.

• [Functional equation] There exist an integer k, a complex number ϵ (usually of modulus 1), and an attached sequence a^* defining both an L-function $L(a^*,s)$ satisfying the above two assumptions and a completed function $\Lambda(a^*,s) = N^{s/2}\gamma_A(s) \cdot L(a^*,s)$, such that

$$\Lambda(a, k-s) = \epsilon\Lambda(a^*, s)$$

for all regular points.

More often than not in number theory we have $a^* = \overline{a}$ (which forces $|\epsilon| = 1$), but this needs not be the case. If a is a real sequence and $a = a^*$, we say that L is *self-dual*. We do not assume that the a_n are multiplicative, nor equivalently that $L(s)$ has an Euler product.

Remark. Of course, a determines the L-function, but the (redundant) datum $a, a^*, A, N, k, \epsilon$ describes the situation in a form more suitable for fast computations; knowing the polar part r of $\Lambda(s)$ (a rational function such that $\Lambda - r$ is holomorphic) is also useful. A subset of these, including only finitely many a_n-values will still completely determine L (in suitable families), and we provide routines to try and compute missing invariants from whatever information is available.

Important Caveat. We currently assume that we can take the growth exponent $k_1 = (k-1)/2$ if L is entire and $k_1 = k-1$ otherwise, and that the implied constants in the O_ϵ are small. This may be changed and made user-configurable in future versions but the essential point remains that it is impossible to return proven results in such a generic framework, without more detailed information about the L function. The intended use of the L-function package is not to prove theorems, but to experiment and formulate conjectures, so all numerical results should be taken with a grain of salt. One can always increase `realbitprecision` and recompute: the difference estimates the actual absolute error in the original output.

Note. The requested precision has a major impact on runtimes. Because of this, most L-function routines, in particular `lfun` itself, specify the requested precision in *bits*, not in decimal digits. This is transparent for the user once `realprecision` or `realbitprecision` are set. We advise to manipulate precision via `realbitprecision` as it allows finer granularity: `realprecision` increases by increments of 64 bits, i.e. 19 decimal digits at a time.

3.6.1 Theta functions.

Given an L-function as above, we define an attached theta function via Mellin inversion: for any positive real $t > 0$, we let

$$\theta(a,t) := \frac{1}{2\pi i}\int_{\Re(s)=c} t^{-s}\Lambda(s)\,ds$$

where c is any positive real number $c > k_1 + 1$ such that $c + \Re(a) > 0$ for all $a \in A$. In fact, we have

$$\theta(a,t) = \sum_{n\geq 1} a_n K(nt/N^{1/2}) \quad \text{where} \quad K(t) := \frac{1}{2\pi i}\int_{\Re(s)=c} t^{-s}\gamma_A(s)\,ds.$$

Note that this function is analytic and actually makes sense for complex t, such that $\Re(t^{2/d}) > 0$, i.e. in a cone containing the positive real half-line. The functional equation for Λ translates into

$$\theta(a,1/t) - \epsilon t^k \theta(a^*,t) = P_\Lambda(t),$$

where P_Λ is an explicit polynomial in t and $\log t$ given by the Taylor development of the polar part of Λ: there are no log's if all poles are simple, and $P = 0$ if Λ is entire. The values $\theta(t)$ are generally easier to compute than the $L(s)$, and this functional equation provides a fast way to guess possible values for missing invariants in the L-function definition.

3.6.2 Data structures describing L and theta functions.

We have 3 levels of description:

• an `Lmath` is an arbitrary description of the underlying mathematical situation (to which e.g., we associate the a_p as traces of Frobenius elements); this is done via constructors to be described in the subsections below.

• an `Ldata` is a computational description of situation, containing the complete datum $(a, a^*, A, k, N, \epsilon, r)$. Where a and a^* describe the coefficients (given n, b we must be able to compute $[a_1, \ldots, a_n]$ with bit accuracy b), A describes the Euler factor, the (classical) weight is k, N is the conductor, and r describes the polar part of $L(s)$. This is obtained via the function `lfuncreate`. N.B. For motivic L-functions, the motivic weight w is $w = k-1$; but we also support non-motivic L-functions.

Design problem. All components of an `Ldata` should be given exactly since the accuracy to which they must be computed is not bounded a priori; but this is not always possible, in particular for ϵ and r.

• an `Linit` contains an `Ldata` and everything needed for fast *numerical* computations. It specifies the functions to be considered (either $L^{(j)}(s)$ or $\theta^{(j)}(t)$ for derivatives of order $j \leq m$, where m is now fixed) and specifies a *domain* which limits the range of arguments (t or s, respectively to certain cones and rectangular regions) and the output accuracy. This is obtained via the functions `lfuninit` or `lfunthetainit`.

All the functions which are specific to L or theta functions share the prefix `lfun`. They take as first argument either an `Lmath`, an `Ldata`, or an `Linit`. If a single value is to be computed, this makes no difference, but when many values are needed (e.g. for plots or when searching for zeros), one should first construct an `Linit` attached to the search range and use it in all subsequent calls. If you attempt to use an `Linit` outside the range for which it was initialized, a warning is issued, because the initialization is performed again, a major inefficiency:

```
? Z = lfuncreate(1); \\ Riemann zeta
? L = lfuninit(Z, [1/2, 0, 100]); \\ zeta(1/2+it), |t| < 100
? lfun(L, 1/2)    \\ OK, within domain
%3 = -1.460354508809586812889499152515298012 5
? lfun(L, 0)      \\ not on critical strip !
  *** lfun: Warning: lfuninit: insufficient initialization.
%4 = -0.500000000000000000000000000000000000000
? lfun(L, 1/2, 1) \\ attempt first derivative !
*** lfun: Warning: lfuninit: insufficient initialization.
%5 = -3.922646139209151727471531446714599513 7
```

For many L-functions, passing from `Lmath` to an `Ldata` is inexpensive: in that case one may use `lfuninit` directly from the `Lmath` even when evaluations in different domains are needed. The above example could equally have skipped the `lfuncreate`:

```
? L = lfuninit(1, [1/2, 0, 100]); \\ zeta(1/2+it), |t| < 100
```

In fact, when computing a single value, you can even skip `lfuninit`:

```
? L = lfun(1, 1/2, 1); \\ zeta'(1/2)
? L = lfun(1, 1+x+0(x^5)); \\ first 5 terms of Taylor development at 1
```

Both give the desired results with no warning.

Complexity. The implementation requires $O(N(|t|+1))^{1/2}$ coefficients a_n to evaluate L of conductor N at $s = \sigma + it$.

We now describe the available high-level constructors, for built-in L functions.

3.6.3 Dirichlet L-functions.

Given a Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}$, we let

$$L(\chi, s) = \sum_{n \geq 1} \chi(n) n^{-s}.$$

Only primitive characters are supported. Given a fundamental discriminant D, the function $L((D/.), s)$, for the quadratic Kronecker symbol, is encoded by the `t_INT` D. This includes Riemann ζ function via the special case $D = 1$.

More general characters can be represented in a variety of ways:

• via Conrey notation (see `znconreychar`): $\chi_N(m, \cdot)$ is given as the `t_INTMOD Mod(m,N)`.

• via a *bid* structure describing the abelian group $(\mathbf{Z}/N\mathbf{Z})^*$, where the character is given in terms of the *bid* generators:

```
? bid = idealstar(,100,2); \\ (Z/100Z)^*
? bid.cyc \\ ~ Z/20 . g1  + Z/2 . g2 for some generators g1 and g2
%2 = [20, 2]
? bid.gen
%3 = [77, 51]
? chi = [a, b]  \\ maps g1 to e(a/20) and g2 to e(b/2);  e(x) = exp(2ipi x)
```

More generally, let $(\mathbf{Z}/N\mathbf{Z})^* = \oplus(\mathbf{Z}/d_i\mathbf{Z})g_i$ be given via a *bid* structure G (`G.cyc` gives the d_i and `G.gen` the g_i). A *character* χ on G is given by a row vector $v = [a_1, \ldots, a_n]$ such that $\chi(\prod g_i^{n_i}) = \exp(2\pi i \sum a_i n_i / d_i)$. The pair $[bid, v]$ encodes the *primitive* character attached to χ.

• in fact, this construction $[bid, m]$ describing a character is more general: m is also allowed to be a Conrey index as seen above, or a Conrey logarithm (see `znconreylog`), and the latter format is actually the fastest one.

• it is also possible to view Dirichlet characters as Hecke characters over $K = \mathbf{Q}$ (see below), for a modulus $[N, [1]]$ but this is both more complicated and less efficient.

3.6.4 Hecke L-functions.

The Dedekind zeta function of a number field $K = \mathbf{Q}[X]/(T)$ is encoded either by the defining polynomial T, or any absolute number fields structure (preferably at least a *bnf*).

Given a finite order Hecke character $\chi : Cl_f(K) \to \mathbf{C}$, we let

$$L(\chi, s) = \sum_{A \subset O_K} \chi(A) \left(N_{K/\mathbf{Q}} A\right)^{-s}.$$

Let $Cl_f(K) = \oplus(\mathbf{Z}/d_i\mathbf{Z})g_i$ given by a *bnr* structure with generators: the d_i are given by `K.cyc` and the g_i by `K.gen`. A *character* χ on the ray class group is given by a row vector $v = [a_1, \ldots, a_n]$ such that $\chi(\prod g_i^{n_i}) = \exp(2\pi i \sum a_i n_i / d_i)$. The pair $[bnr, v]$ encodes the *primitive* character attached to χ.

```
? K  = bnfinit(x^2-60);
? Cf = bnrinit(K, [7, [1,1]], 1); \\ f = 7 oo_1 oo_2
? Cf.cyc
```

```
%3 = [6, 2, 2]
? Cf.gen
%4 = [[2, 1; 0, 1], [22, 9; 0, 1], [-6, 7]~]
? lfuncreate([Cf, [1,0,0]]); \\ χ(g1) = ζ6, χ(g2) = χ(g3) = 1
```

Dirichlet characters on $(\mathbf{Z}/N\mathbf{Z})^*$ are a special case, where $K = \mathbf{Q}$:

```
? Q  = bnfinit(x);
? Cf = bnrinit(Q, [100, [1]]); \\ for odd characters on (Z/100Z)*
```

For even characters, replace by `bnrinit(K, N)`. Note that the simpler direct construction in the previous section will be more efficient.

3.6.5 Artin L functions.

Given a Galois number field $N/\mathbf{Q}$ with group $G =$ `galoisinit`(N), a representation ρ of G over the cyclotomic field $\mathbf{Q}(\zeta_n)$ is specified by the matrices giving the images of `G.gen` by ρ. The corresponding Artin L function is created using `lfunartin`.

```
P = quadhilbert(-47); \\  degree 5, Galois group D_5
N = nfinit(nfsplitting(P)); \\ Galois closure
G = galoisinit(N);
[s,t] = G.gen; \\ order 5 and 2
L = lfunartin(N,G, [[a,0;0,a^-1],[0,1;1,0]], 5); \\ irr. degree 2
```

In the above, the polynomial variable (here `a`) represents $\zeta_5 := \exp(2i\pi/5)$ and the two matrices give the images of s and t. Here, priority of `a` must be lower than the priority of `x`.

3.6.6 L-functions of algebraic varieties.

L-function of elliptic curves over number fields are supported.

```
? E = ellinit([1,1]);
? L = lfuncreate(E);  \\ L-function of E/Q
? E2 = ellinit([1,a], nfinit(a^2-2));
? L2 = lfuncreate(E2);  \\ L-function of E/Q(sqrt(2))
```

L-function of hyperelliptic genus-2 curve can be created with `lfungenus2`. To create the L function of the curve $y^2 + (x^3 + x^2 + 1)y = x^2 + x$:

```
? L = lfungenus2([x^2+x, x^3+x^2+1]);
```

Currently, the model needs to be minimal at 2, and if the conductor is even, its valuation at 2 might be incorrect (a warning is issued).

3.6.7 Eta quotients / Modular forms.

An eta quotient is created by applying `lfunetaquo` to a matrix with 2 columns $[m, r_m]$ representing

$$f(\tau) := \prod_m \eta(m\tau)^{r_m}.$$

It is currently assumed that f is a self-dual cuspidal form on $\Gamma_0(N)$ for some N. For instance, the L-function $\sum \tau(n)n^{-s}$ attached to Ramanujan's Δ function is encoded as follows

```
? L = lfunetaquo(Mat([1,24]));
? lfunan(L, 100)  \\ first 100 values of tau(n)
```

More general modular forms defined by modular symbols will be added later.

3.6.8 Low-level Ldata format.

When no direct constructor is available, you can still input an L function directly by supplying $[a, a^*, A, k, N, \epsilon, r]$ to `lfuncreate` (see `??lfuncreate` for details).

It is *strongly* suggested to first check consistency of the created L-function:

```
? L = lfuncreate([a, as, A, k, N, eps, r]);
? lfuncheckfeq(L)  \\ check functional equation
```

3.6.9 lfun$(L, s, \{D = 0\})$. Compute the L-function value $L(s)$, or if `D` is set, the derivative of order `D` at s. The parameter `L` is either an Lmath, an Ldata (created by `lfuncreate`, or an Linit (created by `lfuninit`), preferrably the latter if many values are to be computed.

The argument s is also allowed to be a power series; for instance, if $s = \alpha + x + O(x^n)$, the function returns the Taylor expansion of order n around α. The result is given with absolute error less than 2^{-B}, where $B =$ realbitprecision.

Caveat. The requested precision has a major impact on runtimes. It is advised to manipulate precision via `realbitprecision` as explained above instead of `realprecision` as the latter allows less granularity: `realprecision` increases by increments of 64 bits, i.e. 19 decimal digits at a time.

```
? lfun(x^2+1, 2)  \\ Lmath: Dedekind zeta for Q(i) at 2
%1 = 1.506703009922985030886565048182071396O
```

```
? L = lfuncreate(ellinit("5077a1")); \\ Ldata: Hasse-Weil zeta function
? lfun(L, 1+x+O(x^4))  \\ zero of order 3 at the central point
%3 = 0.E-58 - 5.[...] E-40*x + 9.[...] E-40*x^2 + 1.7318[...]*x^3 + O(x^4)
```

```
\\ Linit: zeta(1/2+it), |t| < 100, and derivative
? L = lfuninit(1, [100], 1);
? T = lfunzeros(L, [1,25]);
%5 = [14.134725[...], 21.022039[...]]
? z = 1/2 + I*T[1];
? abs( lfun(L, z) )
%7 = 8.706686553341220742078039299112513619O E-39
? abs( lfun(L, z, 1) )
%8 = 0.793160433356506116013897565274352114l2  \\ simple zero
```

The library syntax is `GEN lfun0(GEN L, GEN s, long D, long bitprec)`.

3.6.10 lfunabelianrelinit$(bnfL, bnfK, polrel, sdom, \{der = 0\})$. Returns the `Linit` structure attached to the Dedekind zeta function of the number field L (see `lfuninit`), given a subfield K such that L/K is abelian. Here `polrel` defines L over K, as usual with the priority of the variable of `bnfK` lower than that of `polrel`. `sdom` and `der` are as in `lfuninit`.

```
? D = -47; K = bnfinit(y^2-D);
? rel = quadhilbert(D); T = rnfequation(K.pol, rel); \\ degree 10
? L = lfunabelianrelinit(T,K,rel, [2,0,0]); \\ at 2
time = 84 ms.
? lfun(L, 2)
%4 = 1.015421339440244392988066689446818265O
? lfun(T, 2) \\ using parisize > 300MB
time = 652 ms.
```

```
%5 = 1.0154213394402443929880666894468182656
```

As the example shows, using the (abelian) relative structure is more efficient than a direct computation. The difference becomes drastic as the absolute degree increases while the subfield degree remains constant.

The library syntax is `GEN lfunabelianrelinit(GEN bnfL, GEN bnfK, GEN polrel, GEN sdom, long der, long bitprec)`.

3.6.11 lfunan(L, n). Compute the first n terms of the Dirichlet series attached to the L-function given by `L` (`Lmath`, `Ldata` or `Linit`).

```
? lfunan(1, 10)  \\ Riemann zeta
%1 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
? lfunan(5, 10)  \\ Dirichlet L-function for kronecker(5,.)
%2 = [1, -1, -1, 1, 0, 1, -1, -1, 1, 0]
```

The library syntax is `GEN lfunan(GEN L, long n, long prec)`.

3.6.12 lfunartin(nf, gal, M, n). Returns the `Ldata` structure attached to the Artin L-function attached to the representation ρ of the Galois group of the extension $K/\mathbf{Q}$, defined over the cyclotomic field $\mathbf{Q}(\zeta_n)$, where nf is the nfinit structure attached to K, gal is the galoisinit structure attached to $K/\mathbf{Q}$, and M is the vector of the image of the generators gal`.gen` by ρ. The elements of M are matrices with polynomial entries, whose variable is understood as the complex number $\exp(2i\pi/n)$.

In the following example we build the Artin L-functions attached to the two irreducible degree 2 representations of the dihedral group D_{10} defined over $\mathbf{Q}(\zeta_5)$, for the extension $H/\mathbf{Q}$ where H is the Hilbert class field of $\mathbf{Q}(\sqrt{-47})$. We show numerically some identities involving Dedekind ζ functions and Hecke L series.

```
? P = quadhilbert(-47);
? N = nfinit(nfsplitting(P));
? G = galoisinit(N);
? L1 = lfunartin(N,G, [[a,0;0,a^-1],[0,1;1,0]], 5);
? L2 = lfunartin(N,G, [[a^2,0;0,a^-2],[0,1;1,0]], 5);
? s = 1 + x + O(x^4);
? lfun(1,s)*lfun(-47,s)*lfun(L1,s)^2*lfun(L2,s)^2 - lfun(N,s)
%6 ~ 0
? lfun(1,s)*lfun(L1,s)*lfun(L2,s) - lfun(P,s)
%7 ~ 0
? bnr = bnrinit(bnfinit(x^2+47),1,1);
? lfun([bnr,[1]], s) - lfun(L1, s)
%9 ~ 0
? lfun([bnr,[1]], s) - lfun(L1, s)
%10 ~ 0
```

The first identity is the factorisation of the regular representation of D_{10}, the second the factorisation of the natural representation of $D_{10} \subset S_5$, the next two are the expressions of the degree 2 representations as induced from degree 1 representations.

The library syntax is `GEN lfunartin(GEN nf, GEN gal, GEN M, long n)`.

3.6.13 lfuncheckfeq$(L, \{t\})$. Given the data attached to an L-function (`Lmath`, `Ldata` or `Linit`), check whether the functional equation is satisfied. This is most useful for an `Ldata` constructed "by hand", via `lfuncreate`, to detect mistakes.

If the function has poles, the polar part must be specified. The routine returns a bit accuracy b such that $|w - \hat{w}| < 2^b$, where w is the root number contained in `data`, and $\hat{w}$ is a computed value derived from $\overline{\theta}(t)$ and $\theta(1/t)$ at some $t \neq 0$ and the assumed functional equation. Of course, a large negative value of the order of `realbitprecision` is expected.

If t is given, it should be close to the unit disc for efficiency and such that $\overline{\theta}(t) \neq 0$. We then check the functional equation at that t.

```
? \pb 128       \\ 128 bits of accuracy
? default(realbitprecision)
%1 = 128
? L = lfuncreate(1);  \\ Riemann zeta
? lfuncheckfeq(L)
%3 = -124
```

i.e. the given data is consistent to within 4 bits for the particular check consisting of estimating the root number from all other given quantities. Checking away from the unit disc will either fail with a precision error, or give disappointing results (if $\theta(1/t)$ is large it will be computed with a large absolute error)

```
? lfuncheckfeq(L, 2+I)
%4 = -115
? lfuncheckfeq(L,10)
 ***   at top-level: lfuncheckfeq(L,10)
 ***                 ^------------------
 *** lfuncheckfeq: precision too low in lfuncheckfeq.
```

The library syntax is `long lfuncheckfeq(GEN L, GEN t = NULL, long bitprec)`.

3.6.14 lfunconductor$(L, \{ab = [1, 10000]\}, \{flag = 0\})$. Compute the conductor of the given L-function (if the structure contains a conductor, it is ignored); `ab` $= [a, b]$ is the interval where we expect to find the conductor; it may be given as a single scalar b, in which case we look in $[1, b]$. Increasing `ab` slows down the program but gives better accuracy for the result.

If `flag` is 0 (default), give either the conductor found as an integer, or a vector (possibly empty) of conductors found. If `flag` is 1, same but give the computed floating point approximations to the conductors found, without rounding to integers. It `flag` is 2, give all the conductors found, even those far from integers.

Caveat. This is a heuristic program and the result is not proven in any way:

```
? L = lfuncreate(857); \\ Dirichlet L function for kronecker(857,.)
? \p19
  realprecision = 19 significant digits
? lfunconductor(L)
%2 = [17, 857]
? lfunconductor(L,,1) \\ don't round
%3 = [16.99999999999999999, 857.0000000000000000]
? \p38
  realprecision = 38 significant digits
? lfunconductor(L)
%4 = 857
```

Note. This program should only be used if the primes dividing the conductor are unknown, which is rare. If they are known, a direct search through possible prime exponents using `lfuncheckfeq` will be more efficient and rigorous:

```
? E = ellinit([0,0,0,4,0]); /* Elliptic curve y^2 = x^3+4x */
? E.disc  \\ |disc E| = 2^12
%2 = -4096
\\ create Ldata by hand. Guess that root number is 1 and conductor N
? L(N) = lfuncreate([n->ellan(E,n), 0, [0,1], 1, N, 1]);
? fordiv(E.disc, d, print(d,": ",lfuncheckfeq(L(d))))
1: 0
2: 0
4: -1
8: -2
16: -3
32: -127
64: -3
128: -2
256: -2
512: -1
1024: -1
2048: 0
4096: 0
? lfunconductor(L(1)) \\ lfunconductor ignores conductor = 1 in Ldata !
%5 = 32
```

The above code assumed that root number was 1; had we set it to -1, none of the `lfuncheckfeq` values would have been acceptable:

```
? L2(N) = lfuncreate([n->ellan(E,n), 0, [0,1], 1, N, -1]);
? [ lfuncheckfeq(L2(d)) | d<-divisors(E.disc) ]
%7 = [0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, -1, -1]
```

The library syntax is `GEN lfunconductor(GEN L, GEN ab = NULL, long 10000], long bitprec)`.

3.6.15 lfuncost$(L, \{sdom\}, \{der = 0\})$. Estimate the cost of running `lfuninit(L,sdom,der)` at current bit precision. Returns $[t, b]$, to indicate that t coefficients a_n will be computed, as well as t values of `gammamellininv`, all at bit accuracy b. A subsequent call to `lfun` at s evaluates a polynomial of degree t at $\exp(hs)$ for some real parameter h, at the same bit accuracy b. If L is already an `Linit`, then *sdom* and *der* are ignored and are best left omitted; the bit accuracy is also inferred from L: in short we get an estimate of the cost of using that particular `Linit`.

```
? \pb 128
? lfuncost(1, [100]) \\ for zeta(1/2+I*t), |t| < 100
%1 = [7, 242]  \\ 7 coefficients, 242 bits
? lfuncost(1, [1/2, 100]) \\ for zeta(s) in the critical strip, |Im s| < 100
%2 = [7, 246]  \\ now 246 bits
? lfuncost(1, [100], 10) \\ for zeta(1/2+I*t), |t| < 100
%3 = [8, 263]  \\ 10th derivative increases the cost by a small amount
? lfuncost(1, [10^5])
%3 = [158, 113438]  \\ larger imaginary part: huge accuracy increase
? L = lfuncreate(polcyclo(5)); \\ Dedekind zeta for Q(zeta_5)
? lfuncost(L, [100]) \\ at s = 1/2+I*t), |t| < 100
%5 = [11457, 582]
? lfuncost(L, [200]) \\ twice higher
%6 = [36294, 1035]
? lfuncost(L, [10^4])  \\ much higher: very costly !
%7 = [70256473, 45452]
? \pb 256
? lfuncost(L, [100]); \\ doubling bit accuracy
%8 = [17080, 710]
```

In fact, some L functions can be factorized algebraically by the `lfuninit` call, e.g. the Dedekind zeta function of abelian fields, leading to much faster evaluations than the above upper bounds. In that case, the function returns a vector of costs as above for each individual function in the product actually evaluated:

```
? L = lfuncreate(polcyclo(5)); \\ Dedekind zeta for Q(zeta_5)
? lfuncost(L, [100])  \\ a priori cost
%2 = [11457, 582]
? L = lfuninit(L, [100]); \\ actually perform all initializations
? lfuncost(L)
%4 = [[16, 242], [16, 242], [7, 242]]
```

The Dedekind function of this abelian quartic field is the product of four Dirichlet L-functions attached to the trivial character, a non-trivial real character and two complex conjugate characters. The non-trivial characters happen to have the same conductor (hence same evaluation costs), and correspond to two evaluations only since the two conjugate characters are evaluated simultaneously. For a total of three L-functions evaluations, which explains the three components above. Note that the actual cost is much lower than the a priori cost in this case.

The library syntax is `GEN lfuncost0(GEN L, GEN sdom = NULL, long der, long bitprec)` . Also available is `GEN lfuncost(GEN L, GEN dom, long der, long bitprec)` when L is *not* an `Linit`; the return value is a `t_VECSMALL` in this case.

3.6.16 lfuncreate(obj). This low-level routine creates `Ldata` structures, needed by *lfun* functions, describing an L-function and its functional equation. You are urged to use a high-level constructor when one is available, and this function accepts them, see `??lfun`:

```
? L = lfuncreate(1); \\ Riemann zeta
? L = lfuncreate(5); \\ Dirichlet L-function for quadratic character (5/.)
? L = lfuncreate(x^2+1); \\ Dedekind zeta for Q(i)
? L = lfuncreate(ellinit([0,1])); \\ L-function of E/Q: y^2=x^3+1
```

One can then use, e.g., `Lfun(L,s)` to directly evaluate the respective L-functions at s, or `lfuninit(L, [c,w,h]` to initialize computations in the rectangular box $\Re(s-c) \leq w$, $\Im(s) \leq h$.

We now describe the low-level interface, used to input non-builtin L-functions. The input is now a 6 or 7 component vector $V = [a, astar, Vga, k, N, eps, poles]$, whose components are as follows:

• `V[1]=a` encodes the Dirichlet series coefficients. The preferred format is a closure of arity 1: `n->vector(n,i,a(i))` giving the vector of the first n coefficients. The closure is allowed to return a vector of more than n coefficients (only the first n will be considered) or even less than n, in which case loss of accuracy will occur and a warning that `#an` is less than expected is issued. This allows to precompute and store a fixed large number of Dirichlet coefficients in a vector v and use the closure `n->v`, which does not depend on n. As a shorthand for this latter case, you can input the vector v itself instead of the closure.

A second format is limited to multiplicative L functions affording an Euler product. It is a closure of arity 2 `(p,d)->L(p)` giving the local factor L_p at p as a rational function, to be evaluated at p^{-s} as in `direuler`; d is set to the floor of $\log_p(n)$, where n is the total number of Dirichlet coefficients $(a_1, \dots, a_n)$ that will be computed in this way. This parameter d allows to compute only part of L_p when p is large and L_p expensive to compute, but it can of course be ignored by the closure.

Finally one can describe separately the generic Dirichlet coefficients and the bad local factors by setting `dir` $= [an, [p_1, L_{p_1}^{-1}], \dots, [p_k, L_{p_k}^{-1}]]$, where `an` describes the generic coefficients in one of the two formats above, except that coefficients a_n with $p_i \mid n$ for some $i \leq k$ will be ignored. The subsequent pairs $[p, L_p^{-1}]$ give the bad primes and corresponding *inverse* local factors.

• `V[2]=astar` is the Dirichlet series coefficients of the dual function, encoded as `a` above. The sentinel values 0 and 1 may be used for the special cases where $a = a^*$ and $a = \overline{a^*}$, respectively.

• `V[3]=Vga` is the vector of α_j such that the gamma factor of the L-function is equal to

$$\gamma_A(s) = \prod_{1 \leq j \leq d} \Gamma_{\mathbf{R}}(s + \alpha_j),$$

where $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. This same syntax is used in the `gammamellininv` functions. In particular the length d of `Vga` is the degree of the L-function. In the present implementation, the α_j are assumed to be exact rational numbers. However when calling theta functions with *complex* (as opposed to real) arguments, determination problems occur which may give wrong results when the α_j are not integral.

• `V[4]=k` is a positive integer k. The functional equation relates values at s and $k - s$. For instance, for an Artin L-series such as a Dedekind zeta function we have $k = 1$, for an elliptic curve $k = 2$, and for a modular form, k is its weight. For motivic L-functions, the *motivic* weight w is $w = k - 1$.

• `V[5]=N` is the conductor, an integer $N \geq 1$, such that $\Lambda(s) = N^{s/2}\gamma_A(s)L(s)$ with $\gamma_A(s)$ as above.

• `V[6]=eps` is the root number ε, i.e., the complex number (usually of modulus 1) such that $\Lambda(a, k-s) = \varepsilon\Lambda(a^*, s)$.

• The last optional component `V[7]=poles` encodes the poles of the L or Λ-functions, and is omitted if they have no poles. A polar part is given by a list of 2-component vectors $[\beta, P_\beta(x)]$, where β is a pole and the power series $P_\beta(x)$ describes the attached polar part, such that $L(s) - P_\beta(s-\beta)$ is holomorphic in a neighbourhood of β. For instance $P_\beta = r/x + O(1)$ for a simple pole at β or $r_1/x^2 + r_2/x + O(1)$ for a double pole. The type of the list describing the polar part allows to distinguish between L and Λ: a `t_VEC` is attached to L, and a `t_COL` is attached to Λ.

The latter is mandatory unless $a = \overline{a^*}$ (coded by `astar` equal to 0 or 1): otherwise, the poles of L^* cannot be infered from the poles of L ! (Whereas the functional equation allows to deduce the polar part of Λ^* from the polar part of Λ.) The special coding `poles` $= r$ a complex scalar is available in this case, to describe a L function with at most a single simple pole at $s = k$ and residue r. (This is the usual situation, for instance for Dedekind zeta functions.) This value r can be set to 0 if unknown, and it will be computed.

The library syntax is `GEN lfuncreate(GEN obj)`.

3.6.17 lfundiv$(L1, L2)$. Creates the `Ldata` structure (without initialization) corresponding to the quotient of the Dirichlet series L_1 and L_2 given by `L1` and `L2`. Assume that $v_z(L_1) \geq v_z(L_2)$ at all complex numbers z: the construction may not create new poles, nor increase the order of existing ones.

The library syntax is `GEN lfundiv(GEN L1, GEN L2, long bitprec)`.

3.6.18 lfunetaquo(M). Returns the `Ldata` structure attached to the L function attached to the modular form $z \mapsto \prod_{i=1}^{n} \eta(M_{i,1}z)^{M_{i,2}}$ It is currently assumed that f is a self-dual cuspidal form on $\Gamma_0(N)$ for some N. For instance, the L-function $\sum \tau(n)n^{-s}$ attached to Ramanujan's Δ function is encoded as follows

```
? L = lfunetaquo(Mat([1,24]));
? lfunan(L, 100)  \\ first 100 values of tau(n)
```

The library syntax is `GEN lfunetaquo(GEN M)`.

3.6.19 lfungenus2(F). Returns the `Ldata` structure attached to the L function attached to the genus-2 curve defined by $y^2 = F(x)$ or $y^2 + Q(x)y = P(x)$ if $F = [P, Q]$. Currently, the model needs to be minimal at 2, and if the conductor is even, its valuation at 2 might be incorrect (a warning is issued).

The library syntax is `GEN lfungenus2(GEN F)`.

3.6.20 lfunhardy(L, t). Variant of the Hardy Z-function given by `L`, used for plotting or locating zeros of $L(k/2 + it)$ on the critical line. The precise definition is as follows: if as usual $k/2$ is the center of the critical strip, d is the degree, α_j the entries of `Vga` giving the gamma factors, and ε the root number, then if we set $s = k/2 + it = \rho e^{i\theta}$ and $E = (d(k/2 - 1) + \sum_{1 \le j \le d} \alpha_j)/2$, the computed function at t is equal to

$$Z(t) = \varepsilon^{-1/2} \Lambda(s) \cdot |s|^{-E} e^{dt\theta/2} ,$$

which is a real function of t for self-dual Λ, vanishing exactly when $L(k/2 + it)$ does on the critical line. The normalizing factor $|s|^{-E} e^{dt\theta/2}$ compensates the exponential decrease of $\gamma_A(s)$ as $t \to \infty$ so that $Z(t) \approx 1$.

```
? T = 100; \\ maximal height
? L = lfuninit(1, [T]); \\ initialize for zeta(1/2+it), |t|<T
? \p19 \\ no need for large accuracy
? ploth(t = 0, T, lfunhardy(L,t))
```

Using `lfuninit` is critical for this particular applications since thousands of values are computed. Make sure to initialize up to the maximal t needed: otherwise expect to see many warnings for unsufficient initialization and suffer major slowdowns.

The library syntax is `GEN lfunhardy(GEN L, GEN t, long bitprec)`.

3.6.21 lfuninit$(L, sdom, \{der = 0\})$. Initalization function for all functions linked to the computation of the L-function $L(s)$ encoded by `L`, where s belongs to the rectangular domain `sdom` $= [center, w, h]$ centered on the real axis, $|\Re(s) - center| \le w$, $|\Im(s)| \le h$, where all three components of `sdom` are real and w, h are non-negative. `der` is the maximum order of derivation that will be used. The subdomain $[k/2, 0, h]$ on the critical line (up to height h) can be encoded as $[h]$ for brevity. The subdomain $[k/2, w, h]$ centered on the critical line can be encoded as $[w, h]$ for brevity.

The argument `L` is an `Lmath`, an `Ldata` or an `Linit`. See `??Ldata` and `??lfuncreate` for how to create it.

The height h of the domain is a *crucial* parameter: if you only need $L(s)$ for real s, set h to 0. The running time is roughly proportional to

$$(B/d + \pi h/4)^{d/2+3} N^{1/2},$$

where B is the default bit accuracy, d is the degree of the L-function, and N is the conductor (the exponent $d/2 + 3$ is reduced to $d/2 + 2$ when $d = 1$ and $d = 2$). There is also a dependency on w, which is less crucial, but make sure to use the smallest rectangular domain that you need.

```
? L0 = lfuncreate(1); \\ Riemann zeta
? L = lfuninit(L0, [1/2, 0, 100]); \\ for zeta(1/2+it), |t| < 100
? lfun(L, 1/2 + I)
? L = lfuninit(L0, [100]); \\ same as above !
```

The library syntax is `GEN lfuninit0(GEN L, GEN sdom, long der, long bitprec)`.

3.6.22 lfunlambda$(L, s, \{D = 0\})$. Compute the completed L-function $\Lambda(s) = N^{s/2}\gamma(s)L(s)$, or if `D` is set, the derivative of order `D` at s. The parameter `L` is either an `Lmath`, an `Ldata` (created by `lfuncreate`, or an `Linit` (created by `lfuninit`), preferrably the latter if many values are to be computed.

The result is given with absolute error less than $2^{-B}|\gamma(s)N^{s/2}|$, where B = realbitprecision.

The library syntax is `GEN lfunlambda0(GEN L, GEN s, long D, long bitprec)`.

3.6.23 lfunmfspec(L). Returns `[valeven,valodd,omminus,omplus]`, where `valeven` (resp., `valodd`) is the vector of even (resp., odd) periods of the modular form given by `L`, and `omminus` and `omplus` the corresponding real numbers ω^- and ω^+ normalized in a noncanonical way. For the moment, only for modular forms of even weight.

The library syntax is `GEN lfunmfspec(GEN L, long bitprec)`.

3.6.24 lfunmul$(L1, L2)$. Creates the `Ldata` structure (without initialization) corresponding to the product of the Dirichlet series given by `L1` and `L2`.

The library syntax is `GEN lfunmul(GEN L1, GEN L2, long bitprec)`.

3.6.25 lfunorderzero$(L, \{m = -1\})$. Computes the order of the possible zero of the L-function at the center $k/2$ of the critical strip; return 0 if $L(k/2)$ does not vanish.

If m is given and has a non-negative value, assumes the order is at most m. Otherwise, the algorithm chooses a sensible default:

• if the L argument is an `Linit`, assume that a multiple zero at $s = k/2$ has order less than or equal to the maximal allowed derivation order.

• else assume the order is less than 4.

You may explicitly increase this value using optional argument m; this overrides the default value above. (Possibly forcing a recomputation of the `Linit`.)

The library syntax is `long lfunorderzero(GEN L, long m, long bitprec)`.

3.6.26 lfunqf(Q). Returns the `Ldata` structure attached to the Θ function of the lattice attached to the definite positive quadratic form Q.

```
? L = lfunqf(matid(2));
? lfunqf(L,2)
%2 = 6.026812039691940123546260192728285583 9
? lfun(x^2+1,2)*4
%3 = 6.026812039691940123546260192728285583 9
```

The library syntax is `GEN lfunqf(GEN Q, long prec)`.

3.6.27 lfunrootres($data$). Given the `Ldata` attached to an L-function (or the output of `lfunthetainit`), compute the root number and the residues. The output is a 3-component vector $[r, R, w]$, where r is the residue of $L(s)$ at the unique pole, R is the residue of $\Lambda(s)$, and w is the root number. In the present implementation,

• either the polar part must be completely known (and is then arbitrary): the function determines the root number,

```
? L = lfunmul(1,1); \\ zeta^2
? [r,R,w] = lfunrootres(L);
? r  \\ single pole at 1, double
%3 = [[1, 1.[...]*x^-2 + 1.1544[...]*x^-1 + O(x^0)]]
? w
%4 = 1
? R \\ double pole at 0 and 1
%5 = [[1,[...]], [0,[...]]
```

• or at most a single pole is allowed: the function computes both the root number and the residue (0 if no pole).

The library syntax is `GEN lfunrootres(GEN data, long bitprec)`.

3.6.28 lfuntheta($data, t, \{m = 0\}$). Compute the value of the m-th derivative at t of the theta function attached to the L-function given by `data`. `data` can be either the standard L-function data, or the output of `lfunthetainit`. The theta function is defined by the formula $\Theta(t) = \sum_{n\geq 1} a(n)K(nt/\sqrt{(N)})$, where $a(n)$ are the coefficients of the Dirichlet series, N is the conductor, and K is the inverse Mellin transform of the gamma product defined by the `Vga` component. Its Mellin transform is equal to $\Lambda(s) - P(s)$, where $\Lambda(s)$ is the completed L-function and the rational function $P(s)$ its polar part. In particular, if the L-function is the L-function of a modular form $f(\tau) = \sum_{n\geq 0} a(n)q^n$ with $q = \exp(2\pi i\tau)$, we have $\Theta(t) = 2(f(it/\sqrt{N}) - a(0))$. Note that an easy theorem on modular forms implies that $a(0)$ can be recovered by the formula $a(0) = -L(f, 0)$.

The library syntax is `GEN lfuntheta(GEN data, GEN t, long m, long bitprec)`.

3.6.29 lfunthetacost($L, \{tdom\}, \{m = 0\}$). This function estimates the cost of running `lfunthetainit(L,tdom,m)` at current bit precision. Returns the number of coefficients a_n that would be computed. This also estimates the cost of a subsequent evaluation `lfuntheta`, which must compute that many values of `gammamellininv` at the current bit precision. If L is already an `Linit`, then $tdom$ and m are ignored and are best left omitted: we get an estimate of the cost of using that particular `Linit`.

```
? \pb 1000
? L = lfuncreate(1); \\ Riemann zeta
? lfunthetacost(L); \\ cost for theta(t), t real >= 1
%1 = 15
? lfunthetacost(L, 1 + I); \\ cost for theta(1+I). Domain error !
 ***   at top-level: lfunthetacost(1,1+I)
 ***                 ^--------------------
 *** lfunthetacost: domain error in lfunthetaneed: arg t > 0.785
? lfunthetacost(L, 1 + I/2) \\ for theta(1+I/2).
%2 = 23
? lfunthetacost(L, 1 + I/2, 10) \\ for theta^((10))(1+I/2).
```

```
%3 = 24
? lfunthetacost(L, [2, 1/10]) \\ cost for theta(t), |t| >= 2, |arg(t)| < 1/10
%4 = 8
? L = lfuncreate( ellinit([1,1]) );
? lfunthetacost(L)  \\ for t >= 1
%6 = 2471
```

The library syntax is `long lfunthetacost0(GEN L, GEN tdom = NULL, long m, long bitprec)`.

3.6.30 lfunthetainit$(L, \{tdom\}, \{m = 0\})$. Initalization function for evaluating the m-th derivative of theta functions with argument t in domain $tdom$. By default ($tdom$ omitted), t is real, $t \geq 1$. Otherwise, $tdom$ may be

• a positive real scalar ρ: t is real, $t \geq \rho$.

• a non-real complex number: compute at this particular t; this allows to compute $\theta(z)$ for any complex z satisfying $|z| \geq |t|$ and $|\arg z| \leq |\arg t|$; we must have $|2 \arg z/d| < \pi/2$, where d is the degree of the Γ factor.

• a pair $[\rho, \alpha]$: assume that $|t| \geq \rho$ and $|\arg t| \leq \alpha$; we must have $|2\alpha/d| < \pi/2$, where d is the degree of the Γ factor.

```
? \p500
? L = lfuncreate(1); \\ Riemann zeta
? t = 1+I/2;
? lfuntheta(L, t); \\ direct computation
time = 30 ms.
? T = lfunthetainit(L, 1+I/2);
time = 30 ms.
? lfuntheta(T, t); \\ instantaneous
```

The T structure would allow to quickly compute $\theta(z)$ for any z in the cone delimited by t as explained above. On the other hand

```
? lfuntheta(T,I)
 ***   at top-level: lfuntheta(T,I)
 ***                 ^--------------
 *** lfuntheta: domain error in lfunthetaneed: arg t > 0.785398163397448
```

The initialization is equivalent to

```
? lfunthetainit(L, [abs(t), arg(t)])
```

The library syntax is `GEN lfunthetainit(GEN L, GEN tdom = NULL, long m, long bitprec)`.

3.6.31 lfunzeros$(L, lim, \{divz = 8\})$. `lim` being either a positive upper limit or a non-empty real interval inside $[0, +\infty[$, computes an ordered list of zeros of $L(s)$ on the critical line up to the given upper limit or in the given interval. Use a naive algorithm which may miss some zeros: it assumes that two consecutive zeros at height $T \geq 1$ differ at least by $2\pi/\omega$, where

$$\omega := \texttt{divz} \cdot \big(d \log(T/2\pi) + d + 2\log(N/(\pi/2)^d)\big).$$

To use a finer search mesh, set divz to some integral value larger than the default (= 8).

```
? lfunzeros(1, 30) \\ zeros of Rieman zeta up to height 30
%1 = [14.134[...], 21.022[...], 25.010[...]]
? #lfunzeros(1, [100,110])  \\ count zeros with 100 <= Im(s) <= 110
%2 = 4
```

The algorithm also assumes that all zeros are simple except possibly on the real axis at $s = k/2$ and that there are no poles in the search interval. (The possible zero at $s = k/2$ is repeated according to its multiplicity.)

Should you pass an `Linit` argument to the function, beware that the algorithm needs at least

```
L = lfuninit(Ldata, T+1)
```

where T is the upper bound of the interval defined by `lim`: this allows to detect zeros near T. Make sure that your `Linit` domain contains this one. The algorithm assumes that a multiple zero at $s = k/2$ has order less than or equal to the maximal derivation order allowed by the `Linit`. You may increase that value in the `Linit` but this is costly: only do it for zeros of low height or in `lfunorderzero` instead.

The library syntax is `GEN lfunzeros(GEN L, GEN lim, long divz, long bitprec)`.

3.7 Modular symbols.

Let $\Delta := \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ be the abelian group of divisors of degree 0 on the rational projective line. The standard $\mathrm{GL}(2, \mathbf{Q})$ action on $\mathbf{P}^1(\mathbf{Q})$ via homographies naturally extends to Δ. Given

- G a finite index subgroup of $\mathrm{SL}(2, \mathbf{Z})$,
- a field F and a finite dimensional representation V/F of $\mathrm{GL}(2, \mathbf{Q})$,

we consider the space of *modular symbols* $M := \mathrm{Hom}_G(\Delta, V)$. This finite dimensional F-vector space is a G-module, canonically isomorphic to $H^1_c(X(G), V)$, and allows to compute modular forms for G.

Currently, we only support the groups $\Gamma_0(N)$ ($N > 1$ an integer) and the representations $V_k = \mathbf{Q}[X, Y]_{k-2}$ ($k \geq 2$ an integer) over $\mathbf{Q}$. We represent a space of modular symbols by an *ms* structure, created by the function `msinit`. It encodes basic data attached to the space: chosen $\mathbf{Z}[G]$-generators (g_i) for Δ (and relations among those) and an F-basis of M. A modular symbol s is thus given either in terms of this fixed basis, or as a collection of values $s(g_i)$ satisfying certain relations.

A subspace of M (e.g. the cuspidal or Eisenstein subspaces, the new or old modular symbols, etc.) is given by a structure allowing quick projection and restriction of linear operators; its first component is a matrix whose columns form an F-basis of the subspace.

3.7.1 msatkinlehner$(M, Q, \{H\})$. Let M be a full modular symbol space of level N, as given by `msinit`, let $Q \mid N$, $(Q, N/Q) = 1$, and let H be a subspace stable under the Atkin-Lehner involution w_Q. Return the matrix of w_Q acting on H (M if omitted).

```
? M = msinit(36,2); \\ M_2(Gamma_0(36))
? w = msatkinlehner(M,4); w^2 == 1
%2 = 1
? #w   \\ involution acts on a 13-dimensional space
%3 = 13
? M = msinit(36,2, -1); \\ M_2(Gamma_0(36))^-
? w = msatkinlehner(M,4); w^2 == 1
%5 = 1
? #w
%6 = 4
```

The library syntax is `GEN msatkinlehner(GEN M, long Q, GEN H = NULL)`.

3.7.2 mscuspidal$(M, \{flag = 0\})$. M being a full modular symbol space, as given by `msinit`, return its cuspidal part S. If $flag = 1$, return $[S, E]$ its decomposition into cuspidal and Eisenstein parts.

A subspace is given by a structure allowing quick projection and restriction of linear operators; its first component is a matrix with integer coefficients whose columns form a $\mathbf{Q}$-basis of the subspace.

```
? M = msinit(2,8, 1); \\ M_8(Gamma_0(2))^+
? [S,E] = mscuspidal(M, 1);
? E[1]  \\ 2-dimensional
%3 =
[0 -10]

[0 -15]

[0  -3]

[1   0]

? S[1]  \\ 1-dimensional
%4 =
[ 3]

[30]

[ 6]

[-8]
```

The library syntax is `GEN mscuspidal(GEN M, long flag)`.

3.7.3 mseisenstein(M). M being a full modular symbol space, as given by `msinit`, return its Eisenstein subspace. A subspace is given by a structure allowing quick projection and restriction of linear operators; its first component is a matrix with integer coefficients whose columns form a $\mathbf{Q}$-basis of the subspace. This is the same basis as given by the second component of `mscuspidal`$(M, 1)$.

```
? M = msinit(2,8, 1); \\ M_8(Gamma_0(2))^+
? E = mseisenstein(M);
? E[1]  \\ 2-dimensional
%3 =
[0 -10]

[0 -15]

[0  -3]

[1   0]

? E == mscuspidal(M,1)[2]
%4 = 1
```

The library syntax is `GEN mseisenstein(GEN M)`.

3.7.4 mseval$(M, s, \{p\})$. Let $\Delta := \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$. Let M be a full modular symbol space, as given by `msinit`, let s be a modular symbol from M, i.e. an element of $\mathrm{Hom}_G(\Delta, V)$, and let $p = [a, b] \in \Delta$ be a path between two elements in $\mathbf{P}^1(\mathbf{Q})$, return $s(p) \in V$. The path extremities a and b may be given as `t_INT`, `t_FRAC` or `oo` $= (1 : 0)$. The symbol s is either

• a `t_COL` coding an element of a modular symbol subspace in terms of the fixed basis of $\mathrm{Hom}_G(\Delta, V)$ chosen in M; if M was initialized with a non-zero *sign* ($+$ or $-$), then either the basis for the full symbol space or the $\pm$-part can be used (the dimension being used to distinguish the two).

• a `t_VEC` (v_i) of elements of V, where the $v_i = s(g_i)$ give the image of the generators g_i of Δ, see `mspathgens`. We assume that s is a proper symbol, i.e. that the v_i satisfy the `mspathgens` relations.

If p is omitted, convert the symbol s to the second form: a vector of the $s(g_i)$.

```
? M = msinit(2,8,1); \\ M_8(Gamma_0(2))^+
? g = mspathgens(M)[1]
%2 = [[+oo, 0], [0, 1]]
? N = msnew(M)[1]; #N \\ Q-basis of new subspace, dimension 1
%3 = 1
? s = N[,1]          \\ t_COL representation
%4 = [-3, 6, -8]~
? S = mseval(M, s)   \\ t_VEC representation
%5 = [64*x^6-272*x^4+136*x^2-8, 384*x^5+960*x^4+192*x^3-672*x^2-432*x-72]
? mseval(M,s, g[1])
%6 = 64*x^6 - 272*x^4 + 136*x^2 - 8
? mseval(M,S, g[1])
%7 = 64*x^6 - 272*x^4 + 136*x^2 - 8
```

Note that the symbol should have values in $V = \mathbf{Q}[x, y]_{k-2}$, we return the de-homogenized values corresponding to $y = 1$ instead.

The library syntax is `GEN mseval(GEN M, GEN s, GEN p = NULL)`.

3.7.5 msfromcusp(M, c). Returns the modular symbol attached to the cusp c, where M is a modular symbol space of level N, attached to $G = \Gamma_0(N)$. The cusp c in $\mathbf{P}^1(\mathbf{Q})/G$ can be given either as `oo` $(= (1:0))$, as a rational number a/b $(= (a:b))$. The attached symbol maps the path $[b] - [a] \in \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ to $E_c(b) - E_c(a)$, where $E_c(r)$ is 0 when $r \neq c$ and $X^{k-2} \mid \gamma_r$ otherwise, where $\gamma_r \cdot r = (1:0)$. These symbol span the Eisenstein subspace of M.

```
? M = msinit(2,8);  \\  M_8(Gamma_0(2))
? E =  mseisenstein(M);
? E[1] \\ two-dimensional
%3 =
[0 -10]

[0 -15]

[0  -3]

[1   0]

? s = msfromcusp(M,oo)
%4 = [0, 0, 0, 1]~
? mseval(M, s)
%5 = [1, 0]
? s = msfromcusp(M,1)
%6 = [-5/16, -15/32, -3/32, 0]~
? mseval(M,s)
%7 = [-x^6, -6*x^5 - 15*x^4 - 20*x^3 - 15*x^2 - 6*x - 1]
```

In case M was initialized with a non-zero *sign*, the symbol is given in terms of the fixed basis of the whole symbol space, not the + or − part (to which it need not belong).

```
? M = msinit(2,8, 1);  \\  M_8(Gamma_0(2))^+
? E =  mseisenstein(M);
? E[1] \\ still two-dimensional, in a smaller space
%3 =
[ 0 -10]

[ 0   3]

[-1   0]

? s = msfromcusp(M,oo) \\ in terms of the basis for M_8(Gamma_0(2)) !
%4 = [0, 0, 0, 1]~
? mseval(M, s) \\ same symbol as before
%5 = [1, 0]
```

The library syntax is `GEN msfromcusp(GEN M, GEN c)`.

3.7.6 msfromell$(E, \{sign = 0\})$. Let $E/\mathbf{Q}$ be an elliptic curve of conductor N. For $\varepsilon = \pm 1$, we define the (cuspidal, new) modular symbol x^ε in $H_c^1(X_0(N), \mathbf{Q})^\varepsilon$ attached to E. For all primes p not dividing N we have $T_p(x^\varepsilon) = a_p x^\varepsilon$, where $a_p = p + 1 - \#E(\mathbf{F}_p)$.

Let $\Omega^+ =$ `E.omega[1]` be the real period of E (integration of the Néron differential $dx/(2y + a_1x + a3)$ on the connected component of $E(\mathbf{R})$, i.e. the generator of $H_1(E, \mathbf{Z})^+$) normalized by $\Omega^+ > 0$. Let $i\Omega^-$ the integral on a generator of $H_1(E, \mathbf{Z})^-$ with $\Omega^- \in \mathbf{R}_{>0}$. If c_∞ is the number of connected components of $E(\mathbf{R})$, Ω^- is equal to $(-2/c_\infty) \times$ `imag(E.omega[2])`. The complex modular symbol is defined by

$$F : \delta \to 2i\pi \int_\delta f(z)dz$$

The modular symbols x^ε are normalized so that $F = x^+\Omega^+ + x^- i\Omega^-$. In particular, we have

$$x^+([0] - [\infty]) = L(E, 1)/\Omega^+,$$

which defines $x^\pm$ unless $L(E, 1) = 0$. Furthermore, for all fundamental discriminants D such that $\varepsilon \cdot D > 0$, we also have

$$\sum_{0 \le a < |D|} (D|a) x^\varepsilon([a/|D|] - [\infty]) = L(E, (D|.), 1)/\Omega^\varepsilon,$$

where $(D|.)$ is the Kronecker symbol. The period Ω^- is also $2/c_\infty \times$ the real period of the twist $E^{(-4)} =$ `elltwist(E, -4)`.

This function returns the pair $[M, x]$, where M is `msinit`$(N, 2)$ and x is x^{sign} as above when $sign = \pm 1$, and $x = [x^+, x^-]$ when $sign$ is 0. The modular symbols $x^\pm$ are given as a `t_COL` (in terms of the fixed basis of $\mathrm{Hom}_G(\Delta, \mathbf{Q})$ chosen in M).

```
? E=ellinit([0,-1,1,-10,-20]);  \\ X_0(11)
? [M,xp]= msfromell(E,1);
? xp
%3 = [1/5, -1/2, -1/2]~
? [M,x]= msfromell(E);
? x    \\ both x^+ and x^-
%5 = [[1/5, -1/2, -1/2]~, [0, 1/2, -1/2]~]
? p = 23; (mshecke(M,p) - ellap(E,p))*x[1]
%6 = [0, 0, 0]~ \\ true at all primes, including p = 11; same for x[2]
```

The library syntax is `GEN msfromell(GEN E, long sign)`.

3.7.7 msfromhecke$(M, v, \{H\})$. Given a msinit M and a vector v of pairs $[p, P]$ (where p is prime and P is a polynomial with integer coefficients), return a basis of all modular symbols such that $P(T_p)(s) = 0$. If H is present, it must be a Hecke-stable subspace and we restrict to $s \in H$. When T_p has a rational eigenvalue and $P(x) = x - a_p$ has degree 1, we also accept the integer a_p instead of P.

```
? E = ellinit([0,-1,1,-10,-20]) \\11a1
? ellap(E,2)
%2 = -2
? ellap(E,3)
%3 = -1
```

```
? M = msinit(11,2);
? S = msfromhecke(M, [[2,-2],[3,-1]])
%5 =
[ 1  1]

[-5  0]

[ 0 -5]
? mshecke(M, 2, S)
%6 =
[-2  0]

[ 0 -2]

? M = msinit(23,4);
? S = msfromhecke(M, [[5, x^4-14*x^3-244*x^2+4832*x-19904]]);
? factor( charpoly(mshecke(M,5,S)) )
%9 =
[x^4 - 14*x^3 - 244*x^2 + 4832*x - 19904 2]
```

The library syntax is `GEN msfromhecke(GEN M, GEN v, GEN H = NULL)`.

3.7.8 msgetlevel(M). M being a full modular symbol space, as given by `msinit`, return its level N.

The library syntax is `long msgetlevel(GEN M)`.

3.7.9 msgetsign(M). M being a full modular symbol space, as given by `msinit`, return its sign: ± 1 or 0 (unset).

```
? M = msinit(11,4, 1);
? msgetsign(M)
%2 = 1
? M = msinit(11,4);
? msgetsign(M)
%4 = 0
```

The library syntax is `long msgetsign(GEN M)`.

3.7.10 msgetweight(M). M being a full modular symbol space, as given by `msinit`, return its weight k.

```
? M = msinit(11,4);
? msgetweight(M)
%2 = 4
```

The library syntax is `long msgetweight(GEN M)`.

3.7.11 mshecke$(M, p, \{H\})$. M being a full modular symbol space, as given by `msinit`, p being a prime number, and H being a Hecke-stable subspace (M if omitted) return the matrix of T_p acting on H (U_p if p divides N). Result is undefined if H is not stable by T_p (resp. U_p).

```
? M = msinit(11,2); \\ M_2(Gamma_0(11))
? T2 = mshecke(M,2)
%2 =
[3  0  0]

[1 -2  0]

[1  0 -2]
? M = msinit(11,2, 1); \\ M_2(Gamma_0(11))^+
? T2 = mshecke(M,2)
%4 =
[ 3  0]

[-1 -2]

? N = msnew(M)[1] \\ Q-basis of new cuspidal subspace
%5 =
[-2]

[-5]

? p = 1009; mshecke(M, p, N) \\ action of T_1009 on N
%6 =
[-10]
? ellap(ellinit("11a1"), p)
%7 = -10
```

The library syntax is `GEN mshecke(GEN M, long p, GEN H = NULL)`.

3.7.12 msinit$(G, V, \{sign = 0\})$. Given G a finite index subgroup of $\mathrm{SL}(2, \mathbf{Z})$ and a finite dimensional representation V of $\mathrm{GL}(2, \mathbf{Q})$, creates a space of modular symbols, the G-module $\mathrm{Hom}_G(\mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q})), V)$. This is canonically isomorphic to $H^1_c(X(G), V)$, and allows to compute modular forms for G. If *sign* is present and non-zero, it must be ± 1 and we consider the subspace defined by $\mathrm{Ker}(\sigma - sign)$, where σ is induced by `[-1,0;0,1]`. Currently the only supported groups are the $\Gamma_0(N)$, coded by the integer $N > 1$. The only supported representation is $V_k = \mathbf{Q}[X, Y]_{k-2}$, coded by the integer $k \geq 2$.

The library syntax is `GEN msinit(GEN G, GEN V, long sign)`.

3.7.13 msissymbol(M, s). M being a full modular symbol space, as given by `msinit`, check whether s is a modular symbol attached to M.

```
? M = msinit(7,8, 1); \\ M_8(Gamma_0(7))^+
? N = msnew(M)[1];
? s = N[,1];
? msissymbol(M, s)
%4 = 1
? S = mseval(M,s);
? msissymbol(M, S)
%6 = 1
? [g,R] = mspathgens(M); g
```

```
%7 = [[+oo, 0], [0, 1/2], [1/2, 1]]
? #R  \\ 3 relations among the generators g_i
%8 = 3
? T = S; T[3]++; \\ randomly perturb S(g_3)
? msissymbol(M, T)
%10 = 0  \\ no longer satisfies the relations
```

The library syntax is `long msissymbol(GEN M, GEN s)`.

3.7.14 msnew(M). M being a full modular symbol space, as given by `msinit`, return the *new* part of its cuspidal subspace. A subspace is given by a structure allowing quick projection and restriction of linear operators; its first component is a matrix with integer coefficients whose columns form a $\mathbf{Q}$-basis of the subspace.

```
? M = msinit(11,8, 1); \\ M_8(Gamma_0(11))^+
? N = msnew(M);
? #N[1]  \\ 6-dimensional
%3 = 6
```

The library syntax is `GEN msnew(GEN M)`.

3.7.15 msomseval$(Mp, PHI, path)$. Return the vectors of moments of the p-adic distribution attached to the path `path` by the overconvergent modular symbol `PHI`.

```
? M = msinit(3,6,1);
? Mp= mspadicinit(M,5,10);
? phi = [5,-3,-1]~;
? msissymbol(M,phi)
%4 = 1
? PHI = mstooms(Mp,phi);
? ME = msomseval(Mp,PHI,[oo, 0]);
```

The library syntax is `GEN msomseval(GEN Mp, GEN PHI, GEN path)`.

3.7.16 mspadicL$(mu, \{s = 0\}, \{r = 0\})$. Returns the value (or r-th derivative) on a character χ^s of $\mathbf{Z}_p^*$ of the p-adic L-function attached to `mu`.

Let Φ be the p-adic distribution-valued overconvergent symbol attached to a modular symbol ϕ for $\Gamma_0(N)$ (eigenvector for $T_N(p)$ for the eigenvalue a_p). Then $L_p(\Phi, \chi^s) = L_p(\mu, s)$ is the p-adic L function defined by

$$L_p(\Phi, \chi^s) = \int_{\mathbf{Z}_p^*} \chi^s(z) d\mu(z)$$

where μ is the distribution on $\mathbf{Z}_p^*$ defined by the restriction of $\Phi([\infty]-[0])$ to $\mathbf{Z}_p^*$. The r-th derivative is taken in direction $\langle\chi\rangle$:

$$L_p^{(r)}(\Phi, \chi^s) = \int_{\mathbf{Z}_p^*} \chi^s(z)(\log z)^r d\mu(z).$$

In the argument list,

• `mu` is as returned by `mspadicmoments` (distributions attached to Φ by restriction to discs $a + p^\nu \mathbf{Z}_p$, $(a, p) = 1$).

• $s = [s_1, s_2]$ with $s_1 \in \mathbf{Z} \subset \mathbf{Z}_p$ and $s_2 \bmod p-1$ or $s_2 \bmod 2$ for $p = 2$, encoding the p-adic character $\chi^s := \langle\chi\rangle^{s_1}\tau^{s_2}$; here χ is the cyclotomic character from $\mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ to $\mathbf{Z}_p^*$, and τ is the Teichmüller character (for $p > 2$ and the character of order 2 on $(\mathbf{Z}/4\mathbf{Z})^*$ if $p = 2$); for convenience, the character $[s, s]$ can also be represented by the integer s.

When a_p is a p-adic unit, L_p takes its values in $\mathbf{Q}_p$. When a_p is not a unit, it takes its values in the two-dimensional $\mathbf{Q}_p$-vector space $D_{cris}(M(\phi))$ where $M(\phi)$ is the "motive" attached to ϕ, and we return the two p-adic components with respect to some fixed $\mathbf{Q}_p$-basis.

```
? M = msinit(3,6,1); phi=[5, -3, -1]~;
? msissymbol(M,phi)
%2 = 1
? Mp = mspadicinit(M, 5, 4);
? mu = mspadicmoments(Mp, phi); \\ no twist
\\ End of initializations
? mspadicL(mu,0) \\ L_p(chi^0)
%5 = 5 + 2*5^2 + 2*5^3 + 2*5^4 + ...
? mspadicL(mu,1) \\ L_p(chi), zero for parity reasons
%6 = [O(5^13)]~
? mspadicL(mu,2) \\ L_p(chi^2)
%7 = 3 + 4*5 + 4*5^2 + 3*5^5 + ...
? mspadicL(mu,[0,2]) \\ L_p(tau^2)
%8 = 3 + 5 + 2*5^2 + 2*5^3 + ...
? mspadicL(mu, [1,0]) \\ L_p(<chi>)
%9 = 3*5 + 2*5^2 + 5^3 + 2*5^7 + 5^8 + 5^10 + 2*5^11 + O(5^13)
? mspadicL(mu,0,1) \\ L_p'(chi^0)
%10 = 2*5 + 4*5^2 + 3*5^3 + ...
? mspadicL(mu, 2, 1) \\ L_p'(chi^2)
%11 = 4*5 + 3*5^2 + 5^3 + 5^4 + ...
```

Now several quadratic twists: `mstooms` is indicated.

```
? PHI = mstooms(Mp,phi);
? mu = mspadicmoments(Mp, PHI, 12); \\ twist by 12
? mspadicL(mu)
%14 = 5 + 5^2 + 5^3 + 2*5^4 + ...
? mu = mspadicmoments(Mp, PHI, 8); \\ twist by 8
? mspadicL(mu)
%16 = 2 + 3*5 + 3*5^2 + 2*5^4 + ...
? mu = mspadicmoments(Mp, PHI, -3); \\ twist by -3 < 0
? mspadicL(mu)
%18 = O(5^13) \\ always 0, phi is in the + part and D < 0
```

One can locate interesting symbols of level N and weight k with `msnew` and `mssplit`. Note that instead of a symbol, one can input a 1-dimensional Hecke-subspace from `mssplit`: the function will automatically use the underlying basis vector.

```
? M=msinit(5,4,1); \\ M_4(Gamma_0(5))^+
? L = mssplit(M, msnew(M)); \\ list of irreducible Hecke-subspaces
? phi = L[1]; \\ one Galois orbit of newforms
? #phi[1] \\... this one is rational
```

```
%4 = 1
? Mp = mspadicinit(M, 3, 4);
? mu = mspadicmoments(Mp, phi);
? mspadicL(mu)
%7 = 1 + 3 + 3^3 + 3^4 + 2*3^5 + 3^6 + O(3^9)
? M = msinit(11,8, 1); \\ M_8(Gamma_0(11))^+
? Mp = mspadicinit(M, 3, 4);
? L = mssplit(M, msnew(M));
? phi = L[1]; #phi[1] \\ ... this one is two-dimensional
%11 = 2
? mu = mspadicmoments(Mp, phi);
 ***   at top-level: mu=mspadicmoments(Mp,ph
 ***                       ^--------------------
 *** mspadicmoments: incorrect type in mstooms [dim_Q (eigenspace) > 1]
```

The library syntax is `GEN mspadicL(GEN mu, GEN s = NULL, long r)`.

3.7.17 mspadicinit$(M, p, n, \{flag\})$. M being a full modular symbol space, as given by `msinit`, and p a prime, initialize technical data needed to compute with overconvergent modular symbols, modulo p^n. If *flag* is unset, allow all symbols; else initialize only for a restricted range of symbols depending on *flag*: if $flag = 0$ restrict to ordinary symbols, else restrict to symbols ϕ such that $T_p(\phi) = a_p\phi$, with $v_p(a_p) \geq flag$, which is faster as *flag* increases. (The fastest initialization is obtained for $flag = 0$ where we only allow ordinary symbols.) For supersingular eigensymbols, such that $p \mid a_p$, we must further assume that p does not divide the level.

```
? E = ellinit("11a1");
? [M,phi] = msfromell(E,1);
? ellap(E,3)
%3 = -1
? Mp = mspadicinit(M, 3, 10, 0); \\ commit to ordinary symbols
? PHI = mstooms(Mp,phi);
```

If we restrict the range of allowed symbols with *flag*(for faster initialization), exceptions will occur if $v_p(a_p)$ violates this bound:

```
? E = ellinit("15a1");
? [M,phi] = msfromell(E,1);
? ellap(E,7)
%3 = 0
? Mp = mspadicinit(M,7,5,0); \\ restrict to ordinary symbols
? PHI = mstooms(Mp,phi)
***   at top-level: PHI=mstooms(Mp,phi)
***                       ^---------------
*** mstooms: incorrect type in mstooms [v_p(ap) > mspadicinit flag] (t_VEC).
? Mp = mspadicinit(M,7,5); \\ no restriction
? PHI = mstooms(Mp,phi);
```

This function uses $O(N^2(n+k)^2p)$ memory, where N is the level of M.

The library syntax is `GEN mspadicinit(GEN M, long p, long n, long flag)`.

3.7.18 mspadicmoments$(Mp, PHI, \{D = 1\})$. Given `Mp` from `mspadicinit`, an overconvergent eigensymbol `PHI` from `mstooms` and a fundamental discriminant D coprime to p, let $\texttt{PHI}^D$ denote the twisted symbol. This function computes the distribution $\mu = \texttt{PHI}^D([0] - \infty]) \mid \mathbf{Z}_p^*$ restricted to $\mathbf{Z}_p^*$. More precisely, it returns the moments of the $p-1$ distributions $\texttt{PHI}^D([0] - [\infty]) \mid (a + p\mathbf{Z}_p)$, $0 < a < p$. We also allow `PHI` to be given as a classical symbol, which is then lifted to an overconvergent symbol by `mstooms`; but this is wasteful if more than one twist is later needed.

The returned data μ (p-adic distributions attached to `PHI`) can then be used in `mspadicL` or `mspadicseries`. This precomputation allows to quickly compute derivatives of different orders or values at different characters.

```
? M = msinit(3,6, 1);
? phi = [5,-3,-1]~;
? msissymbol(M, phi)
%3 = 1
? p = 5; mshecke(M,p) * phi  \\ eigenvector of T_5, a_5 = 6
%4 = [30, -18, -6]~
? Mp = mspadicinit(M, p, 10, 0); \\ restrict to ordinary symbols, mod p^10
? PHI = mstooms(Mp, phi);
? mu = mspadicmoments(Mp, PHI);
? mspadicL(mu)
%8 = 5 + 2*5^2 + 2*5^3 + ...
? mu = mspadicmoments(Mp, PHI, 12); \\ twist by 12
? mspadicL(mu)
%10 = 5 + 5^2 + 5^3 + 2*5^4 + ...
```

The library syntax is `GEN mspadicmoments(GEN Mp, GEN PHI, long D)`.

3.7.19 mspadicseries$(mu, \{i = 0\})$. Let Φ be the p-adic distribution-valued overconvergent symbol attached to a modular symbol ϕ for $\Gamma_0(N)$ (eigenvector for $T_N(p)$ for the eigenvalue a_p). If μ is the distribution on $\mathbf{Z}_p^*$ defined by the restriction of $\Phi([\infty] - [0])$ to $\mathbf{Z}_p^*$, let

$$\hat{L}_p(\mu, \tau^i)(x) = \int_{\mathbf{Z}_p^*} \tau^i(t)(1+x)^{\log_p(t)/\log_p(u)} d\mu(t)$$

Here, τ is the Teichmüller character and u is a specific multiplicative generator of $1+2p\mathbf{Z}_p$. (Namely $1+p$ if $p > 2$ or 5 if $p = 2$.) To explain the formula, let $G_\infty := \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$, let $\chi : G_\infty \to \mathbf{Z}_p^*$ be the cyclotomic character (isomorphism) and γ the element of G_∞ such that $\chi(\gamma) = u$; then $\chi(\gamma)^{\log_p(t)/\log_p(u)} = \langle t \rangle$.

The p-padic precision of individual terms is maximal given the precision of the overconvergent symbol μ.

```
? [M,phi] = msfromell(ellinit("17a1"),1);
? Mp = mspadicinit(M, 5,7);
? mu = mspadicmoments(Mp, phi,1); \\ overconvergent symbol
? mspadicseries(mu)
%4 = (4 + 3*5 + 4*5^2 + 2*5^3 + 2*5^4 + 5^5 + 4*5^6 + 3*5^7 + O(5^9)) \
 + (3 + 3*5 + 5^2 + 5^3 + 2*5^4 + 5^6 + O(5^7))*x \
 + (2 + 3*5 + 5^2 + 4*5^3 + 2*5^4 + O(5^5))*x^2 \
 + (3 + 4*5 + 4*5^2 + O(5^3))*x^3 \
```

```
 + (3 + O(5))*x^4 + O(x^5)
```

An example with non-zero Teichmüller:

```
? [M,phi] = msfromell(ellinit("11a1"),1);
? Mp = mspadicinit(M, 3,10);
? mu = mspadicmoments(Mp, phi,1);
? mspadicseries(mu, 2)
%4 = (2 + 3 + 3^2 + 2*3^3 + 2*3^5 + 3^6 + 3^7 + 3^10 + 3^11 + O(3^12)) \
 + (1 + 3 + 2*3^2 + 3^3 + 3^5 + 2*3^6 + 2*3^8 + O(3^9))*x \
 + (1 + 2*3 + 3^4 + 2*3^5 + O(3^6))*x^2 \
 + (3 + O(3^2))*x^3 + O(x^4)
```

Supersingular example (not checked)

```
? E = ellinit("17a1"); ellap(E,3)
%1 = 0
? [M,phi] = msfromell(E,1);
? Mp = mspadicinit(M, 3,7);
? mu = mspadicmoments(Mp, phi,1);
? mspadicseries(mu)
%5 = [(2*3^-1 + 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + O(3^7)) \
 + (2 + 3^3 + O(3^5))*x \
 + (1 + 2*3 + O(3^2))*x^2 + O(x^3),\
 (3^-1 + 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + O(3^7)) \
 + (1 + 2*3 + 2*3^2 + 3^3 + 2*3^4 + O(3^5))*x \
 + (3^-2 + 3^-1 + O(3^2))*x^2 + O(3^-2)*x^3 + O(x^4)]
```

Example with a twist:

```
? E = ellinit("11a1");
? [M,phi] = msfromell(E,1);
? Mp = mspadicinit(M, 3,10);
? mu = mspadicmoments(Mp, phi,5); \\ twist by 5
? L = mspadicseries(mu)
%5 = (2*3^2 + 2*3^4 + 3^5 + 3^6 + 2*3^7 + 2*3^10 + O(3^12)) \
 + (2*3^2 + 2*3^6 + 3^7 + 3^8 + O(3^9))*x \
 + (3^3 + O(3^6))*x^2 + O(3^2)*x^3 + O(x^4)
? mspadicL(mu)
%6 = [2*3^2 + 2*3^4 + 3^5 + 3^6 + 2*3^7 + 2*3^10 + O(3^12)]~
? ellpadicL(E,3,10,,5)
%7 = 2 + 2*3^2 + 3^3 + 2*3^4 + 2*3^5 + 3^6 + 2*3^7 + O(3^10)
? mspadicseries(mu,1) \\ must be 0
%8 = O(3^12) + O(3^9)*x + O(3^6)*x^2 + O(3^2)*x^3 + O(x^4)
```

The library syntax is `GEN mspadicseries(GEN mu, long i)`.

3.7.20 mspathgens(M). Let $\Delta := \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$. Let M being a full modular symbol space, as given by `msinit`, return a set of $\mathbf{Z}[G]$-generators for Δ. The output is $[g, R]$, where g is a minimal system of generators and R the vector of $\mathbf{Z}[G]$-relations between the given generators. A relation is coded by a vector of pairs $[a_i, i]$ with $a_i \in \mathbf{Z}[G]$ and i the index of a generator, so that $\sum_i a_i g[i] = 0$.

An element $[v] - [u]$ in Δ is coded by the "path" $[u, v]$, where `oo` denotes the point at infinity $(1 : 0)$ on the projective line. An element of $\mathbf{Z}[G]$ is coded by a "factorization matrix": the first column contains distinct elements of G, and the second integers:

```
? M = msinit(11,8); \\ M_8(Gamma_0(11))
? [g,R] = mspathgens(M);
? g
%3 = [[+oo, 0], [0, 1/3], [1/3, 1/2]] \\ 3 paths
? #R  \\ a single relation
%4 = 1
? r = R[1]; #r \\ ...involving all 3 generators
%5 = 3
? r[1]
%6 = [[1, 1; [1, 1; 0, 1], -1], 1]
? r[2]
%7 = [[1, 1; [7, -2; 11, -3], -1], 2]
? r[3]
%8 = [[1, 1; [8, -3; 11, -4], -1], 3]
```

The given relation is of the form $\sum_i (1 - \gamma_i) g_i = 0$, with $\gamma_i \in \Gamma_0(11)$. There will always be a single relation involving all generators (corresponding to a round trip along all cusps), then relations involving a single generator (corresponding to 2 and 3-torsion elements in the group:

```
? M = msinit(2,8); \\ M_8(Gamma_0(2))
? [g,R] = mspathgens(M);
? g
%3 = [[+oo, 0], [0, 1]]
```

Note that the output depends only on the group G, not on the representation V.

The library syntax is `GEN mspathgens(GEN M)`.

3.7.21 mspathlog(M, p). Let $\Delta := \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$. Let M being a full modular symbol space, as given by `msinit`, encoding fixed $\mathbf{Z}[G]$-generators (g_i) of Δ (see `mspathgens`). A path $p = [a, b]$ between two elements in $\mathbf{P}^1(\mathbf{Q})$ corresponds to $[b] - [a] \in \Delta$. The path extremities a and b may be given as `t_INT`, `t_FRAC` or `oo` $= (1 : 0)$.

Returns (p_i) in $\mathbf{Z}[G]$ such that $p = \sum_i p_i g_i$.

```
? M = msinit(2,8); \\ M_8(Gamma_0(2))
? [g,R] = mspathgens(M);
? g
%3 = [[+oo, 0], [0, 1]]
? p = mspathlog(M, [1/2,2/3]);
? p[1]
%5 =
[[1, 0; 2, 1] 1]
? p[2]
```

```
%6 =
[[1, 0; 0, 1] 1]

[[3, -1; 4, -1] 1]
```

Note that the output depends only on the group G, not on the representation V.

The library syntax is `GEN mspathlog(GEN M, GEN p)`.

3.7.22 msqexpansion$(M, projH, \{B = seriesprecision\})$. M being a full modular symbol space, as given by `msinit`, and $projH$ being a projector on a Hecke-simple subspace (as given by `mssplit`), return the Fourier coefficients a_n, $n \leq B$ of the corresponding normalized newform. If B is omitted, use `seriesprecision`.

This function uses a naive $O(B^2 d^3)$ algorithm, where $d = O(kN)$ is the dimension of $M_k(\Gamma_0(N))$.

```
? M = msinit(11,2, 1); \\ M_2(Gamma_0(11))^+
? L = mssplit(M, msnew(M));
? msqexpansion(M,L[1], 20)
%3 = [1, -2, -1, 2, 1, 2, -2, 0, -2, -2, 1, -2, 4, 4, -1, -4, -2, 4, 0, 2]
? ellan(ellinit("11a1"), 20)
%4 = [1, -2, -1, 2, 1, 2, -2, 0, -2, -2, 1, -2, 4, 4, -1, -4, -2, 4, 0, 2]
```

The shortcut `msqexpansion(M, s, B)` is available for a symbol s, provided it is a Hecke eigenvector:

```
? E = ellinit("11a1");
? [M,s]=msfromell(E);
? msqexpansion(M,s,10)
%3 = [1, -2, -1, 2, 1, 2, -2, 0, -2, -2]
? ellan(E, 10)
%4 = [1, -2, -1, 2, 1, 2, -2, 0, -2, -2]
```

The library syntax is `GEN msqexpansion(GEN M, GEN projH, long precdl)`.

3.7.23 mssplit$(M, H, \{dimlim\})$. Let M denote a full modular symbol space, as given by `msinit`$(N, k, 1)$ or `msinit`$(N, k, -1)$ and let H be a Hecke-stable subspace of `msnew`(M). This function split H into Hecke-simple subspaces. If `dimlim` is present and positive, restrict to subspaces of dimension $\leq$ `dimlim`. A subspace is given by a structure allowing quick projection and restriction of linear operators; its first component is a matrix with integer coefficients whose columns form a $\mathbf{Q}$-basis of the subspace.

```
? M = msinit(11,8, 1); \\ M_8(Gamma_0(11))^+
? L = mssplit(M, msnew(M));
? #L
%3 = 2
? f = msqexpansion(M,L[1],5); f[1].mod
%4 = x^2 + 8*x - 44
? lift(f)
%5 = [1, x, -6*x - 27, -8*x - 84, 20*x - 155]
? g = msqexpansion(M,L[2],5); g[1].mod
%6 = x^4 - 558*x^2 + 140*x + 51744
```

To a Hecke-simple subspace corresponds an orbit of (normalized) newforms, defined over a number field. In the above example, we printed the polynomials defining the said fields, as well as the first 5 Fourier coefficients (at the infinite cusp) of one such form.

The library syntax is `GEN mssplit(GEN M, GEN H, long dimlim)`.

3.7.24 msstar$(M, \{H\})$. M being a full modular symbol space, as given by `msinit`, return the matrix of the `*` involution, induced by complex conjugation, acting on the (stable) subspace H (M if omitted).

```
? M = msinit(11,2); \\ M_2(Gamma_0(11))
? w = msstar(M);
? w^2 == 1
%3 = 1
```

The library syntax is `GEN msstar(GEN M, GEN H = NULL)`.

3.7.25 mstooms(Mp, phi). Given `Mp` from `mspadicinit`, lift the (classical) eigen symbol `phi` to a p-adic distribution-valued overconvergent symbol in the sense of Pollack and Stevens. More precisely, let ϕ belong to the space W of modular symbols of level N, $v_p(N) \leq 1$, and weight k which is an eigenvector for the Hecke operator $T_N(p)$ for a non-zero eigenvalue a_p and let $N_0 = \mathrm{lcm}(N, p)$.

Under the action of $T_{N_0}(p)$, ϕ generates a subspace W_ϕ of dimension 1 (if $p \mid N$) or 2 (if p does not divide N) in the space of modular symbols of level N_0.

Let $V_p = [p, 0; 0, 1]$ and $C_p = [a_p, p^{k-1}; -1, 0]$. When p does not divide N and a_p is divisible by p, `mstooms` returns the lift Φ of $(\phi, \phi|_k V_p)$ such that

$$T_{N_0}(p)\Phi = C_p\Phi$$

When p does not divide N and a_p is not divisible by p, `mstooms` returns the lift Φ of $\phi - \alpha^{-1}\phi|_k V_p$ which is an eigenvector of $T_{N_0}(p)$ for the unit eigenvalue where $\alpha^2 - a_p\alpha + p^{k-1} = 0$.

The resulting overconvergent eigensymbol can then be used in `mspadicmoments`, then `mspadicL` or `mspadicseries`.

```
? M = msinit(3,6, 1); p = 5;
? Tp = mshecke(M, p); factor(charpoly(Tp))
%2 =
[x - 3126 2]

[   x - 6 1]
? phi = matker(Tp - 6)[,1] \\ generator of p-Eigenspace, a_p = 6
%3 = [5, -3, -1]~
? Mp = mspadicinit(M, p, 10, 0); \\ restrict to ordinary symbols, mod p^10
? PHI = mstooms(Mp, phi);
? mu = mspadicmoments(Mp, PHI);
? mspadicL(mu)
%7 = 5 + 2*5^2 + 2*5^3 + ...
```

A non ordinary symbol.

```
? M = msinit(4,6,1); p = 3;
? Tp = mshecke(M, p); factor(charpoly(Tp))
```

```
%2 =
[x - 244 3]

[ x + 12 1]
? phi = matker(Tp + 12)[,1] \\ a_p = -12 is divisible by p = 3
%3 = [-1/32, -1/4, -1/32, 1]~
? msissymbol(M,phi)
%4 = 1
? Mp = mspadicinit(M,3,5,0);
? PHI = mstooms(Mp,phi);
 ***   at top-level: PHI=mstooms(Mp,phi)
 ***                       ^---------------
 *** mstooms: incorrect type in mstooms [v_p(ap) > mspadicinit flag] (t_VEC).
? Mp = mspadicinit(M,3,5,1);
? PHI = mstooms(Mp,phi);
```

The library syntax is `GEN mstooms(GEN Mp, GEN phi)`.

3.8 General number fields.

In this section, we describe functions related to general number fields. Functions related to quadratic number fields are found in Section 3.4 (Arithmetic functions).

3.8.1 Number field structures.

Let $K = \mathbf{Q}[X]/(T)$ a number field, $\mathbf{Z}_K$ its ring of integers, $T \in \mathbf{Z}[X]$ is monic. Three basic number field structures can be attached to K in GP:

• *nf* denotes a number field, i.e. a data structure output by `nfinit`. This contains the basic arithmetic data attached to the number field: signature, maximal order (given by a basis `nf.zk`), discriminant, defining polynomial T, etc.

• *bnf* denotes a "Buchmann's number field", i.e. a data structure output by `bnfinit`. This contains *nf* and the deeper invariants of the field: units $U(K)$, class group $\mathrm{Cl}(K)$, as well as technical data required to solve the two attached discrete logarithm problems.

• *bnr* denotes a "ray number field", i.e. a data structure output by `bnrinit`, corresponding to the ray class group structure of the field, for some modulus f. It contains a *bnf*, the modulus f, the ray class group $\mathrm{Cl}_f(K)$ and data attached to the discrete logarithm problem therein.

3.8.2 Algebraic numbers and ideals.

An *algebraic number* belonging to $K = \mathbf{Q}[X]/(T)$ is given as

• a `t_INT`, `t_FRAC` or `t_POL` (implicitly modulo T), or

• a `t_POLMOD` (modulo T), or

• a `t_COL v` of dimension $N = [K : \mathbf{Q}]$, representing the element in terms of the computed integral basis, as `sum(i = 1, N, v[i] * nf.zk[i])`. Note that a `t_VEC` will not be recognized.

An *ideal* is given in any of the following ways:

• an algebraic number in one of the above forms, defining a principal ideal.

• a prime ideal, i.e. a 5-component vector in the format output by `idealprimedec` or `idealfactor`.

• a `t_MAT`, square and in Hermite Normal Form (or at least upper triangular with non-negative coefficients), whose columns represent a $\mathbf{Z}$-basis of the ideal.

One may use `idealhnf` to convert any ideal to the last (preferred) format.

• an *extended ideal* is a 2-component vector $[I, t]$, where I is an ideal as above and t is an algebraic number, representing the ideal $(t)I$. This is useful whenever `idealred` is involved, implicitly working in the ideal class group, while keeping track of principal ideals. Ideal operations suitably update the principal part when it makes sense (in a multiplicative context), e.g. using `idealmul` on $[I, t]$, $[J, u]$, we obtain $[IJ, tu]$. When it does not make sense, the extended part is silently discarded, e.g. using `idealadd` with the above input produces $I + J$.

The "principal part" t in an extended ideal may be represented in any of the above forms, and *also* as a factorization matrix (in terms of number field elements, not ideals!), possibly the empty matrix `[;]` representing 1. In the latter case, elements stay in factored form, or *famat* for *fa*ctorization *mat*rix, which is a convenient way to avoid coefficient explosion. To recover the conventional expanded form, try `nffactorback`; but many functions already accept *famat*s as input, for instance `ideallog`, so expanding huge elements should never be necessary.

3.8.3 Finite abelian groups.

A finite abelian group G in user-readable format is given by its Smith Normal Form as a pair $[h, d]$ or triple $[h, d, g]$. Here h is the cardinality of G, (d_i) is the vector of elementary divisors, and (g_i) is a vector of generators. In short, $G = \oplus_{i \leq n}(\mathbf{Z}/d_i\mathbf{Z})g_i$, with $d_n \mid \ldots \mid d_2 \mid d_1$ and $\prod d_i = h$. This information can also be retrieved as G.`no`, G.`cyc` and G.`gen`.

• a *character* on the abelian group $\oplus(\mathbf{Z}/d_j\mathbf{Z})g_j$ is given by a row vector $\chi = [a_1, \ldots, a_n]$ such that $\chi(\prod g_j^{n_j}) = \exp(2\pi i \sum a_j n_j/d_j)$.

• given such a structure, a *subgroup* H is input as a square matrix in HNF, whose columns express generators of H on the given generators g_i. Note that the determinant of that matrix is equal to the index $(G : H)$.

3.8.4 Relative extensions.

We now have a look at data structures attached to relative extensions of number fields L/K, and to projective $\mathbf{Z}_K$-modules. When defining a relative extension L/K, the *nf* attached to the base field K must be defined by a variable having a lower priority (see Section 2.5.3) than the variable defining the extension. For example, you may use the variable name y to define the base field K, and x to define the relative extension L/K.

3.8.4.1 Basic definitions.

• *rnf* denotes a relative number field, i.e. a data structure output by `rnfinit`, attached to the extension L/K. The *nf* attached to be base field K is `rnf.nf`.

• A *relative matrix* is an $m \times n$ matrix whose entries are elements of K, in any form. Its m columns A_j represent elements in K^n.

• An *ideal list* is a row vector of fractional ideals of the number field *nf*.

• A *pseudo-matrix* is a 2-component row vector (A, I) where A is a relative $m \times n$ matrix and I an ideal list of length n. If $I = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ and the columns of A are $(A_1, \ldots, A_n)$, this data defines the torsion-free (projective) $\mathbf{Z}_K$-module $\mathfrak{a}_1 A_1 \oplus \mathfrak{a}_n A_n$.

• An *integral pseudo-matrix* is a 3-component row vector $\mathrm{w}(A, I, J)$ where $A = (a_{i,j})$ is an $m \times n$ relative matrix and $I = (\mathfrak{b}_1, \ldots, \mathfrak{b}_m)$, $J = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ are ideal lists, such that $a_{i,j} \in \mathfrak{b}_i \mathfrak{a}_j^{-1}$ for all i, j. This data defines two abstract projective $\mathbf{Z}_K$-modules $N = \mathfrak{a}_1\omega_1 \oplus \cdots \oplus \mathfrak{a}_n\omega_n$ in K^n, $P = \mathfrak{b}_1\eta_1 \oplus \cdots \oplus \mathfrak{b}_m\eta_m$ in K^m, and a $\mathbf{Z}_K$-linear map $f : N \to P$ given by

$$f(\sum \alpha_j \omega_j) = \sum_i \Big(a_{i,j}\alpha_j\Big)\eta_i.$$

This data defines the $\mathbf{Z}_K$-module $M = P/f(N)$.

• Any *projective* $\mathbf{Z}_K$-module M of finite type in K^m can be given by a pseudo matrix (A, I).

• An arbitrary $\mathbf{Z}_K$ modules of finite type in K^m, with non-trivial torsion, is given by an integral pseudo-matrix (A, I, J)

3.8.4.2 Pseudo-bases, determinant.

• The pair (A, I) is a *pseudo-basis* of the module it generates if the $\mathfrak{a}_j$ are non-zero, and the A_j are K-linearly independent. We call n the *size* of the pseudo-basis. If A is a relative matrix, the latter condition means it is square with non-zero determinant; we say that it is in Hermite Normal Form (HNF) if it is upper triangular and all the elements of the diagonal are equal to 1.

• For instance, the relative integer basis `rnf.zk` is a pseudo-basis (A, I) of $\mathbf{Z}_L$, where $A =$ `rnf.zk`[1] is a vector of elements of L, which are K-linearly independent. Most *rnf* routines return and handle $\mathbf{Z}_K$-modules contained in L (e.g. $\mathbf{Z}_L$-ideals) via a pseudo-basis (A', I'), where A' is a relative matrix representing a vector of elements of L in terms of the fixed basis `rnf.zk[1]`

• The *determinant* of a pseudo-basis (A, I) is the ideal equal to the product of the determinant of A by all the ideals of I. The determinant of a pseudo-matrix is the determinant of any pseudo-basis of the module it generates.

3.8.5 Class field theory.

A *modulus*, in the sense of class field theory, is a divisor supported on the non-complex places of K. In PARI terms, this means either an ordinary ideal I as above (no Archimedean component), or a pair $[I, a]$, where a is a vector with r_1 $\{0, 1\}$-components, corresponding to the infinite part of the divisor. More precisely, the i-th component of a corresponds to the real embedding attached to the i-th real root of `K.roots`. (That ordering is not canonical, but well defined once a defining polynomial for K is chosen.) For instance, `[1, [1,1]]` is a modulus for a real quadratic field, allowing ramification at any of the two places at infinity, and nowhere else.

A *bid* or "big ideal" is a structure output by `idealstar` needed to compute in $(\mathbf{Z}_K/I)^*$, where I is a modulus in the above sense. It is a finite abelian group as described above, supplemented by technical data needed to solve discrete log problems.

Finally we explain how to input ray number fields (or *bnr*), using class field theory. These are defined by a triple A, B, C, where the defining set $[A, B, C]$ can have any of the following forms: $[bnr]$, $[bnr, subgroup]$, $[bnr, character]$, $[bnf, mod]$, $[bnf, mod, subgroup]$. The last two forms are kept for backward compatibility, but no longer serve any real purpose (see example below); no newly written function will accept them.

• *bnf* is as output by `bnfinit`, where units are mandatory unless the modulus is trivial; *bnr* is as output by `bnrinit`. This is the ground field K.

• *mod* is a modulus $\mathfrak{f}$, as described above.

• *subgroup* a subgroup of the ray class group modulo $\mathfrak{f}$ of K. As described above, this is input as a square matrix expressing generators of a subgroup of the ray class group *bnr*`.clgp` on the given generators.

The corresponding *bnr* is the subfield of the ray class field of K modulo $\mathfrak{f}$, fixed by the given subgroup.

```
? K = bnfinit(y^2+1);
? bnr = bnrinit(K, 13)
? %.clgp
%3 = [36, [12, 3]]
? bnrdisc(bnr); \\ discriminant of the full ray class field
? bnrdisc(bnr, [3,1;0,1]); \\ discriminant of cyclic cubic extension of K
? bnrconductor(bnr, [3,1]); \\ conductor of chi: g1->zeta_12^3, g2->zeta_3
```

We could have written directly

```
? bnrdisc(K, 13);
? bnrdisc(K, 13, [3,1;0,1]);
```

avoiding one `bnrinit`, but this would actually be slower since the `bnrinit` is called internally anyway. And now twice!

3.8.6 General use.

All the functions which are specific to relative extensions, number fields, Buchmann's number fields, Buchmann's number rays, share the prefix `rnf`, `nf`, `bnf`, `bnr` respectively. They take as first argument a number field of that precise type, respectively output by `rnfinit`, `nfinit`, `bnfinit`, and `bnrinit`.

However, and even though it may not be specified in the descriptions of the functions below, it is permissible, if the function expects a *nf*, to use a *bnf* instead, which contains much more information. On the other hand, if the function requires a `bnf`, it will *not* launch `bnfinit` for you, which is a costly operation. Instead, it will give you a specific error message. In short, the types

$$\texttt{nf} \leq \texttt{bnf} \leq \texttt{bnr}$$

are ordered, each function requires a minimal type to work properly, but you may always substitute a larger type.

The data types corresponding to the structures described above are rather complicated. Thus, as we already have seen it with elliptic curves, GP provides "member functions" to retrieve data from these structures (once they have been initialized of course). The relevant types of number fields are indicated between parentheses:

`bid` (*bnr*) : bid ideal structure.
`bnf` (*bnr*, *bnf*) : Buchmann's number field.
`clgp` (*bnr*, *bnf*) : classgroup. This one admits the following three subclasses:
  `cyc` : cyclic decomposition (SNF).
  `gen` : generators.

`no`		: number of elements.
`diff`	(*bnr*, *bnf*, *nf*)	: the different ideal.
`codiff`	(*bnr*, *bnf*, *nf*)	: the codifferent (inverse of the different in the ideal group).
`disc`	(*bnr*, *bnf*, *nf*)	: discriminant.
`fu`	(*bnr*, *bnf*)	: fundamental units.
`index`	(*bnr*, *bnf*, *nf*)	: index of the power order in the ring of integers.
`mod`	(*bnr*)	: modulus.
`nf`	(*bnr*, *bnf*, *nf*)	: number field.
`pol`	(*bnr*, *bnf*, *nf*)	: defining polynomial.
`r1`	(*bnr*, *bnf*, *nf*)	: the number of real embeddings.
`r2`	(*bnr*, *bnf*, *nf*)	: the number of pairs of complex embeddings.
`reg`	(*bnr*, *bnf*)	: regulator.
`roots`	(*bnr*, *bnf*, *nf*)	: roots of the polynomial generating the field.
`sign`	(*bnr*, *bnf*, *nf*)	: signature $[r1, r2]$.
`t2`	(*bnr*, *bnf*, *nf*)	: the T_2 matrix (see `nfinit`).
`tu`	(*bnr*, *bnf*)	: a generator for the torsion units.
`zk`	(*bnr*, *bnf*, *nf*)	: integral basis, i.e. a $\mathbf{Z}$-basis of the maximal order.
`zkst`	(*bnr*)	: structure of $(\mathbf{Z}_K/m)^*$.

Deprecated. The following member functions are still available, but deprecated and should not be used in new scripts :

`futu`	(*bnr*, *bnf*,)	: $[u_1, ..., u_r, w]$, (u_i) is a vector of fundamental units, w generates the torsion units.
`tufu`	(*bnr*, *bnf*,)	: $[w, u_1, ..., u_r]$, (u_i) is a vector of fundamental units, w generates the torsion units.

For instance, assume that *bnf* = `bnfinit`(*pol*), for some polynomial. Then *bnf*`.clgp` retrieves the class group, and *bnf*`.clgp.no` the class number. If we had set *bnf* = `nfinit`(*pol*), both would have output an error message. All these functions are completely recursive, thus for instance *bnr*`.bnf.nf.zk` will yield the maximal order of *bnr*, which you could get directly with a simple *bnr*`.zk`.

3.8.7 Class group, units, and the GRH.

Some of the functions starting with `bnf` are implementations of the sub-exponential algorithms for finding class and unit groups under GRH, due to Hafner-McCurley, Buchmann and Cohen-Diaz-Olivier. The general call to the functions concerning class groups of general number fields (i.e. excluding `quadclassunit`) involves a polynomial P and a technical vector

$$tech = [c_1, c_2, nrpid],$$

where the parameters are to be understood as follows:

P is the defining polynomial for the number field, which must be in $\mathbf{Z}[X]$, irreducible and monic. In fact, if you supply a non-monic polynomial at this point, `gp` issues a warning, then *transforms your polynomial* so that it becomes monic. The `nfinit` routine will return a different result in this case: instead of `res`, you get a vector `[res,Mod(a,Q)]`, where `Mod(a,Q) = Mod(X,P)` gives the change of variables. In all other routines, the variable change is simply lost.

The *tech* interface is obsolete and you should not tamper with these parameters. Indeed, from version 2.4.0 on,

• the results are always rigorous under GRH (before that version, they relied on a heuristic strengthening, hence the need for overrides).

• the influence of these parameters on execution time and stack size is marginal. They *can* be useful to fine-tune and experiment with the `bnfinit` code, but you will be better off modifying all tuning parameters in the C code (there are many more than just those three). We nevertheless describe it for completeness.

The numbers $c_1 \leq c_2$ are non-negative real numbers. By default they are chosen so that the result is correct under GRH. For $i = 1, 2$, let $B_i = c_i(\log|d_K|)^2$, and denote by $S(B)$ the set of maximal ideals of K whose norm is less than B. We want $S(B_1)$ to generate $\mathrm{Cl}(K)$ and hope that $S(B_2)$ can be *proven* to generate $\mathrm{Cl}(K)$.

More precisely, $S(B_1)$ is a factorbase used to compute a tentative $\mathrm{Cl}(K)$ by generators and relations. We then check explicitly, using essentially `bnfisprincipal`, that the elements of $S(B_2)$ belong to the span of $S(B_1)$. Under the assumption that $S(B_2)$ generates $\mathrm{Cl}(K)$, we are done. User-supplied c_i are only used to compute initial guesses for the bounds B_i, and the algorithm increases them until one can *prove* under GRH that $S(B_2)$ generates $\mathrm{Cl}(K)$. A uniform result of Bach says that $c_2 = 12$ is always suitable, but this bound is very pessimistic and a direct algorithm due to Belabas-Diaz-Friedman is used to check the condition, assuming GRH. The default values are $c_1 = c_2 = 0$. When c_1 is equal to 0 the algorithm takes it equal to c_2.

nrpid is the maximal number of small norm relations attached to each ideal in the factor base. Set it to 0 to disable the search for small norm relations. Otherwise, reasonable values are between 4 and 20. The default is 4.

Warning. Make sure you understand the above! By default, most of the `bnf` routines depend on the correctness of the GRH. In particular, any of the class number, class group structure, class group generators, regulator and fundamental units may be wrong, independently of each other. Any result computed from such a `bnf` may be wrong. The only guarantee is that the units given generate a subgroup of finite index in the full unit group. You must use `bnfcertify` to certify the computations unconditionally.

Remarks.

You do not need to supply the technical parameters (under the library you still need to send at least an empty vector, coded as `NULL`). However, should you choose to set some of them, they *must* be given in the requested order. For example, if you want to specify a given value of *nrpid*, you must give some values as well for c_1 and c_2, and provide a vector $[c_1, c_2, nrpid]$.

Note also that you can use an *nf* instead of P, which avoids recomputing the integral basis and analogous quantities.

3.8.8 bnfcertify$(bnf, \{flag = 0\})$. *bnf* being as output by `bnfinit`, checks whether the result is correct, i.e. whether it is possible to remove the assumption of the Generalized Riemann Hypothesis. It is correct if and only if the answer is 1. If it is incorrect, the program may output some error message, or loop indefinitely. You can check its progress by increasing the debug level. The *bnf* structure must contain the fundamental units:

```
? K = bnfinit(x^3+2^2^3+1); bnfcertify(K)
  ***   at top-level: K=bnfinit(x^3+2^2^3+1);bnfcertify(K)
  ***                                        ^-------------
  *** bnfcertify: missing units in bnf.
```

```
? K = bnfinit(x^3+2^2^3+1, 1); \\ include units
? bnfcertify(K)
%3 = 1
```

If flag is present, only certify that the class group is a quotient of the one computed in bnf (much simpler in general); likewise, the computed units may form a subgroup of the full unit group. In this variant, the units are no longer needed:

```
? K = bnfinit(x^3+2^2^3+1); bnfcertify(K, 1)
%4 = 1
```

The library syntax is `long bnfcertify0(GEN bnf, long flag)`. Also available is `GEN bnfcertify(GEN bnf)` (*flag* = 0).

3.8.9 bnfcompress(*bnf*). Computes a compressed version of *bnf* (from `bnfinit`), a "small Buchmann's number field" (or *sbnf* for short) which contains enough information to recover a full *bnf* vector very rapidly, but which is much smaller and hence easy to store and print. Calling `bnfinit` on the result recovers a true `bnf`, in general different from the original. Note that an *snbf* is useless for almost all purposes besides storage, and must be converted back to *bnf* form before use; for instance, no `nf*`, `bnf*` or member function accepts them.

An *sbnf* is a 12 component vector v, as follows. Let `bnf` be the result of a full `bnfinit`, complete with units. Then $v[1]$ is `bnf.pol`, $v[2]$ is the number of real embeddings `bnf.sign[1]`, $v[3]$ is `bnf.disc`, $v[4]$ is `bnf.zk`, $v[5]$ is the list of roots `bnf.roots`, $v[7]$ is the matrix `W` = `bnf`[1], $v[8]$ is the matrix `matalpha` = `bnf`[2], $v[9]$ is the prime ideal factor base `bnf[5]` coded in a compact way, and ordered according to the permutation `bnf[6]`, $v[10]$ is the 2-component vector giving the number of roots of unity and a generator, expressed on the integral basis, $v[11]$ is the list of fundamental units, expressed on the integral basis, $v[12]$ is a vector containing the algebraic numbers alpha corresponding to the columns of the matrix `matalpha`, expressed on the integral basis.

All the components are exact (integral or rational), except for the roots in $v[5]$.

The library syntax is `GEN bnfcompress(GEN bnf)`.

3.8.10 bnfdecodemodule(nf, m). If m is a module as output in the first component of an extension given by `bnrdisclist`, outputs the true module.

```
? K = bnfinit(x^2+23); L = bnrdisclist(K, 10); s = L[1][2]
%1 = [[Mat([8, 1]), [[0, 0, 0]]], [Mat([9, 1]), [[0, 0, 0]]]]
? bnfdecodemodule(K, s[1][1])
%2 =
[2 0]

[0 1]
```

The library syntax is `GEN decodemodule(GEN nf, GEN m)`.

3.8.11 bnfinit$(P, \{flag = 0\}, \{tech = []\})$. Initializes a `bnf` structure. Used in programs such as `bnfisprincipal`, `bnfisunit` or `bnfnarrow`. By default, the results are conditional on the GRH, see 3.8.7. The result is a 10-component vector *bnf*.

This implements Buchmann's sub-exponential algorithm for computing the class group, the regulator and a system of fundamental units of the general algebraic number field K defined by the irreducible polynomial P with integer coefficients.

If the precision becomes insufficient, `gp` does not strive to compute the units by default ($flag = 0$).

When $flag = 1$, we insist on finding the fundamental units exactly. Be warned that this can take a very long time when the coefficients of the fundamental units on the integral basis are very large. If the fundamental units are simply too large to be represented in this form, an error message is issued. They could be obtained using the so-called compact representation of algebraic numbers as a formal product of algebraic integers. The latter is implemented internally but not publicly accessible yet.

tech is a technical vector (empty by default, see 3.8.7). Careful use of this parameter may speed up your computations, but it is mostly obsolete and you should leave it alone.

The components of a *bnf* or *sbnf* are technical and never used by the casual user. In fact: *never access a component directly, always use a proper member function.* However, for the sake of completeness and internal documentation, their description is as follows. We use the notations explained in the book by H. Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Maths **138**, Springer-Verlag, 1993, Section 6.5, and subsection 6.5.5 in particular.

bnf[1] contains the matrix W, i.e. the matrix in Hermite normal form giving relations for the class group on prime ideal generators $(\mathfrak{p}_i)_{1\le i\le r}$.

bnf[2] contains the matrix B, i.e. the matrix containing the expressions of the prime ideal factorbase in terms of the $\mathfrak{p}_i$. It is an $r \times c$ matrix.

bnf[3] contains the complex logarithmic embeddings of the system of fundamental units which has been found. It is an $(r_1 + r_2) \times (r_1 + r_2 - 1)$ matrix.

bnf[4] contains the matrix M''_C of Archimedean components of the relations of the matrix $(W|B)$.

bnf[5] contains the prime factor base, i.e. the list of prime ideals used in finding the relations.

bnf[6] used to contain a permutation of the prime factor base, but has been obsoleted. It contains a dummy 0.

bnf[7] or *bnf*`.nf` is equal to the number field data *nf* as would be given by `nfinit`.

bnf[8] is a vector containing the classgroup *bnf*`.clgp` as a finite abelian group, the regulator *bnf*`.reg`, a 1 (used to contain an obsolete "check number"), the number of roots of unity and a generator *bnf*`.tu`, the fundamental units *bnf*`.fu`.

bnf[9] is a 3-element row vector used in `bnfisprincipal` only and obtained as follows. Let $D = UWV$ obtained by applying the Smith normal form algorithm to the matrix W (= *bnf*[1]) and let U_r be the reduction of U modulo D. The first elements of the factorbase are given (in terms of `bnf.gen`) by the columns of U_r, with Archimedean component g_a; let also GD_a be the Archimedean components of the generators of the (principal) ideals defined by the `bnf.gen[i]^ bnf.cyc[i]`. Then *bnf*[9] $= [U_r, g_a, GD_a]$.

bnf[10] is by default unused and set equal to 0. This field is used to store further information about the field as it becomes available, which is rarely needed, hence would be too expensive to compute during the initial `bnfinit` call. For instance, the generators of the principal ideals `bnf.gen[i]^bnf.cyc[i]` (during a call to `bnrisprincipal`), or those corresponding to the relations in W and B (when the `bnf` internal precision needs to be increased).

The library syntax is `GEN bnfinit0(GEN P, long flag, GEN tech = NULL, long prec)`.

Also available is `GEN Buchall(GEN P, long flag, long prec)`, corresponding to `tech = NULL`, where `flag` is either 0 (default) or `nf_FORCE` (insist on finding fundamental units). The function `GEN Buchall_param(GEN P, double c1, double c2, long nrpid, long flag, long prec)` gives direct access to the technical parameters.

3.8.12 bnfisintnorm(bnf, x). Computes a complete system of solutions (modulo units of positive norm) of the absolute norm equation $\mathrm{Norm}(a) = x$, where a is an integer in *bnf*. If *bnf* has not been certified, the correctness of the result depends on the validity of GRH.

See also `bnfisnorm`.

The library syntax is `GEN bnfisintnorm(GEN bnf, GEN x)`. The function `GEN bnfisintnormabs(GEN bnf, GEN a)` returns a complete system of solutions modulo units of the absolute norm equation $|\mathrm{Norm}(x)| = |a|$. As fast as `bnfisintnorm`, but solves the two equations $\mathrm{Norm}(x) = \pm a$ simultaneously.

3.8.13 bnfisnorm$(bnf, x, \{flag = 1\})$. Tries to tell whether the rational number x is the norm of some element y in *bnf*. Returns a vector $[a, b]$ where $x = Norm(a) * b$. Looks for a solution which is an S-unit, with S a certain set of prime ideals containing (among others) all primes dividing x. If *bnf* is known to be Galois, set *flag* $= 0$ (in this case, x is a norm iff $b = 1$). If *flag* is non zero the program adds to S the following prime ideals, depending on the sign of *flag*. If *flag* > 0, the ideals of norm less than *flag*. And if *flag* < 0 the ideals dividing *flag*.

Assuming GRH, the answer is guaranteed (i.e. x is a norm iff $b = 1$), if S contains all primes less than $12\log(\mathrm{disc}(Bnf))^2$, where *Bnf* is the Galois closure of *bnf*.

See also `bnfisintnorm`.

The library syntax is `GEN bnfisnorm(GEN bnf, GEN x, long flag)`.

3.8.14 bnfisprincipal$(bnf, x, \{flag = 1\})$. *bnf* being the number field data output by `bnfinit`, and x being an ideal, this function tests whether the ideal is principal or not. The result is more complete than a simple true/false answer and solves general discrete logarithm problem. Assume the class group is $\oplus(\mathbf{Z}/d_i\mathbf{Z})g_i$ (where the generators g_i and their orders d_i are respectively given by `bnf.gen` and `bnf.cyc`). The routine returns a row vector $[e, t]$, where e is a vector of exponents $0 \le e_i < d_i$, and t is a number field element such that

$$x = (t)\prod_i g_i^{e_i}.$$

For *given* g_i (i.e. for a given `bnf`), the e_i are unique, and t is unique modulo units.

In particular, x is principal if and only if e is the zero vector. Note that the empty vector, which is returned when the class number is 1, is considered to be a zero vector (of dimension 0).

```
? K = bnfinit(y^2+23);
? K.cyc
%2 = [3]
? K.gen
%3 = [[2, 0; 0, 1]]          \\ a prime ideal above 2
? P = idealprimedec(K,3)[1]; \\ a prime ideal above 3
? v = bnfisprincipal(K, P)
%5 = [[2]~, [3/4, 1/4]~]
? idealmul(K, v[2], idealfactorback(K, K.gen, v[1]))
%6 =
[3 0]

[0 1]
? % == idealhnf(K, P)
%7 = 1
```

The binary digits of *flag* mean:

• 1: If set, outputs $[e, t]$ as explained above, otherwise returns only e, which is much easier to compute. The following idiom only tests whether an ideal is principal:

```
is_principal(bnf, x) = !bnfisprincipal(bnf,x,0);
```

• 2: It may not be possible to recover t, given the initial accuracy to which the `bnf` structure was computed. In that case, a warning is printed and t is set equal to the empty vector `[]~`. If this bit is set, increase the precision and recompute needed quantities until t can be computed. Warning: setting this may induce *lengthy* computations.

The library syntax is `GEN bnfisprincipal0(GEN bnf, GEN x, long flag)`. Instead of the above hardcoded numerical flags, one should rather use an or-ed combination of the symbolic flags `nf_GEN` (include generators, possibly a place holder if too difficult) and `nf_FORCE` (insist on finding the generators).

3.8.15 bnfissunit(bnf, sfu, x). *bnf* being output by `bnfinit`, *sfu* by `bnfsunit`, gives the column vector of exponents of x on the fundamental S-units and the roots of unity. If x is not a unit, outputs an empty vector.

The library syntax is `GEN bnfissunit(GEN bnf, GEN sfu, GEN x)`.

3.8.16 bnfisunit(bnf, x). *bnf* being the number field data output by `bnfinit` and x being an algebraic number (type integer, rational or polmod), this outputs the decomposition of x on the fundamental units and the roots of unity if x is a unit, the empty vector otherwise. More precisely, if $u_1, \ldots, u_r$ are the fundamental units, and ζ is the generator of the group of roots of unity (`bnf.tu`), the output is a vector $[x_1, \ldots, x_r, x_{r+1}]$ such that $x = u_1^{x_1} \cdots u_r^{x_r} \cdot \zeta^{x_{r+1}}$. The x_i are integers for $i \leq r$ and is an integer modulo the order of ζ for $i = r + 1$.

Note that *bnf* need not contain the fundamental unit explicitly:

```
? setrand(1); bnf = bnfinit(x^2-x-100000);
? bnf.fu
  ***   at top-level: bnf.fu
  ***                       ^--
  *** _.fu: missing units in .fu.
? u = [11983616564425078999046283595002287166517812761131616131167, \
```

```
        379554884019013781006303254896369154068336082609238336]~;
? bnfisunit(bnf, u)
%3 = [-1, Mod(0, 2)]~
```

The given u is the inverse of the fundamental unit implicitly stored in *bnf*. In this case, the fundamental unit was not computed and stored in algebraic form since the default accuracy was too low. (Re-run the command at "g1 or higher to see such diagnostics.)

The library syntax is `GEN bnfisunit(GEN bnf, GEN x)`.

3.8.17 bnflog(bnf, l). Let *bnf* be attached to a number field F and let l be a prime number (hereafter denoted ℓ for typographical reasons). Return the logarithmic ℓ-class group $\widetilde{Cl}_F$ of F. This is an abelian group, conjecturally finite (known to be finite if $F/\mathbf{Q}$ is abelian). The function returns if and only if the group is indeed finite (otherwise it would run into an infinite loop). Let $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ be the set of ℓ-adic places (maximal ideals containing ℓ). The function returns $[D, G(\ell), G']$, where

- D is the vector of elementary divisors for $\widetilde{Cl}_F$;
- $G(\ell)$ is the vector of elementary divisors for the (conjecturally finite) abelian group

$$\widetilde{\mathrm{Cl}}(\ell) = \{\mathfrak{a} = \sum_{i \le k} a_i \mathfrak{p}_i : \ \deg_F \mathfrak{a} = 0\},$$

where the $\mathfrak{p}_i$ are the ℓ-adic places of F; this is a subgroup of $\widetilde{\mathrm{Cl}}$.

- G' is the vector of elementary divisors for the ℓ-Sylow Cl' of the S-class group of F; the group $\widetilde{\mathrm{Cl}}$ maps to Cl' with a simple co-kernel.

The library syntax is `GEN bnflog(GEN bnf, GEN l)`.

3.8.18 bnflogdegree(nf, A, l). Let *nf* be the number field data output by `nfinit`, attached to the field F, and let l be a prime number (hereafter denoted ℓ). The ℓ-adified group of idèles of F quotiented by the group of logarithmic units is identified to the ℓ-group of logarithmic divisors $\oplus \mathbf{Z}_\ell[\mathfrak{p}]$, generated by the maximal ideals of F.

The *degree* map $\deg_F$ is additive with values in $\mathbf{Z}_\ell$, defined by $\deg_F \mathfrak{p} = \tilde{f}_\mathfrak{p} \deg_\ell p$, where the integer $\tilde{f}$ is as in `bnflogef` and $\deg_\ell p$ is $\log_\ell p$ for $p \neq \ell$, $\log_\ell(1+\ell)$ for $p = \ell \neq 2$ and $\log_\ell(1+2^2)$ for $p = \ell = 2$.

Let $A = \prod \mathfrak{p}^{n_\mathfrak{p}}$ be an ideal and let $\tilde{A} = \sum n_\mathfrak{p}[\mathfrak{p}]$ be the attached logarithmic divisor. Return the exponential of the ℓ-adic logarithmic degree $\deg_F A$, which is a natural number.

The library syntax is `GEN bnflogdegree(GEN nf, GEN A, GEN l)`.

3.8.19 bnflogef(nf, pr). Let F be a number field represented by the *nf* structure, and let *pr* be a `prid` structure attached to the maximal ideal $\mathfrak{p}/p$. Return $[\tilde{e}(F_\mathfrak{p}/\mathbf{Q}_p), \tilde{f}(F_\mathfrak{p}/\mathbf{Q}_p)]$ the logarithmic ramification and residue degrees. Let $\mathbf{Q}_p^c/\mathbf{Q}_p$ be the cyclotomic $\mathbf{Z}_p$-extension, then $\tilde{e} = [F_\mathfrak{p}\colon F_\mathfrak{p} \cap \mathbf{Q}_p^c]$ $\tilde{f} = [F_\mathfrak{p} \cap \mathbf{Q}_p^c\colon \mathbf{Q}_p]$. Note that $\tilde{e}\tilde{f} = e(\mathfrak{p}/p)f(\mathfrak{p}/p)$, where e, f denote the usual ramification and residue degrees.

```
? F = nfinit(y^6 - 3*y^5 + 5*y^3 - 3*y + 1);
? bnflogef(F, idealprimedec(F,2)[1])
%2 = [6, 1]
? bnflogef(F, idealprimedec(F,5)[1])
%3 = [1, 2]
```

The library syntax is `GEN bnflogef(GEN nf, GEN pr)`.

3.8.20 bnfnarrow(bnf). *bnf* being as output by `bnfinit`, computes the narrow class group of *bnf*. The output is a 3-component row vector v analogous to the corresponding class group component *bnf*`.clgp`: the first component is the narrow class number v`.no`, the second component is a vector containing the SNF cyclic components v`.cyc` of the narrow class group, and the third is a vector giving the generators of the corresponding v`.gen` cyclic groups. Note that this function is a special case of `bnrinit`; the *bnf* need not contain fundamental units.

The library syntax is `GEN buchnarrow(GEN bnf)`.

3.8.21 bnfsignunit(bnf). *bnf* being as output by `bnfinit`, this computes an $r_1 \times (r_1 + r_2 - 1)$ matrix having ± 1 components, giving the signs of the real embeddings of the fundamental units. The following functions compute generators for the totally positive units:

```
/* exponents of totally positive units generators on bnf.tufu */
tpuexpo(bnf)=
{ my(K, S = bnfsignunit(bnf), [m,n] = matsize(S));
  \\ m = bnf.r1, n = r1+r2-1
  S = matrix(m,n, i,j, if (S[i,j] < 0, 1,0));
  S = concat(vectorv(m,i,1), S);   \\ add sign(-1)
  K = matker(S * Mod(1,2));
  if (K, mathnfmodid(lift(K), 2), 2*matid(n+1))
}

/* totally positive fundamental units */
tpu(bnf)=
{ my(ex = tpuexpo(bnf)[,2..-1]); \\ remove ex[,1], corresponds to 1 or -1
  vector(#ex, i, nffactorback(bnf, bnf.tufu, ex[,i]));
}
```

The library syntax is `GEN signunits(GEN bnf)`.

3.8.22 bnfsunit(bnf, S). Computes the fundamental S-units of the number field *bnf* (output by `bnfinit`), where S is a list of prime ideals (output by `idealprimedec`). The output is a vector v with 6 components.

$v[1]$ gives a minimal system of (integral) generators of the S-unit group modulo the unit group.

$v[2]$ contains technical data needed by `bnfissunit`.

$v[3]$ is an empty vector (used to give the logarithmic embeddings of the generators in $v[1]$ in version 2.0.16).

$v[4]$ is the S-regulator (this is the product of the regulator, the determinant of $v[2]$ and the natural logarithms of the norms of the ideals in S).

$v[5]$ gives the S-class group structure, in the usual format (a row vector whose three components give in order the S-class number, the cyclic components and the generators).

$v[6]$ is a copy of S.

The library syntax is `GEN bnfsunit(GEN bnf, GEN S, long prec)`.

3.8.23 bnrL1$(bnr, \{H\}, \{flag = 0\})$. Let *bnr* be the number field data output by `bnrinit(,,1)` and H be a square matrix defining a congruence subgroup of the ray class group corresponding to *bnr* (the trivial congruence subgroup if omitted). This function returns, for each character χ of the ray class group which is trivial on H, the value at $s = 1$ (or $s = 0$) of the abelian L-function attached to χ. For the value at $s = 0$, the function returns in fact for each χ a vector $[r_\chi, c_\chi]$ where

$$L(s, \chi) = c \cdot s^r + O(s^{r+1})$$

near 0.

The argument *flag* is optional, its binary digits mean 1: compute at $s = 0$ if unset or $s = 1$ if set, 2: compute the primitive L-function attached to χ if unset or the L-function with Euler factors at prime ideals dividing the modulus of *bnr* removed if set (that is $L_S(s, \chi)$, where S is the set of infinite places of the number field together with the finite prime ideals dividing the modulus of *bnr*), 3: return also the character if set.

```
K = bnfinit(x^2-229);
bnr = bnrinit(K,1,1);
bnrL1(bnr)
```

returns the order and the first non-zero term of $L(s, \chi)$ at $s = 0$ where χ runs through the characters of the class group of $K = \mathbf{Q}(\sqrt{229})$. Then

```
bnr2 = bnrinit(K,2,1);
bnrL1(bnr2,,2)
```

returns the order and the first non-zero terms of $L_S(s, \chi)$ at $s = 0$ where χ runs through the characters of the class group of K and S is the set of infinite places of K together with the finite prime 2. Note that the ray class group modulo 2 is in fact the class group, so `bnrL1(bnr2,0)` returns the same answer as `bnrL1(bnr,0)`.

This function will fail with the message

```
*** bnrL1: overflow in zeta_get_N0 [need too many primes].
```

if the approximate functional equation requires us to sum too many terms (if the discriminant of K is too large).

The library syntax is `GEN bnrL1(GEN bnr, GEN H = NULL, long flag, long prec)`.

3.8.24 bnrchar$(bnr, g, \{v\})$. Returns all characters χ on `bnr.clgp` such that $\chi(g_i) = e(v_i)$, where $e(x) = \exp(2i\pi x)$. If v is omitted, returns all characters that are trivial on the g_i. Else the vectors g and v must have the same length, the g_i must be ideals in any form, and each v_i is a rational number whose denominator must divide the order of g_i in the ray class group. For convenience, the vector of the g_i can be replaced by a matrix whose columns give their discrete logarithm, as given by `bnrisprincipal`; this allows to specify abstractly a subgroup of the ray class group.

```
? bnr = bnrinit(bnfinit(x), [160,[1]], 1); /* (Z/160Z)^* */
? bnr.cyc
%2 = [8, 4, 2]
? g = bnr.gen;
? bnrchar(bnr, g, [1/2,0,0])
%4 = [[4, 0, 0]]  \\ a unique character
? bnrchar(bnr, [g[1],g[3]]) \\ all characters trivial on g[1] and g[3]
%5 = [[0, 1, 0], [0, 2, 0], [0, 3, 0], [0, 0, 0]]
? bnrchar(bnr, [1,0,0;0,1,0;0,0,2])
%6 = [[0, 0, 1], [0, 0, 0]]  \\ characters trivial on given subgroup
```

The library syntax is `GEN bnrchar(GEN bnr, GEN g, GEN v = NULL)`.

3.8.25 bnrclassno$(A, \{B\}, \{C\})$. Let A, B, C define a class field L over a ground field K (of type `[`*bnr*`]`, `[`*bnr*`,` *subgroup*`]`, or `[`*bnf*`,` *modulus*`]`, or `[`*bnf*`,` *modulus*`,`*subgroup*`]`, Section 3.8.5); this function returns the relative degree $[L : K]$.

In particular if A is a *bnf* (with units), and B a modulus, this function returns the corresponding ray class number modulo B. One can input the attached *bid* (with generators if the subgroup C is non trivial) for B instead of the module itself, saving some time.

This function is faster than `bnrinit` and should be used if only the ray class number is desired. See `bnrclassnolist` if you need ray class numbers for all moduli less than some bound.

The library syntax is `GEN bnrclassno0(GEN A, GEN B = NULL, GEN C = NULL)`. Also available is `GEN bnrclassno(GEN bnf, GEN f)` to compute the ray class number modulo f.

3.8.26 bnrclassnolist$(bnf, list)$. *bnf* being as output by `bnfinit`, and *list* being a list of moduli (with units) as output by `ideallist` or `ideallistarch`, outputs the list of the class numbers of the corresponding ray class groups. To compute a single class number, `bnrclassno` is more efficient.

```
? bnf = bnfinit(x^2 - 2);
? L = ideallist(bnf, 100, 2);
? H = bnrclassnolist(bnf, L);
? H[98]
%4 = [1, 3, 1]
? l = L[1][98]; ids = vector(#l, i, l[i].mod[1])
%5 = [[98, 88; 0, 1], [14, 0; 0, 7], [98, 10; 0, 1]]
```

The weird `l[i].mod[1]`, is the first component of `l[i].mod`, i.e. the finite part of the conductor. (This is cosmetic: since by construction the Archimedean part is trivial, I do not want to see it). This tells us that the ray class groups modulo the ideals of norm 98 (printed as `%5`) have respectively order 1, 3 and 1. Indeed, we may check directly:

```
? bnrclassno(bnf, ids[2])
%6 = 3
```

The library syntax is `GEN bnrclassnolist(GEN bnf, GEN list)`.

3.8.27 bnrconductor$(A, \{B\}, \{C\}, \{flag = 0\})$. Conductor f of the subfield of a ray class field as defined by $[A, B, C]$ (of type `[`*bnr*`]`, `[`*bnr*`,` *subgroup*`]`, `[`*bnf*`,` *modulus*`]` or `[`*bnf*`,` *modulus*`,` *subgroup*`]`, Section 3.8.5)

If *flag* $= 0$, returns f.

If *flag* $= 1$, returns $[f, Cl_f, H]$, where Cl_f is the ray class group modulo f, as a finite abelian group; finally H is the subgroup of Cl_f defining the extension.

If *flag* $= 2$, returns $[f, bnr(f), H]$, as above except Cl_f is replaced by a `bnr` structure, as output by `bnrinit`$(, f, 1)$.

In place of a subgroup H, this function also accepts a character `chi` $= (a_j)$, expressed as usual in terms of the generators `bnr.gen`: $\chi(g_j) = \exp(2i\pi a_j/d_j)$, where g_j has order $d_j =$ `bnr.cyc[j]`. In which case, the function returns respectively

If *flag* $= 0$, the conductor f of Kerχ.

If *flag* $= 1$, $[f, Cl_f, \chi_f]$, where χ_f is χ expressed on the minimal ray class group, whose modulus is the conductor.

If *flag* $= 2$, $[f, bnr(f), \chi_f]$.

The library syntax is `GEN bnrconductor0(GEN A, GEN B = NULL, GEN C = NULL, long flag)`.

Also available is `GEN bnrconductor(GEN bnr, GEN H, long flag)`

3.8.28 bnrconductorofchar(bnr, chi). This function is obsolete, use *bnrconductor*.

The library syntax is `GEN bnrconductorofchar(GEN bnr, GEN chi)`.

3.8.29 bnrdisc$(A, \{B\}, \{C\}, \{flag = 0\})$. A, B, C defining a class field L over a ground field K (of type `[`*bnr*`]`, `[`*bnr*`,` *subgroup*`]`, `[`*bnr*`,` *character*`]`, `[`*bnf*`,` *modulus*`]` or `[`*bnf*`,` *modulus*`,` *subgroup*`]`, Section 3.8.5), outputs data $[N, r_1, D]$ giving the discriminant and signature of L, depending on the binary digits of *flag*:

- 1: if this bit is unset, output absolute data related to $L/\mathbf{Q}$: N is the absolute degree $[L : \mathbf{Q}]$, r_1 the number of real places of L, and D the discriminant of $L/\mathbf{Q}$. Otherwise, output relative data for L/K: N is the relative degree $[L : K]$, r_1 is the number of real places of K unramified in L (so that the number of real places of L is equal to r_1 times N), and D is the relative discriminant ideal of L/K.

- 2: if this bit is set and if the modulus is not the conductor of L, only return 0.

The library syntax is `GEN bnrdisc0(GEN A, GEN B = NULL, GEN C = NULL, long flag)`.

3.8.30 bnrdisclist($bnf, bound, \{arch\}$). *bnf* being as output by `bnfinit` (with units), computes a list of discriminants of Abelian extensions of the number field by increasing modulus norm up to bound *bound*. The ramified Archimedean places are given by *arch*; all possible values are taken if *arch* is omitted.

The alternative syntax `bnrdisclist`($bnf, list$) is supported, where *list* is as output by `idealliststarch` (with units), in which case *arch* is disregarded.

The output v is a vector of vectors, where $v[i][j]$ is understood to be in fact $V[2^{15}(i-1)+j]$ of a unique big vector V. (This awkward scheme allows for larger vectors than could be otherwise represented.)

$V[k]$ is itself a vector W, whose length is the number of ideals of norm k. We consider first the case where *arch* was specified. Each component of W corresponds to an ideal m of norm k, and gives invariants attached to the ray class field L of *bnf* of conductor $[m, arch]$. Namely, each contains a vector $[m, d, r, D]$ with the following meaning: m is the prime ideal factorization of the modulus, $d = [L : \mathbf{Q}]$ is the absolute degree of L, r is the number of real places of L, and D is the factorization of its absolute discriminant. We set $d = r = D = 0$ if m is not the finite part of a conductor.

If *arch* was omitted, all $t = 2^{r_1}$ possible values are taken and a component of W has the form $[m, [[d_1, r_1, D_1], \ldots, [d_t, r_t, D_t]]]$, where m is the finite part of the conductor as above, and $[d_i, r_i, D_i]$ are the invariants of the ray class field of conductor $[m, v_i]$, where v_i is the i-th Archimedean component, ordered by inverse lexicographic order; so $v_1 = [0, \ldots, 0]$, $v_2 = [1, 0 \ldots, 0]$, etc. Again, we set $d_i = r_i = D_i = 0$ if $[m, v_i]$ is not a conductor.

Finally, each prime ideal $pr = [p, \alpha, e, f, \beta]$ in the prime factorization m is coded as the integer $p \cdot n^2 + (f-1) \cdot n + (j-1)$, where n is the degree of the base field and j is such that

`pr = idealprimedec(`*nf*`,p)[j]`.

m can be decoded using `bnfdecodemodule`.

Note that to compute such data for a single field, either `bnrclassno` or `bnrdisc` is more efficient.

The library syntax is `GEN bnrdisclist0(GEN bnf, GEN bound, GEN arch = NULL)`.

3.8.31 bnrgaloisapply(bnr, mat, H). Apply the automorphism given by its matrix *mat* to the congruence subgroup H given as a HNF matrix. The matrix *mat* can be computed with `bnrgaloismatrix`.

The library syntax is `GEN bnrgaloisapply(GEN bnr, GEN mat, GEN H)`.

3.8.32 bnrgaloismatrix(bnr, aut). Return the matrix of the action of the automorphism *aut* of the base field `bnf.nf` on the generators of the ray class field `bnr.gen`. *aut* can be given as a polynomial, an algebraic number, or a vector of automorphisms or a Galois group as output by `galoisinit`, in which case a vector of matrices is returned (in the later case, only for the generators `aut.gen`).

See `bnrisgalois` for an example.

The library syntax is `GEN bnrgaloismatrix(GEN bnr, GEN aut)`. When *aut* is a polynomial or an algebraic number, `GEN bnrautmatrix(GEN bnr, GEN aut)` is available.

3.8.33 bnrinit($bnf, f, \{flag = 0\}$). *bnf* is as output by `bnfinit` (including fundamental units), f is a modulus, initializes data linked to the ray class group structure corresponding to this module, a so-called `bnr` structure. One can input the attached *bid* with generators for f instead of the module itself, saving some time. (As in `idealstar`, the finite part of the conductor may be given by a factorization into prime ideals, as produced by `idealfactor`.)

The following member functions are available on the result: `.bnf` is the underlying *bnf*, `.mod` the modulus, `.bid` the `bid` structure attached to the modulus; finally, `.clgp`, `.no`, `.cyc`, `.gen` refer to the ray class group (as a finite abelian group), its cardinality, its elementary divisors, its generators (only computed if $flag = 1$).

The last group of functions are different from the members of the underlying *bnf*, which refer to the class group; use *bnr*`.bnf.`*xxx* to access these, e.g. *bnr*`.bnf.cyc` to get the cyclic decomposition of the class group.

They are also different from the members of the underlying *bid*, which refer to $(\mathbf{Z}_K/f)^*$; use *bnr*`.bid.`*xxx* to access these, e.g. *bnr*`.bid.no` to get $\phi(f)$.

If $flag = 0$ (default), the generators of the ray class group are not computed, which saves time. Hence *bnr*`.gen` would produce an error.

If $flag = 1$, as the default, except that generators are computed.

The library syntax is `GEN bnrinit0(GEN bnf, GEN f, long flag)`. Instead the above hardcoded numerical flags, one should rather use `GEN Buchray(GEN bnf, GEN module, long flag)` where flag is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (if omitted, return just the cardinality of the ray class group and its structure), possibly 0.

3.8.34 bnrisconductor($A, \{B\}, \{C\}$). Fast variant of `bnrconductor`(A, B, C); A, B, C represent an extension of the base field, given by class field theory (see Section 3.8.5). Outputs 1 if this modulus is the conductor, and 0 otherwise. This is slightly faster than `bnrconductor` when the character or subgroup is not primitive.

The library syntax is `long bnrisconductor0(GEN A, GEN B = NULL, GEN C = NULL)`.

3.8.35 bnrisgalois(bnr, gal, H). Check whether the class field attached to the subgroup H is Galois over the subfield of `bnr.nf` fixed by the group *gal*, which can be given as output by `galoisinit`, or as a matrix or a vector of matrices as output by `bnrgaloismatrix`, the second option being preferable, since it saves the recomputation of the matrices. Note: The function assumes that the ray class field attached to bnr is Galois, which is not checked.

In the following example, we lists the congruence subgroups of subextension of degree at most 3 of the ray class field of conductor 9 which are Galois over the rationals.

```
K=bnfinit(a^4-3*a^2+253009);
G=galoisinit(K);
B=bnrinit(K,9,1);
L1=[H|H<-subgrouplist(B,3), bnrisgalois(B,G,H)]
##
M=bnrgaloismatrix(B,G)
L2=[H|H<-subgrouplist(B,3), bnrisgalois(B,M,H)]
##
```

The second computation is much faster since `bnrgaloismatrix(B,G)` is computed only once.

The library syntax is `long bnrisgalois(GEN bnr, GEN gal, GEN H)`.

3.8.36 bnrisprincipal$(bnr, x, \{flag = 1\})$. *bnr* being the number field data which is output by `bnrinit(,,1)` and x being an ideal in any form, outputs the components of x on the ray class group generators in a way similar to `bnfisprincipal`. That is a 2-component vector v where $v[1]$ is the vector of components of x on the ray class group generators, $v[2]$ gives on the integral basis an element α such that $x = \alpha \prod_i g_i^{x_i}$.

If $flag = 0$, outputs only v_1. In that case, *bnr* need not contain the ray class group generators, i.e. it may be created with `bnrinit(,,0)` If x is not coprime to the modulus of *bnr* the result is undefined.

The library syntax is `GEN bnrisprincipal(GEN bnr, GEN x, long flag)`. Instead of hard-coded numerical flags, one should rather use `GEN isprincipalray(GEN bnr, GEN x)` for `flag` $= 0$, and if you want generators:

```
bnrisprincipal(bnr, x, nf_GEN)
```

3.8.37 bnrrootnumber$(bnr, chi, \{flag = 0\})$. If $\chi = chi$ is a character over *bnr*, not necessarily primitive, let $L(s, \chi) = \sum_{id} \chi(id)N(id)^{-s}$ be the attached Artin L-function. Returns the so-called Artin root number, i.e. the complex number $W(\chi)$ of modulus 1 such that

$$\Lambda(1 - s, \chi) = W(\chi)\Lambda(s, \overline{\chi})$$

where $\Lambda(s, \chi) = A(\chi)^{s/2}\gamma_\chi(s)L(s, \chi)$ is the enlarged L-function attached to L.

The generators of the ray class group are needed, and you can set $flag = 1$ if the character is known to be primitive. Example:

```
bnf = bnfinit(x^2 - x - 57);
bnr = bnrinit(bnf, [7,[1,1]], 1);
bnrrootnumber(bnr, [2,1])
```

returns the root number of the character χ of $\mathrm{Cl}_{7\infty_1\infty_2}(\mathbf{Q}(\sqrt{229}))$ defined by $\chi(g_1^a g_2^b) = \zeta_1^{2a}\zeta_2^b$. Here g_1, g_2 are the generators of the ray-class group given by `bnr.gen` and $\zeta_1 = e^{2i\pi/N_1}, \zeta_2 = e^{2i\pi/N_2}$ where N_1, N_2 are the orders of g_1 and g_2 respectively ($N_1 = 6$ and $N_2 = 3$ as `bnr.cyc` readily tells us).

The library syntax is `GEN bnrrootnumber(GEN bnr, GEN chi, long flag, long prec)` .

3.8.38 bnrstark$(bnr, \{subgroup\})$. *bnr* being as output by `bnrinit(,,1)`, finds a relative equation for the class field corresponding to the modulus in *bnr* and the given congruence subgroup (as usual, omit *subgroup* if you want the whole ray class group).

The main variable of *bnr* must not be x, and the ground field and the class field must be totally real. When the base field is $\mathbf{Q}$, the vastly simpler `galoissubcyclo` is used instead. Here is an example:

```
bnf = bnfinit(y^2 - 3);
bnr = bnrinit(bnf, 5, 1);
bnrstark(bnr)
```

returns the ray class field of $\mathbf{Q}(\sqrt{3})$ modulo 5. Usually, one wants to apply to the result one of

```
rnfpolredabs(bnf, pol, 16)      \\ compute a reduced relative polynomial
rnfpolredabs(bnf, pol, 16 + 2) \\ compute a reduced absolute polynomial
```

The routine uses Stark units and needs to find a suitable auxiliary conductor, which may not exist when the class field is not cyclic over the base. In this case `bnrstark` is allowed to return a vector of polynomials defining *independent* relative extensions, whose compositum is the requested class field. It was decided that it was more useful to keep the extra information thus made available, hence the user has to take the compositum herself.

Even if it exists, the auxiliary conductor may be so large that later computations become unfeasible. (And of course, Stark's conjecture may simply be wrong.) In case of difficulties, try `rnfkummer`:

```
? bnr = bnrinit(bnfinit(y^8-12*y^6+36*y^4-36*y^2+9,1), 2, 1);
? bnrstark(bnr)
  ***   at top-level: bnrstark(bnr)
  ***                 ^-------------
  *** bnrstark: need 3919350809720744 coefficients in initzeta.
  *** Computation impossible.
? lift( rnfkummer(bnr) )
time = 24 ms.
%2 = x^2 + (1/3*y^6 - 11/3*y^4 + 8*y^2 - 5)
```

The library syntax is `GEN bnrstark(GEN bnr, GEN subgroup = NULL, long prec)`.

3.8.39 dirzetak(nf, b). Gives as a vector the first b coefficients of the Dedekind zeta function of the number field nf considered as a Dirichlet series.

The library syntax is `GEN dirzetak(GEN nf, GEN b)`.

3.8.40 factornf(x, t). This function is obsolete, use `nffactor`.

factorization of the univariate polynomial x over the number field defined by the (univariate) polynomial t. x may have coefficients in $\mathbf{Q}$ or in the number field. The algorithm reduces to factorization over $\mathbf{Q}$ (Trager's trick). The direct approach of `nffactor`, which uses van Hoeij's method in a relative setting, is in general faster.

The main variable of t must be of *lower* priority than that of x (see Section 2.5.3). However if non-rational number field elements occur (as polmods or polynomials) as coefficients of x, the variable of these polmods *must* be the same as the main variable of t. For example

```
? factornf(x^2 + Mod(y, y^2+1), y^2+1);
? factornf(x^2 + y, y^2+1); \\ these two are OK
? factornf(x^2 + Mod(z,z^2+1), y^2+1)
  ***   at top-level: factornf(x^2+Mod(z,z
  ***                 ^--------------------
  *** factornf: inconsistent data in rnf function.
? factornf(x^2 + z, y^2+1)
  ***   at top-level: factornf(x^2+z,y^2+1
  ***                 ^--------------------
  *** factornf: incorrect variable in rnf function.
```

The library syntax is `GEN polfnf(GEN x, GEN t)`.

3.8.41 galoisexport$(gal, \{flag\})$. *gal* being be a Galois group as output by `galoisinit`, export the underlying permutation group as a string suitable for (no flags or $flag = 0$) GAP or ($flag = 1$) Magma. The following example compute the index of the underlying abstract group in the GAP library:

```
? G = galoisinit(x^6+108);
? s = galoisexport(G)
%2 = "Group((1, 2, 3)(4, 5, 6), (1, 4)(2, 6)(3, 5))"
? extern("echo \"IdGroup("s");\" | gap -q")
%3 = [6, 1]
? galoisidentify(G)
%4 = [6, 1]
```

This command also accepts subgroups returned by `galoissubgroups`.

To *import* a GAP permutation into gp (for `galoissubfields` for instance), the following GAP function may be useful:

```
PermToGP := function(p, n)
  return Permuted([1..n],p);
end;
```

```
gap> p:= (1,26)(2,5)(3,17)(4,32)(6,9)(7,11)(8,24)(10,13)(12,15)(14,27)
  (16,22)(18,28)(19,20)(21,29)(23,31)(25,30)
gap> PermToGP(p,32);
[ 26, 5, 17, 32, 2, 9, 11, 24, 6, 13, 7, 15, 10, 27, 12, 22, 3, 28, 20, 19,
  29, 16, 31, 8, 30, 1, 14, 18, 21, 25, 23, 4 ]
```

The library syntax is `GEN galoisexport(GEN gal, long flag)`.

3.8.42 galoisfixedfield$(gal, perm, \{flag\}, \{v = y\})$. *gal* being be a Galois group as output by `galoisinit` and *perm* an element of *gal.group*, a vector of such elements or a subgroup of *gal* as returned by galoissubgroups, computes the fixed field of *gal* by the automorphism defined by the permutations *perm* of the roots *gal.roots*. P is guaranteed to be squarefree modulo *gal.p*.

If no flags or $flag = 0$, output format is the same as for `nfsubfield`, returning $[P, x]$ such that P is a polynomial defining the fixed field, and x is a root of P expressed as a polmod in *gal.pol*.

If $flag = 1$ return only the polynomial P.

If $flag = 2$ return $[P, x, F]$ where P and x are as above and F is the factorization of *gal.pol* over the field defined by P, where variable v (y by default) stands for a root of P. The priority of v must be less than the priority of the variable of *gal.pol* (see Section 2.5.3). Example:

```
? G = galoisinit(x^4+1);
? galoisfixedfield(G,G.group[2],2)
%2 = [x^2 + 2, Mod(x^3 + x, x^4 + 1), [x^2 - y*x - 1, x^2 + y*x - 1]]
```

computes the factorization $x^4 + 1 = (x^2 - \sqrt{-2}x - 1)(x^2 + \sqrt{-2}x - 1)$

The library syntax is `GEN galoisfixedfield(GEN gal, GEN perm, long flag, long v = -1)` where `v` is a variable number.

3.8.43 galoisgetpol$(a, \{b\}, \{s\})$. Query the galpol package for a polynomial with Galois group isomorphic to GAP4(a,b), totally real if $s = 1$ (default) and totally complex if $s = 2$. The output is a vector [`pol`, `den`] where

• `pol` is the polynomial of degree a

• `den` is the denominator of `nfgaloisconj(pol)`. Pass it as an optional argument to `galoisinit` or `nfgaloisconj` to speed them up:

```
? [pol,den] = galoisgetpol(64,4,1);
? G = galoisinit(pol);
time = 352ms
? galoisinit(pol, den);  \\ passing 'den' speeds up the computation
time = 264ms
? % == %‘
%4 = 1  \\ same answer
```

If b and s are omitted, return the number of isomorphism classes of groups of order a.

The library syntax is `GEN galoisgetpol(long a, long b, long s)`. Also available is `GEN galoisnbpol(long a)` when b and s are omitted.

3.8.44 galoisidentify(gal). *gal* being be a Galois group as output by `galoisinit`, output the isomorphism class of the underlying abstract group as a two-components vector $[o, i]$, where o is the group order, and i is the group index in the GAP4 Small Group library, by Hans Ulrich Besche, Bettina Eick and Eamonn O'Brien.

This command also accepts subgroups returned by `galoissubgroups`.

The current implementation is limited to degree less or equal to 127. Some larger "easy" orders are also supported.

The output is similar to the output of the function `IdGroup` in GAP4. Note that GAP4 `IdGroup` handles all groups of order less than 2000 except 1024, so you can use `galoisexport` and GAP4 to identify large Galois groups.

The library syntax is `GEN galoisidentify(GEN gal)`.

3.8.45 galoisinit$(pol, \{den\})$. Computes the Galois group and all necessary information for computing the fixed fields of the Galois extension $K/\mathbf{Q}$ where K is the number field defined by *pol* (monic irreducible polynomial in $\mathbf{Z}[X]$ or a number field as output by `nfinit`). The extension $K/\mathbf{Q}$ must be Galois with Galois group "weakly" super-solvable, see below; returns 0 otherwise. Hence this permits to quickly check whether a polynomial of order strictly less than 36 is Galois or not.

The algorithm used is an improved version of the paper "An efficient algorithm for the computation of Galois automorphisms", Bill Allombert, Math. Comp, vol. 73, 245, 2001, pp. 359–375.

A group G is said to be "weakly" super-solvable if there exists a normal series

$\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_{n-1} \lhd H_n$

such that each H_i is normal in G and for $i < n$, each quotient group H_{i+1}/H_i is cyclic, and either $H_n = G$ (then G is super-solvable) or G/H_n is isomorphic to either A_4 or S_4.

In practice, almost all small groups are WKSS, the exceptions having order 36(1 exception), 48(2), 56(1), 60(1), 72(5), 75(1), 80(1), 96(10) and ≥ 108.

This function is a prerequisite for most of the `galois`xxx routines. For instance:

```
P = x^6 + 108;
G = galoisinit(P);
L = galoissubgroups(G);
vector(#L, i, galoisisabelian(L[i],1))
vector(#L, i, galoisidentify(L[i]))
```

The output is an 8-component vector gal.

gal[1] contains the polynomial pol (gal`.pol`).

gal[2] is a three-components vector $[p, e, q]$ where p is a prime number (gal`.p`) such that pol totally split modulo p , e is an integer and $q = p^e$ (gal`.mod`) is the modulus of the roots in gal`.roots`.

gal[3] is a vector L containing the p-adic roots of pol as integers implicitly modulo gal`.mod`. (gal`.roots`).

gal[4] is the inverse of the Vandermonde matrix of the p-adic roots of pol, multiplied by gal[5].

gal[5] is a multiple of the least common denominator of the automorphisms expressed as polynomial in a root of pol.

gal[6] is the Galois group G expressed as a vector of permutations of L (gal`.group`).

gal[7] is a generating subset $S = [s_1, \ldots, s_g]$ of G expressed as a vector of permutations of L (gal`.gen`).

gal[8] contains the relative orders $[o_1, \ldots, o_g]$ of the generators of S (gal`.orders`).

Let H_n be as above, we have the following properties:

- if $G/H_n \simeq A_4$ then $[o_1, \ldots, o_g]$ ends by $[2, 2, 3]$.

- if $G/H_n \simeq S_4$ then $[o_1, \ldots, o_g]$ ends by $[2, 2, 3, 2]$.

- for $1 \leq i \leq g$ the subgroup of G generated by $[s_1, \ldots, s_g]$ is normal, with the exception of $i = g - 2$ in the A_4 case and of $i = g - 3$ in the S_A case.

- the relative order o_i of s_i is its order in the quotient group $G/\langle s_1, \ldots, s_{i-1}\rangle$, with the same exceptions.

- for any $x \in G$ there exists a unique family $[e_1, \ldots, e_g]$ such that (no exceptions):

– for $1 \leq i \leq g$ we have $0 \leq e_i < o_i$

– $x = g_1^{e_1} g_2^{e_2} \ldots g_n^{e_n}$

If present den must be a suitable value for gal[5].

The library syntax is `GEN galoisinit(GEN pol, GEN den = NULL)`.

3.8.46 galoisisabelian$(gal, \{flag = 0\})$. gal being as output by `galoisinit`, return 0 if gal is not an abelian group, and the HNF matrix of gal over `gal.gen` if $fl = 0$, 1 if $fl = 1$.

This command also accepts subgroups returned by `galoissubgroups`.

The library syntax is `GEN galoisisabelian(GEN gal, long flag)`.

3.8.47 galoisisnormal$(gal, subgrp)$. *gal* being as output by `galoisinit`, and *subgrp* a subgroup of *gal* as output by `galoissubgroups`,return 1 if *subgrp* is a normal subgroup of *gal*, else return 0.

This command also accepts subgroups returned by `galoissubgroups`.

The library syntax is `long galoisisnormal(GEN gal, GEN subgrp)`.

3.8.48 galoispermtopol$(gal, perm)$. *gal* being a Galois group as output by `galoisinit` and *perm* a element of *gal.group*, return the polynomial defining the Galois automorphism, as output by `nfgaloisconj`, attached to the permutation *perm* of the roots *gal.roots*. *perm* can also be a vector or matrix, in this case, `galoispermtopol` is applied to all components recursively.

Note that

```
G = galoisinit(pol);
galoispermtopol(G, G[6])~
```

is equivalent to `nfgaloisconj(pol)`, if degree of *pol* is greater or equal to 2.

The library syntax is `GEN galoispermtopol(GEN gal, GEN perm)`.

3.8.49 galoissubcyclo$(N, H, \{fl = 0\}, \{v\})$. Computes the subextension of $\mathbf{Q}(\zeta_n)$ fixed by the subgroup $H \subset (\mathbf{Z}/n\mathbf{Z})^*$. By the Kronecker-Weber theorem, all abelian number fields can be generated in this way (uniquely if n is taken to be minimal).

The pair (n, H) is deduced from the parameters (N, H) as follows

• N an integer: then $n = N$; H is a generator, i.e. an integer or an integer modulo n; or a vector of generators.

• N the output of `znstar(`n`)`. H as in the first case above, or a matrix, taken to be a HNF left divisor of the SNF for $(\mathbf{Z}/n\mathbf{Z})^*$ (of type N`.cyc`), giving the generators of H in terms of N`.gen`.

• N the output of `bnrinit(bnfinit(y),` m`, 1)` where m is a module. H as in the first case, or a matrix taken to be a HNF left divisor of the SNF for the ray class group modulo m (of type N`.cyc`), giving the generators of H in terms of N`.gen`.

In this last case, beware that H is understood relatively to N; in particular, if the infinite place does not divide the module, e.g if m is an integer, then it is not a subgroup of $(\mathbf{Z}/n\mathbf{Z})^*$, but of its quotient by $\{\pm 1\}$.

If $fl = 0$, compute a polynomial (in the variable v) defining the subfield of $\mathbf{Q}(\zeta_n)$ fixed by the subgroup H of $(\mathbf{Z}/n\mathbf{Z})^*$.

If $fl = 1$, compute only the conductor of the abelian extension, as a module.

If $fl = 2$, output $[pol, N]$, where *pol* is the polynomial as output when $fl = 0$ and N the conductor as output when $fl = 1$.

The following function can be used to compute all subfields of $\mathbf{Q}(\zeta_n)$ (of exact degree `d`, if `d` is set):

```
polsubcyclo(n, d = -1)=
{ my(bnr,L,IndexBound);
  IndexBound = if (d < 0, n, [d]);
  bnr = bnrinit(bnfinit(y), [n,[1]], 1);
  L = subgrouplist(bnr, IndexBound, 1);
```

```
  vector(#L,i, galoissubcyclo(bnr,L[i]));
}
```

Setting `L = subgrouplist(bnr, IndexBound)` would produce subfields of exact conductor $n\infty$.

The library syntax is `GEN galoissubcyclo(GEN N, GEN H = NULL, long fl, long v = -1)` where `v` is a variable number.

3.8.50 galoissubfields$(G, \{flag = 0\}, \{v\})$. Outputs all the subfields of the Galois group G, as a vector. This works by applying `galoisfixedfield` to all subgroups. The meaning of *flag* is the same as for `galoisfixedfield`.

The library syntax is `GEN galoissubfields(GEN G, long flag, long v = -1)` where `v` is a variable number.

3.8.51 galoissubgroups(G). Outputs all the subgroups of the Galois group `gal`. A subgroup is a vector [*gen*, *orders*], with the same meaning as for *gal.gen* and *gal.orders*. Hence *gen* is a vector of permutations generating the subgroup, and *orders* is the relatives orders of the generators. The cardinality of a subgroup is the product of the relative orders. Such subgroup can be used instead of a Galois group in the following command: `galoisisabelian`, `galoissubgroups`, `galoisexport` and `galoisidentify`.

To get the subfield fixed by a subgroup *sub* of *gal*, use

```
galoisfixedfield(gal,sub[1])
```

The library syntax is `GEN galoissubgroups(GEN G)`.

3.8.52 idealadd(nf, x, y). Sum of the two ideals x and y in the number field nf. The result is given in HNF.

```
? K = nfinit(x^2 + 1);
? a = idealadd(K, 2, x + 1)  \\ ideal generated by 2 and 1+I
%2 =
[2 1]

[0 1]
? pr = idealprimedec(K, 5)[1];  \\ a prime ideal above 5
? idealadd(K, a, pr)     \\ coprime, as expected
%4 =
[1 0]

[0 1]
```

This function cannot be used to add arbitrary **Z**-modules, since it assumes that its arguments are ideals:

```
? b = Mat([1,0]~);
? idealadd(K, b, b)     \\ only square t_MATs represent ideals
*** idealadd: non-square t_MAT in idealtyp.
? c = [2, 0; 2, 0]; idealadd(K, c, c)   \\ non-sense
%6 =
[2 0]

[0 2]
? d = [1, 0; 0, 2]; idealadd(K, d, d)   \\ non-sense
```

```
%7 =
[1 0]

[0 1]
```

In the last two examples, we get wrong results since the matrices c and d do not correspond to an ideal: the $\mathbf{Z}$-span of their columns (as usual interpreted as coordinates with respect to the integer basis `K.zk`) is not an O_K-module. To add arbitrary $\mathbf{Z}$-modules generated by the columns of matrices A and B, use `mathnf(concat(A,B))`.

The library syntax is `GEN idealadd(GEN nf, GEN x, GEN y)`.

3.8.53 idealaddtoone$(nf, x, \{y\})$. x and y being two co-prime integral ideals (given in any form), this gives a two-component row vector $[a, b]$ such that $a \in x$, $b \in y$ and $a + b = 1$.

The alternative syntax `idealaddtoone`(nf, v), is supported, where v is a k-component vector of ideals (given in any form) which sum to $\mathbf{Z}_K$. This outputs a k-component vector e such that $e[i] \in x[i]$ for $1 \leq i \leq k$ and $\sum_{1 \leq i \leq k} e[i] = 1$.

The library syntax is `GEN idealaddtoone0(GEN nf, GEN x, GEN y = NULL)`.

3.8.54 idealappr$(nf, x, \{flag\})$. If x is a fractional ideal (given in any form), gives an element α in nf such that for all prime ideals $\mathfrak{p}$ such that the valuation of x at $\mathfrak{p}$ is non-zero, we have $v_{\mathfrak{p}}(\alpha) = v_{\mathfrak{p}}(x)$, and $v_{\mathfrak{p}}(\alpha) \geq 0$ for all other $\mathfrak{p}$.

The argument x may also be given as a prime ideal factorization, as output by `idealfactor`, but allowing zero exponents. This yields an element α such that for all prime ideals $\mathfrak{p}$ occurring in x, $v_{\mathfrak{p}}(\alpha) = v_{\mathfrak{p}}(x)$; for all other prime ideals, $v_{\mathfrak{p}}(\alpha) \geq 0$.

flag is deprecated (ignored), kept for backward compatibility

The library syntax is `GEN idealappr0(GEN nf, GEN x, long flag)`. Use directly `GEN idealappr(GEN nf, GEN x)` since *flag*is ignored.

3.8.55 idealchinese$(nf, x, \{y\})$. x being a prime ideal factorization (i.e. a 2 by 2 matrix whose first column contains prime ideals, and the second column integral exponents), y a vector of elements in nf indexed by the ideals in x, computes an element b such that

$v_{\mathfrak{p}}(b - y_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(x)$ for all prime ideals in x and $v_{\mathfrak{p}}(b) \geq 0$ for all other $\mathfrak{p}$.

```
? K = nfinit(t^2-2);
? x = idealfactor(K, 2^2*3)
%2 =
[[2, [0, 1]~, 2, 1, [0, 2; 1, 0]] 4]

[           [3, [3, 0]~, 1, 2, 1] 1]
? y = [t,1];
? idealchinese(K, x, y)
%4 = [4, -3]~
```

The argument x may also be of the form $[x, s]$ where the first component is as above and s is a vector of signs, with r_1 components s_i in $\{-1, 0, 1\}$: if σ_i denotes the i-th real embedding of the number field, the element b returned satisfies further $s_i \mathtt{sign}(\sigma_i(b)) \geq 0$ for all i. In other words, the sign is fixed to s_i at the i-th embedding whenever s_i is non-zero.

```
? idealchinese(K, [x, [1,1]], y)
```

```
%5 = [16, -3]~
? idealchinese(K, [x, [-1,-1]], y)
%6 = [-20, -3]~
? idealchinese(K, [x, [1,-1]], y)
%7 = [4, -3]~
```

If y is omitted, return a data structure which can be used in place of x in later calls and allows to solve many chinese remainder problems for a given x more efficiently.

```
? C = idealchinese(K, [x, [1,1]]);
? idealchinese(K, C, y) \\ as above
%9 = [16, -3]~
? for(i=1,10^4, idealchinese(K,C,y))  \\ ... but faster !
time = 80 ms.
? for(i=1,10^4, idealchinese(K,[x,[1,1]],y))
time = 224 ms.
```

Finally, this structure is itself allowed in place of x, the new s overriding the one already present in the structure. This allows to initialize for different sign conditions more efficiently when the underlying ideal factorization remains the same.

```
? D = idealchinese(K, [C, [1,-1]]);   \\ replaces [1,1]
? idealchinese(K, D, y)
%13 = [4, -3]~
? for(i=1,10^4,idealchinese(K,[C,[1,-1]]))
time = 40 ms.   \\ faster than starting from scratch
? for(i=1,10^4,idealchinese(K,[x,[1,-1]]))
time = 128 ms.
```

The library syntax is `GEN idealchinese(GEN nf, GEN x, GEN y = NULL)`. Also available is `GEN idealchineseinit(GEN nf, GEN x)` when $y =$ `NULL`.

3.8.56 idealcoprime(nf, x, y). Given two integral ideals x and y in the number field nf, returns a β in the field, such that $\beta \cdot x$ is an integral ideal coprime to y.

The library syntax is `GEN idealcoprime(GEN nf, GEN x, GEN y)`.

3.8.57 idealdiv$(nf, x, y, \{flag = 0\})$. Quotient $x \cdot y^{-1}$ of the two ideals x and y in the number field nf. The result is given in HNF.

If *flag* is non-zero, the quotient $x \cdot y^{-1}$ is assumed to be an integral ideal. This can be much faster when the norm of the quotient is small even though the norms of x and y are large.

The library syntax is `GEN idealdiv0(GEN nf, GEN x, GEN y, long flag)`. Also available are `GEN idealdiv(GEN nf, GEN x, GEN y)` (*flag* $= 0$) and `GEN idealdivexact(GEN nf, GEN x, GEN y)` (*flag* $= 1$).

3.8.58 idealfactor(nf, x). Factors into prime ideal powers the ideal x in the number field nf. The output format is similar to the `factor` function, and the prime ideals are represented in the form output by the `idealprimedec` function.

The library syntax is `GEN idealfactor(GEN nf, GEN x)`.

3.8.59 idealfactorback$(nf, f, \{e\}, \{flag = 0\})$. Gives back the ideal corresponding to a factorization. The integer 1 corresponds to the empty factorization. If e is present, e and f must be vectors of the same length (e being integral), and the corresponding factorization is the product of the $f[i]^{e[i]}$.

If not, and f is vector, it is understood as in the preceding case with e a vector of 1s: we return the product of the $f[i]$. Finally, f can be a regular factorization, as produced by `idealfactor`.

```
? nf = nfinit(y^2+1); idealfactor(nf, 4 + 2*y)
%1 =
[[2, [1, 1]~, 2, 1, [1, 1]~] 2]

[[5, [2, 1]~, 1, 1, [-2, 1]~] 1]

? idealfactorback(nf, %)
%2 =
[10 4]

[0  2]

? f = %1[,1]; e = %1[,2]; idealfactorback(nf, f, e)
%3 =
[10 4]

[0  2]

? % == idealhnf(nf, 4 + 2*y)
%4 = 1
```

If `flag` is non-zero, perform ideal reductions (`idealred`) along the way. This is most useful if the ideals involved are all *extended* ideals (for instance with trivial principal part), so that the principal parts extracted by `idealred` are not lost. Here is an example:

```
? f = vector(#f, i, [f[i], [;]]);  \\ transform to extended ideals
? idealfactorback(nf, f, e, 1)
%6 = [[1, 0; 0, 1], [2, 1; [2, 1]~, 1]]
? nffactorback(nf, %[2])
%7 = [4, 2]~
```

The extended ideal returned in `%6` is the trivial ideal 1, extended with a principal generator given in factored form. We use `nffactorback` to recover it in standard form.

The library syntax is `GEN idealfactorback(GEN nf, GEN f, GEN e = NULL, long flag)`.

3.8.60 idealfrobenius(nf, gal, pr). Let K be the number field defined by nf and assume $K/\mathbf{Q}$ be a Galois extension with Galois group given `gal=galoisinit(nf)`, and that pr is an unramified prime ideal $\mathfrak{p}$ in `prid` format. This function returns a permutation of `gal.group` which defines the Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ attached to $\mathfrak{p}$. If p is the unique prime number in $\mathfrak{p}$, then $\mathrm{Frob}(x) \equiv x^p \bmod \mathfrak{p}$ for all $x \in \mathbf{Z}_K$.

```
? nf = nfinit(polcyclo(31));
? gal = galoisinit(nf);
? pr = idealprimedec(nf,101)[1];
? g = idealfrobenius(nf,gal,pr);
? galoispermtopol(gal,g)
```

```
%5 = x^8
```

This is correct since $101 \equiv 8 \bmod 31$.

The library syntax is `GEN idealfrobenius(GEN nf, GEN gal, GEN pr)`.

3.8.61 idealhnf$(nf, u, \{v\})$. Gives the Hermite normal form of the ideal $u\mathbf{Z}_K + v\mathbf{Z}_K$, where u and v are elements of the number field K defined by nf.

```
? nf = nfinit(y^3 - 2);
? idealhnf(nf, 2, y+1)
%2 =
[1 0 0]

[0 1 0]

[0 0 1]
? idealhnf(nf, y/2, [0,0,1/3]~)
%3 =
[1/3 0 0]

[0 1/6 0]

[0 0 1/6]
```

If b is omitted, returns the HNF of the ideal defined by u: u may be an algebraic number (defining a principal ideal), a maximal ideal (as given by `idealprimedec` or `idealfactor`), or a matrix whose columns give generators for the ideal. This last format is a little complicated, but useful to reduce general modules to the canonical form once in a while:

• if strictly less than $N = [K : \mathbf{Q}]$ generators are given, u is the $\mathbf{Z}_K$-module they generate,

• if N or more are given, it is *assumed* that they form a $\mathbf{Z}$-basis of the ideal, in particular that the matrix has maximal rank N. This acts as `mathnf` since the $\mathbf{Z}_K$-module structure is (taken for granted hence) not taken into account in this case.

```
? idealhnf(nf, idealprimedec(nf,2)[1])
%4 =
[2 0 0]

[0 1 0]

[0 0 1]
? idealhnf(nf, [1,2;2,3;3,4])
%5 =
[1 0 0]

[0 1 0]

[0 0 1]
```

Finally, when K is quadratic with discriminant D_K, we allow $u =$ `Qfb(a,b,c)`, provided $b^2 - 4ac = D_K$. As usual, this represents the ideal $a\mathbf{Z} + (1/2)(-b + \sqrt{D_K})\mathbf{Z}$.

```
? K = nfinit(x^2 - 60); K.disc
%1 = 60
? idealhnf(K, qfbprimeform(60,2))
%2 =
[2 1]
```

```
[0 1]
? idealhnf(K, Qfb(1,2,3))
  ***   at top-level: idealhnf(K,Qfb(1,2,3
  ***                 ^--------------------
  *** idealhnf: Qfb(1, 2, 3) has discriminant != 60 in idealhnf.
```

The library syntax is `GEN idealhnf0(GEN nf, GEN u, GEN v = NULL)`. Also available is `GEN idealhnf(GEN nf, GEN a)`.

3.8.62 idealintersect(nf, A, B). Intersection of the two ideals A and B in the number field nf. The result is given in HNF.

```
? nf = nfinit(x^2+1);
? idealintersect(nf, 2, x+1)
%2 =
[2 0]

[0 2]
```

This function does not apply to general $\mathbf{Z}$-modules, e.g. orders, since its arguments are replaced by the ideals they generate. The following script intersects $\mathbf{Z}$-modules A and B given by matrices of compatible dimensions with integer coefficients:

```
ZM_intersect(A,B) =
{ my(Ker = matkerint(concat(A,B)));
  mathnf( A * Ker[1..#A,] )
}
```

The library syntax is `GEN idealintersect(GEN nf, GEN A, GEN B)`.

3.8.63 idealinv(nf, x). Inverse of the ideal x in the number field nf, given in HNF. If x is an extended ideal, its principal part is suitably updated: i.e. inverting $[I, t]$, yields $[I^{-1}, 1/t]$.

The library syntax is `GEN idealinv(GEN nf, GEN x)`.

3.8.64 ideallist$(nf, bound, \{flag = 4\})$. Computes the list of all ideals of norm less or equal to *bound* in the number field nf. The result is a row vector with exactly *bound* components. Each component is itself a row vector containing the information about ideals of a given norm, in no specific order, depending on the value of *flag*:

The possible values of *flag* are:

0: give the *bid* attached to the ideals, without generators.

1: as 0, but include the generators in the *bid*.

2: in this case, *nf* must be a *bnf* with units. Each component is of the form $[bid, U]$, where *bid* is as case 0 and U is a vector of discrete logarithms of the units. More precisely, it gives the `ideallog`s with respect to *bid* of `bnf.tufu`. This structure is technical, and only meant to be used in conjunction with `bnrclassnolist` or `bnrdisclist`.

3: as 2, but include the generators in the *bid*.

4: give only the HNF of the ideal.

```
? nf = nfinit(x^2+1);
```

```
? L = ideallist(nf, 100);
? L[1]
%3 = [[1, 0; 0, 1]]  \\ A single ideal of norm 1
? #L[65]
%4 = 4               \\ There are 4 ideals of norm 4 in Z[i]
```

If one wants more information, one could do instead:

```
? nf = nfinit(x^2+1);
? L = ideallist(nf, 100, 0);
? l = L[25]; vector(#l, i, l[i].clgp)
%3 = [[20, [20]], [16, [4, 4]], [20, [20]]]
? l[1].mod
%4 = [[25, 18; 0, 1], []]
? l[2].mod
%5 = [[5, 0; 0, 5], []]
? l[3].mod
%6 = [[25, 7; 0, 1], []]
```

where we ask for the structures of the $(\mathbf{Z}[i]/I)^*$ for all three ideals of norm 25. In fact, for all moduli with finite part of norm 25 and trivial Archimedean part, as the last 3 commands show. See `ideallistarch` to treat general moduli.

The library syntax is `GEN ideallist0(GEN nf, long bound, long flag)`.

3.8.65 ideallistarch$(nf, list, arch)$. *list* is a vector of vectors of bid's, as output by `ideallist` with flag 0 to 3. Return a vector of vectors with the same number of components as the original *list*. The leaves give information about moduli whose finite part is as in original list, in the same order, and Archimedean part is now *arch* (it was originally trivial). The information contained is of the same kind as was present in the input; see `ideallist`, in particular the meaning of *flag*.

```
? bnf = bnfinit(x^2-2);
? bnf.sign
%2 = [2, 0]                          \\ two places at infinity
? L = ideallist(bnf, 100, 0);
? l = L[98]; vector(#l, i, l[i].clgp)
%4 = [[42, [42]], [36, [6, 6]], [42, [42]]]
? La = ideallistarch(bnf, L, [1,1]); \\ add them to the modulus
? l = La[98]; vector(#l, i, l[i].clgp)
%6 = [[168, [42, 2, 2]], [144, [6, 6, 2, 2]], [168, [42, 2, 2]]]
```

Of course, the results above are obvious: adding t places at infinity will add t copies of $\mathbf{Z}/2\mathbf{Z}$ to $(\mathbf{Z}_K/f)^*$. The following application is more typical:

```
? L = ideallist(bnf, 100, 2);        \\ units are required now
? La = ideallistarch(bnf, L, [1,1]);
? H = bnrclassnolist(bnf, La);
? H[98];
%4 = [2, 12, 2]
```

The library syntax is `GEN ideallistarch(GEN nf, GEN list, GEN arch)`.

3.8.66 ideallog$(\{nf\}, x, bid)$. *nf* is a number field, *bid* is as output by `idealstar(nf, D, ...)` and x a non-necessarily integral element of *nf* which must have valuation equal to 0 at all prime ideals in the support of `D`. This function computes the discrete logarithm of x on the generators given in *bid*`.gen`. In other words, if g_i are these generators, of orders d_i respectively, the result is a column vector of integers (x_i) such that $0 \leq x_i < d_i$ and

$$x \equiv \prod_i g_i^{x_i} \pmod{{}^*D} \ .$$

Note that when the support of `D` contains places at infinity, this congruence implies also sign conditions on the attached real embeddings. See `znlog` for the limitations of the underlying discrete log algorithms.

When *nf* is omitted, take it to be the rational number field. In that case, x must be a `t_INT` and *bid* must have been initialized by `idealstar(,N)`.

The library syntax is `GEN ideallog(GEN nf = NULL, GEN x, GEN bid)`. Also available is `GEN Zideallog(GEN bid, GEN x)` when `nf` is `NULL`.

3.8.67 idealmin$(nf, ix, \{vdir\})$. *This function is useless and kept for backward compatibility only, use* `idealred`. Computes a pseudo-minimum of the ideal x in the direction *vdir* in the number field *nf*.

The library syntax is `GEN idealmin(GEN nf, GEN ix, GEN vdir = NULL)`.

3.8.68 idealmul$(nf, x, y, \{flag = 0\})$. Ideal multiplication of the ideals x and y in the number field *nf*; the result is the ideal product in HNF. If either x or y are extended ideals, their principal part is suitably updated: i.e. multiplying $[I, t]$, $[J, u]$ yields $[IJ, tu]$; multiplying I and $[J, u]$ yields $[IJ, u]$.

```
? nf = nfinit(x^2 + 1);
? idealmul(nf, 2, x+1)
%2 =
[4 2]

[0 2]
? idealmul(nf, [2, x], x+1)        \\ extended ideal * ideal
%3 = [[4, 2; 0, 2], x]
? idealmul(nf, [2, x], [x+1, x])   \\ two extended ideals
%4 = [[4, 2; 0, 2], [-1, 0]~]
```

If *flag* is non-zero, reduce the result using `idealred`.

The library syntax is `GEN idealmul0(GEN nf, GEN x, GEN y, long flag)`.

See also `GEN idealmul(GEN nf, GEN x, GEN y)` ($flag = 0$) and `GEN idealmulred(GEN nf, GEN x, GEN y)` ($flag \neq 0$).

3.8.69 idealnorm(nf, x). Computes the norm of the ideal x in the number field *nf*.

The library syntax is `GEN idealnorm(GEN nf, GEN x)`.

3.8.70 idealnumden(nf, x). Returns $[A, B]$, where A, B are coprime integer ideals such that $x = A/B$, in the number field nf.

```
? nf = nfinit(x^2+1);
? idealnumden(nf, (x+1)/2)
%2 = [[1, 0; 0, 1], [2, 1; 0, 1]]
```

The library syntax is `GEN idealnumden(GEN nf, GEN x)`.

3.8.71 idealpow$(nf, x, k, \{flag = 0\})$. Computes the k-th power of the ideal x in the number field nf; $k \in \mathbf{Z}$. If x is an extended ideal, its principal part is suitably updated: i.e. raising $[I, t]$ to the k-th power, yields $[I^k, t^k]$.

If *flag* is non-zero, reduce the result using `idealred`, *throughout the (binary) powering process*; in particular, this is *not* the same as `idealpow`(nf, x, k) followed by reduction.

The library syntax is `GEN idealpow0(GEN nf, GEN x, GEN k, long flag)`.

See also `GEN idealpow(GEN nf, GEN x, GEN k)` and `GEN idealpows(GEN nf, GEN x, long k)` (*flag* $= 0$). Corresponding to *flag* $= 1$ is `GEN idealpowred(GEN nf, GEN vp, GEN k)`.

3.8.72 idealprimedec$(nf, p, \{f = 0\})$. Computes the prime ideal decomposition of the (positive) prime number p in the number field K represented by nf. If a non-prime p is given the result is undefined. If f is present and non-zero, restrict the result to primes of residue degree $\leq f$.

The result is a vector of *prid* structures, each representing one of the prime ideals above p in the number field nf. The representation `pr` $= [p, a, e, f, mb]$ of a prime ideal means the following: a is an algebraic integer in the maximal order $\mathbf{Z}_K$ and the prime ideal is equal to `pr` $= p\mathbf{Z}_K + a\mathbf{Z}_K$; e is the ramification index; f is the residual index; finally, mb is the multiplication table attached to the algebraic integer b is such that `pr`$^{-1} = \mathbf{Z}_K + b/p\mathbf{Z}_K$, which is used internally to compute valuations. In other words if p is inert, then mb is the integer 1, and otherwise it is a square `t_MAT` whose j-th column is $b \cdot$ `nf.zk`[j].

The algebraic number a is guaranteed to have a valuation equal to 1 at the prime ideal (this is automatic if $e > 1$).

The components of `pr` should be accessed by member functions: `pr.p`, `pr.e`, `pr.f`, and `pr.gen` (returns the vector $[p, a]$):

```
? K = nfinit(x^3-2);
? P = idealprimedec(K, 5);
? #P       \\ 2 primes above 5 in Q(2^(1/3))
%3 = 2
? [p1,p2] = P;
? [p1.e, p1.f]    \\ the first is unramified of degree 1
%5 = [1, 1]
? [p2.e, p2.f]    \\ the second is unramified of degree 2
%6 = [1, 2]
? p1.gen
%7 = [5, [2, 1, 0]~]
? nfbasistoalg(K, %[2])  \\ a uniformizer for p1
%8 = Mod(x + 2, x^3 - 2)
? #idealprimedec(K, 5, 1) \\ restrict to f = 1
```

```
%9 = 1              \\ now only p1
```

The library syntax is `GEN idealprimedec_limit_f(GEN nf, GEN p, long f)`.

3.8.73 idealprincipalunits(nf, pr, k). Given a prime ideal in `idealprimedec` format, returns the multiplicative group $(1 + pr)/(1 + pr^k)$ as an abelian group. This function is much faster than `idealstar` when the norm of pr is large, since it avoids (useless) work in the multiplicative group of the residue field.

```
? K = nfinit(y^2+1);
? P = idealprimedec(K,2)[1];
? G = idealprincipalunits(K, P, 20);
? G.cyc
%4 = [512, 256, 4]   \\ Z/512 x Z/256 x Z/4
? G.gen
%5 = [[-1, -2]~, 1021, [0, -1]~] \\ minimal generators of given order
```

The library syntax is `GEN idealprincipalunits(GEN nf, GEN pr, long k)`.

3.8.74 idealramgroups(nf, gal, pr). Let K be the number field defined by nf and assume that $K/\mathbf{Q}$ is Galois with Galois group G given by `gal=galoisinit(nf)`. Let pr be the prime ideal $\mathfrak{P}$ in prid format. This function returns a vector g of subgroups of `gal` as follow:

- `g[1]` is the decomposition group of $\mathfrak{P}$,
- `g[2]` is $G_0(\mathfrak{P})$, the inertia group of $\mathfrak{P}$,

and for $i \geq 2$,

- `g[i]` is $G_{i-2}(\mathfrak{P})$, the $i-2$-th ramification group of $\mathfrak{P}$.

The length of g is the number of non-trivial groups in the sequence, thus is 0 if $e = 1$ and $f = 1$, and 1 if $f > 1$ and $e = 1$. The following function computes the cardinality of a subgroup of G, as given by the components of g:

```
card(H) =my(o=H[2]); prod(i=1,#o,o[i]);
```

```
? nf=nfinit(x^6+3); gal=galoisinit(nf); pr=idealprimedec(nf,3)[1];
? g = idealramgroups(nf, gal, pr);
? apply(card,g)
%3 = [6, 6, 3, 3, 3] \\ cardinalities of the G_i
```

```
? nf=nfinit(x^6+108); gal=galoisinit(nf); pr=idealprimedec(nf,2)[1];
? iso=idealramgroups(nf,gal,pr)[2]
%5 = [[Vecsmall([2, 3, 1, 5, 6, 4])], Vecsmall([3])]
? nfdisc(galoisfixedfield(gal,iso,1))
%6 = -3
```

The field fixed by the inertia group of 2 is not ramified at 2.

The library syntax is `GEN idealramgroups(GEN nf, GEN gal, GEN pr)`.

3.8.75 idealred$(nf, I, \{v = 0\})$. LLL reduction of the ideal I in the number field K attached to nf, along the direction v. The v parameter is best left omitted, but if it is present, it must be an `nf.r1` + `nf.r2`-component vector of *non-negative* integers. (What counts is the relative magnitude of the entries: if all entries are equal, the effect is the same as if the vector had been omitted.)

This function finds an $a \in K^*$ such that $J = (a)I$ is "small" and integral (see the end for technical details). The result is the Hermite normal form of the "reduced" ideal J.

```
? K = nfinit(y^2+1);
? P = idealprimedec(K,5)[1];
? idealred(K, P)
%3 =
[1 0]

[0 1]
```

More often than not, a principal ideal yields the unit ideal as above. This is a quick and dirty way to check if ideals are principal, but it is not a necessary condition: a non-trivial result does not prove that the ideal is non-principal. For guaranteed results, see `bnfisprincipal`, which requires the computation of a full `bnf` structure.

If the input is an extended ideal $[I, s]$, the output is $[J, sa]$; in this way, one keeps track of the principal ideal part:

```
? idealred(K, [P, 1])
%5 = [[1, 0; 0, 1], [2, -1]~]
```

meaning that P is generated by $[2, -1]$. The number field element in the extended part is an algebraic number in any form *or* a factorization matrix (in terms of number field elements, not ideals!). In the latter case, elements stay in factored form, which is a convenient way to avoid coefficient explosion; see also `idealpow`.

Technical note. The routine computes an LLL-reduced basis for the lattice $I^(-1)$ equipped with the quadratic form

$$||x||_v^2 = \sum_{i=1}^{r_1+r_2} 2^{v_i} \varepsilon_i |\sigma_i(x)|^2,$$

where as usual the σ_i are the (real and) complex embeddings and $\varepsilon_i = 1$, resp. 2, for a real, resp. complex place. The element a is simply the first vector in the LLL basis. The only reason you may want to try to change some directions and set some $v_i \neq 0$ is to randomize the elements found for a fixed ideal, which is heuristically useful in index calculus algorithms like `bnfinit` and `bnfisprincipal`.

Even more technical note. In fact, the above is a white lie. We do not use $||\cdot||_v$ exactly but a rescaled rounded variant which gets us faster and simpler LLLs. There's no harm since we are not using any theoretical property of a after all, except that it belongs to $I^(-1)$ and that aI is "expected to be small".

The library syntax is `GEN idealred0(GEN nf, GEN I, GEN v = NULL)`.

3.8.76 idealstar$(\{nf\}, N, \{flag = 1\})$. Outputs a `bid` structure, necessary for computing in the finite abelian group $G = (\mathbf{Z}_K/N)^*$. Here, *nf* is a number field and N is a *modulus*: either an ideal in any form, or a row vector whose first component is an ideal and whose second component is a row vector of r_1 0 or 1. Ideals can also be given by a factorization into prime ideals, as produced by `idealfactor`.

This *bid* is used in `ideallog` to compute discrete logarithms. It also contains useful information which can be conveniently retrieved as *bid*`.mod` (the modulus), *bid*`.clgp` (G as a finite abelian group), *bid*`.no` (the cardinality of G), *bid*`.cyc` (elementary divisors) and *bid*`.gen` (generators).

If $flag = 1$ (default), the result is a `bid` structure without generators: they are well defined but not explicitly computed, which saves time.

If $flag = 2$, as $flag = 1$, but including generators.

If $flag = 0$, only outputs $(\mathbf{Z}_K/N)^*$ as an abelian group, i.e as a 3-component vector $[h, d, g]$: h is the order, d is the vector of SNF cyclic components and g the corresponding generators.

If *nf* is omitted, we take it to be the rational number fields, N must be an integer and we return the structure of $(\mathbf{Z}/N\mathbf{Z})^*$. In other words `idealstar(, N, flag)` is short for

```
idealstar(nfinit(x), N, flag)
```

but much faster. The alternative syntax `znstar(N, flag)` is also available for the same effect, but due to an unfortunate historical oversight, the default value of `flag` is different in the two functions (`znstar` does not initialize by default).

The library syntax is `GEN idealstar0(GEN nf = NULL, GEN N, long flag)`. Instead the above hardcoded numerical flags, one should rather use `GEN Idealstar(GEN nf, GEN ideal, long flag)`, where `flag` is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (return a full `bid`, not a group), possibly 0. This offers one more combination: gen, but no init.

3.8.77 idealtwoelt$(nf, x, \{a\})$. Computes a two-element representation of the ideal x in the number field *nf*, combining a random search and an approximation theorem; x is an ideal in any form (possibly an extended ideal, whose principal part is ignored)

• When called as `idealtwoelt(nf,x)`, the result is a row vector $[a, \alpha]$ with two components such that $x = a\mathbf{Z}_K + \alpha\mathbf{Z}_K$ and a is chosen to be the positive generator of $x \cap \mathbf{Z}$, unless x was given as a principal ideal (in which case we may choose $a = 0$). The algorithm uses a fast lazy factorization of $x \cap \mathbf{Z}$ and runs in randomized polynomial time.

• When called as `idealtwoelt(nf,x,a)` with an explicit non-zero a supplied as third argument, the function assumes that $a \in x$ and returns $\alpha \in x$ such that $x = a\mathbf{Z}_K + \alpha\mathbf{Z}_K$. Note that we must factor a in this case, and the algorithm is generally much slower than the default variant.

The library syntax is `GEN idealtwoelt0(GEN nf, GEN x, GEN a = NULL)`. Also available are `GEN idealtwoelt(GEN nf, GEN x)` and `GEN idealtwoelt2(GEN nf, GEN x, GEN a)`.

3.8.78 idealval(nf, x, pr). Gives the valuation of the ideal x at the prime ideal *pr* in the number field *nf*, where *pr* is in `idealprimedec` format. The valuation of the 0 ideal is `+oo`.

The library syntax is `GEN gpidealval(GEN nf, GEN x, GEN pr)`. Also available is `long idealval(GEN nf, GEN x, GEN pr)`, which returns `LONG_MAX` if $x = 0$ and the valuation as a `long` integer.

3.8.79 matalgtobasis(nf, x). This function is deprecated, use `apply`.

nf being a number field in `nfinit` format, and x a (row or column) vector or matrix, apply `nfalgtobasis` to each entry of x.

The library syntax is `GEN matalgtobasis(GEN nf, GEN x)`.

3.8.80 matbasistoalg(nf, x). This function is deprecated, use `apply`.

nf being a number field in `nfinit` format, and x a (row or column) vector or matrix, apply `nfbasistoalg` to each entry of x.

The library syntax is `GEN matbasistoalg(GEN nf, GEN x)`.

3.8.81 modreverse(z). Let $z =$ `Mod(A,T)` be a polmod, and Q be its minimal polynomial, which must satisfy $\deg(Q) = \deg(T)$. Returns a "reverse polmod" `Mod(B, Q)`, which is a root of T.

This is quite useful when one changes the generating element in algebraic extensions:

```
? u = Mod(x, x^3 - x -1); v = u^5;
? w = modreverse(v)
%2 = Mod(x^2 - 4*x + 1, x^3 - 5*x^2 + 4*x - 1)
```

which means that $x^3 - 5x^2 + 4x - 1$ is another defining polynomial for the cubic field

$$\mathbf{Q}(u) = \mathbf{Q}[x]/(x^3 - x - 1) = \mathbf{Q}[x]/(x^3 - 5x^2 + 4x - 1) = \mathbf{Q}(v),$$

and that $u \to v^2 - 4v + 1$ gives an explicit isomorphism. From this, it is easy to convert elements between the $A(u) \in \mathbf{Q}(u)$ and $B(v) \in \mathbf{Q}(v)$ representations:

```
? A = u^2 + 2*u + 3; subst(lift(A), 'x, w)
%3 = Mod(x^2 - 3*x + 3, x^3 - 5*x^2 + 4*x - 1)
? B = v^2 + v + 1;   subst(lift(B), 'x, v)
%4 = Mod(26*x^2 + 31*x + 26, x^3 - x - 1)
```

If the minimal polynomial of z has lower degree than expected, the routine fails

```
? u = Mod(-x^3 + 9*x, x^4 - 10*x^2 + 1)
? modreverse(u)
 *** modreverse: domain error in modreverse: deg(minpoly(z)) < 4
 ***   Break loop: type 'break' to go back to GP prompt
break> Vec( dbg_err() ) \\ ask for more info
["e_DOMAIN", "modreverse", "deg(minpoly(z))", "<", 4,
  Mod(-x^3 + 9*x, x^4 - 10*x^2 + 1)]
break> minpoly(u)
x^2 - 8
```

The library syntax is `GEN modreverse(GEN z)`.

3.8.82 newtonpoly(x, p). Gives the vector of the slopes of the Newton polygon of the polynomial x with respect to the prime number p. The n components of the vector are in decreasing order, where n is equal to the degree of x. Vertical slopes occur iff the constant coefficient of x is zero and are denoted by `+oo`.

The library syntax is `GEN newtonpoly(GEN x, GEN p)`.

3.8.83 nfalgtobasis(nf, x). Given an algebraic number x in the number field nf, transforms it to a column vector on the integral basis nf.`.zk`.

```
? nf = nfinit(y^2 + 4);
? nf.zk
%2 = [1, 1/2*y]
? nfalgtobasis(nf, [1,1]~)
%3 = [1, 1]~
? nfalgtobasis(nf, y)
%4 = [0, 2]~
? nfalgtobasis(nf, Mod(y, y^2+4))
%5 = [0, 2]~
```

This is the inverse function of `nfbasistoalg`.

The library syntax is `GEN algtobasis(GEN nf, GEN x)`.

3.8.84 nfbasis(T). Let $T(X)$ be an irreducible polynomial with integral coefficients. This function returns an integral basis of the number field defined by T, that is a $\mathbf{Z}$-basis of its maximal order. The basis elements are given as elements in $\mathbf{Q}[X]/(T)$:

```
? nfbasis(x^2 + 1)
%1 = [1, x]
```

This function uses a modified version of the round 4 algorithm, due to David Ford, Sebastian Pauli and Xavier Roblot.

Local basis, orders maximal at certain primes.

Obtaining the maximal order is hard: it requires factoring the discriminant D of T. Obtaining an order which is maximal at a finite explicit set of primes is easy, but it may then be a strict suborder of the maximal order. To specify that we are interested in a given set of places only, we can replace the argument T by an argument $[T, listP]$, where *listP* encodes the primes we are interested in: it must be a factorization matrix, a vector of integers or a single integer.

• Vector: we assume that it contains distinct *prime* numbers.

• Matrix: we assume that it is a two-column matrix of a (partial) factorization of D; namely the first column contains distinct *primes* and the second one the valuation of D at each of these primes.

• Integer B: this is replaced by the vector of primes up to B. Note that the function will use at least $O(B)$ time: a small value, about 10^5, should be enough for most applications. Values larger than 2^{32} are not supported.

In all these cases, the primes may or may not divide the discriminant D of T. The function then returns a $\mathbf{Z}$-basis of an order whose index is not divisible by any of these prime numbers. The result is actually a global integral basis if all prime divisors of the *field* discriminant are included! Note that `nfinit` has built-in support for such a check:

```
? K = nfinit([T, listP]);
? nfcertify(K)   \\ we computed an actual maximal order
%2 = [];
```

The first line initializes a number field structure incorporating `nfbasis([T, listP]` in place of a proven integral basis. The second line certifies that the resulting structure is correct. This allows

to create an `nf` structure attached to the number field $K = \mathbf{Q}[X]/(T)$, when the discriminant of T cannot be factored completely, whereas the prime divisors of disc$K$ are known.

Of course, if *listP* contains a single prime number p, the function returns a local integral basis for $\mathbf{Z}_p[X]/(T)$:

```
? nfbasis(x^2+x-1001)
%1 = [1, 1/3*x - 1/3]
? nfbasis( [x^2+x-1001, [2]] )
%2 = [1, x]
```

The Buchmann-Lenstra algorithm.

We now complicate the picture: it is in fact allowed to include *composite* numbers instead of primes in `listP` (Vector or Matrix case), provided they are pairwise coprime. The result will still be a correct integral basis *if* the field discriminant factors completely over the actual primes in the list. Adding a composite C such that C^2 *divides* D may help because when we consider C as a prime and run the algorithm, two good things can happen: either we succeed in proving that no prime dividing C can divide the index (without actually needing to find those primes), or the computation exhibits a non-trivial zero divisor, thereby factoring C and we go on with the refined factorization. (Note that including a C such that C^2 does not divide D is useless.) If neither happen, then the computed basis need not generate the maximal order. Here is an example:

```
? B = 10^5;
? P = factor(poldisc(T), B)[,1]; \\ primes <= B dividing D + cofactor
? basis = nfbasis([T, listP])
? disc = nfdisc([T, listP])
```

We obtain the maximal order and its discriminant if the field discriminant factors completely over the primes less than B (together with the primes contained in the `addprimes` table). This can be tested as follows:

```
check = factor(disc, B);
lastp = check[-1..-1,1];
if (lastp > B && !setsearch(addprimes(), lastp),
  warning("nf may be incorrect!"))
```

This is a sufficient but not a necessary condition, hence the warning, instead of an error. N.B. `lastp` is the last entry in the first column of the `check` matrix, i.e. the largest prime dividing `nf.disc` if $\leq B$ or if it belongs to the prime table.

The function `nfcertify` speeds up and automates the above process:

```
? B = 10^5;
? nf = nfinit([T, B]);
? nfcertify(nf)
%3 = []      \\ nf is unconditionally correct
? basis = nf.zk;
? disc = nf.disc;
```

The library syntax is **nfbasis**(`GEN T, GEN *d, GEN listP = NULL`), which returns the order basis, and where `*d` receives the order discriminant.

3.8.85 nfbasistoalg(nf, x). Given an algebraic number x in the number field nf, transforms it into `t_POLMOD` form.

```
? nf = nfinit(y^2 + 4);
? nf.zk
%2 = [1, 1/2*y]
? nfbasistoalg(nf, [1,1]~)
%3 = Mod(1/2*y + 1, y^2 + 4)
? nfbasistoalg(nf, y)
%4 = Mod(y, y^2 + 4)
? nfbasistoalg(nf, Mod(y, y^2+4))
%5 = Mod(y, y^2 + 4)
```

This is the inverse function of `nfalgtobasis`.

The library syntax is `GEN basistoalg(GEN nf, GEN x)`.

3.8.86 nfcertify(nf). *nf* being as output by `nfinit`, checks whether the integer basis is known unconditionally. This is in particular useful when the argument to `nfinit` was of the form $[T, \texttt{listP}]$, specifying a finite list of primes when p-maximality had to be proven, or a list of coprime integers to which Buchmann-Lenstra algorithm was to be applied.

The function returns a vector of coprime composite integers. If this vector is empty, then `nf.zk` and `nf.disc` are correct. Otherwise, the result is dubious. In order to obtain a certified result, one must completely factor each of the given integers, then `addprime` each of their prime factors, then check whether `nfdisc(nf.pol)` is equal to `nf.disc`.

The library syntax is `GEN nfcertify(GEN nf)`.

3.8.87 nfcompositum$(nf, P, Q, \{flag = 0\})$. Let *nf* be a number field structure attached to the field K and let P and Q be squarefree polynomials in $K[X]$ in the same variable. Outputs the simple factors of the étale K-algebra $A = K[X, Y]/(P(X), Q(Y))$. The factors are given by a list of polynomials R in $K[X]$, attached to the number field $K[X]/(R)$, and sorted by increasing degree (with respect to lexicographic ordering for factors of equal degrees). Returns an error if one of the polynomials is not squarefree.

Note that it is more efficient to reduce to the case where P and Q are irreducible first. The routine will not perform this for you, since it may be expensive, and the inputs are irreducible in most applications anyway. In this case, there will be a single factor R if and only if the number fields defined by P and Q are linearly disjoint (their intersection is K).

The binary digits of *flag* mean

1: outputs a vector of 4-component vectors $[R, a, b, k]$, where R ranges through the list of all possible compositums as above, and a (resp. b) expresses the root of P (resp. Q) as an element of $K[X]/(R)$. Finally, k is a small integer such that $b + ka = X$ modulo R.

2: assume that P and Q define number fields that are linearly disjoint: both polynomials are irreducible and the corresponding number fields have no common subfield besides K. This allows to save a costly factorization over K. In this case return the single simple factor instead of a vector with one element.

A compositum is often defined by a complicated polynomial, which it is advisable to reduce before further work. Here is an example involving the field $K(\zeta_5, 5^{1/10})$, $K = \mathbf{Q}(\sqrt{5})$:

```
? K = nfinit(y^2-5);
? L = nfcompositum(K, x^5 - y, polcyclo(5), 1); \\ list of [R,a,b,k]
? [R, a] = L[1];  \\ pick the single factor, extract R,a (ignore b,k)
? lift(R)         \\ defines the compositum
%4 = x^10 + (-5/2*y + 5/2)*x^9 + (-5*y + 20)*x^8 + (-20*y + 30)*x^7 + \
(-45/2*y + 145/2)*x^6 + (-71/2*y + 121/2)*x^5 + (-20*y + 60)*x^4 +    \
(-25*y + 5)*x^3 + 45*x^2 + (-5*y + 15)*x + (-2*y + 6)
? a^5 - y         \\ a fifth root of y
%5 = 0
? [T, X] = rnfpolredbest(K, R, 1);
? lift(T)     \\ simpler defining polynomial for K[x]/(R)
%7 = x^10 + (-11/2*y + 25/2)
? liftall(X)  \\  root of R in K[x]/(T(x))
%8 = (3/4*y + 7/4)*x^7 + (-1/2*y - 1)*x^5 + 1/2*x^2 + (1/4*y - 1/4)
? a = subst(a.pol, 'x, X);  \\ a in the new coordinates
? liftall(a)
%10 = (-3/4*y - 7/4)*x^7 - 1/2*x^2
? a^5 - y
%11 = 0
```

The main variables of P and Q must be the same and have higher priority than that of nf (see `varhigher` and `varlower`).

The library syntax is `GEN nfcompositum(GEN nf, GEN P, GEN Q, long flag)`.

3.8.88 nfdetint(nf, x). Given a pseudo-matrix x, computes a non-zero ideal contained in (i.e. multiple of) the determinant of x. This is particularly useful in conjunction with `nfhnfmod`.

The library syntax is `GEN nfdetint(GEN nf, GEN x)`.

3.8.89 nfdisc(T). field discriminant of the number field defined by the integral, preferably monic, irreducible polynomial $T(X)$. Returns the discriminant of the number field $\mathbf{Q}[X]/(T)$, using the Round 4 algorithm.

Local discriminants, valuations at certain primes.

As in `nfbasis`, the argument T can be replaced by $[T, listP]$, where `listP` is as in `nfbasis`: a vector of pairwise coprime integers (usually distinct primes), a factorization matrix, or a single integer. In that case, the function returns the discriminant of an order whose basis is given by `nfbasis(T,listP)`, which need not be the maximal order, and whose valuation at a prime entry in `listP` is the same as the valuation of the field discriminant.

In particular, if `listP` is $[p]$ for a prime p, we can return the p-adic discriminant of the maximal order of $\mathbf{Z}_p[X]/(T)$, as a power of p, as follows:

```
? padicdisc(T,p) = p^valuation(nfdisc(T,[p]), p);
? nfdisc(x^2 + 6)
%2 = -24
? padicdisc(x^2 + 6, 2)
%3 = 8
? padicdisc(x^2 + 6, 3)
%4 = 3
```

The library syntax is **nfdisc**(GEN T) (listP = NULL). Also available is GEN nfbasis(GEN T, GEN *d, GEN listP = NULL), which returns the order basis, and where *d receives the order discriminant.

3.8.90 nfeltadd(nf, x, y). Given two elements x and y in nf, computes their sum $x + y$ in the number field nf.

The library syntax is GEN nfadd(GEN nf, GEN x, GEN y).

3.8.91 nfeltdiv(nf, x, y). Given two elements x and y in nf, computes their quotient x/y in the number field nf.

The library syntax is GEN nfdiv(GEN nf, GEN x, GEN y).

3.8.92 nfeltdiveuc(nf, x, y). Given two elements x and y in nf, computes an algebraic integer q in the number field nf such that the components of $x - qy$ are reasonably small. In fact, this is functionally identical to round(nfdiv(nf,x,y)).

The library syntax is GEN nfdiveuc(GEN nf, GEN x, GEN y).

3.8.93 nfeltdivmodpr(nf, x, y, pr). This function is obsolete, use nfmodpr.

Given two elements x and y in nf and pr a prime ideal in modpr format (see nfmodprinit), computes their quotient x/y modulo the prime ideal pr.

The library syntax is GEN nfdivmodpr(GEN nf, GEN x, GEN y, GEN pr). This function is normally useless in library mode. Project your inputs to the residue field using nf_to_Fq, then work there.

3.8.94 nfeltdivrem(nf, x, y). Given two elements x and y in nf, gives a two-element row vector $[q, r]$ such that $x = qy + r$, q is an algebraic integer in nf, and the components of r are reasonably small.

The library syntax is GEN nfdivrem(GEN nf, GEN x, GEN y).

3.8.95 nfeltmod(nf, x, y). Given two elements x and y in nf, computes an element r of nf of the form $r = x - qy$ with q and algebraic integer, and such that r is small. This is functionally identical to

$$\mathtt{x} - \mathtt{nfmul}(nf, \mathtt{round}(\mathtt{nfdiv}(nf, \mathtt{x}, \mathtt{y})), \mathtt{y}).$$

The library syntax is GEN nfmod(GEN nf, GEN x, GEN y).

3.8.96 nfeltmul(nf, x, y). Given two elements x and y in nf, computes their product $x * y$ in the number field nf.

The library syntax is GEN nfmul(GEN nf, GEN x, GEN y).

3.8.97 nfeltmulmodpr(nf, x, y, pr). This function is obsolete, use nfmodpr.

Given two elements x and y in nf and pr a prime ideal in modpr format (see nfmodprinit), computes their product $x * y$ modulo the prime ideal pr.

The library syntax is GEN nfmulmodpr(GEN nf, GEN x, GEN y, GEN pr). This function is normally useless in library mode. Project your inputs to the residue field using nf_to_Fq, then work there.

3.8.98 nfeltnorm(nf, x). Returns the absolute norm of x.

The library syntax is `GEN nfnorm(GEN nf, GEN x)`.

3.8.99 nfeltpow(nf, x, k). Given an element x in nf, and a positive or negative integer k, computes x^k in the number field nf.

The library syntax is `GEN nfpow(GEN nf, GEN x, GEN k)`. `GEN nfinv(GEN nf, GEN x)` correspond to $k = -1$, and `GEN nfsqr(GEN nf, GEN x)` to $k = 2$.

3.8.100 nfeltpowmodpr(nf, x, k, pr). This function is obsolete, use `nfmodpr`.

Given an element x in nf, an integer k and a prime ideal pr in `modpr` format (see `nfmodprinit`), computes x^k modulo the prime ideal pr.

The library syntax is `GEN nfpowmodpr(GEN nf, GEN x, GEN k, GEN pr)`. This function is normally useless in library mode. Project your inputs to the residue field using `nf_to_Fq`, then work there.

3.8.101 nfeltreduce(nf, a, id). Given an ideal id in Hermite normal form and an element a of the number field nf, finds an element r in nf such that $a - r$ belongs to the ideal and r is small.

The library syntax is `GEN nfreduce(GEN nf, GEN a, GEN id)`.

3.8.102 nfeltreducemodpr(nf, x, pr). This function is obsolete, use `nfmodpr`.

Given an element x of the number field nf and a prime ideal pr in `modpr` format compute a canonical representative for the class of x modulo pr.

The library syntax is `GEN nfreducemodpr(GEN nf, GEN x, GEN pr)`. This function is normally useless in library mode. Project your inputs to the residue field using `nf_to_Fq`, then work there.

3.8.103 nfelttrace(nf, x). Returns the absolute trace of x.

The library syntax is `GEN nftrace(GEN nf, GEN x)`.

3.8.104 nfeltval$(nf, x, pr, \{\&y\})$. Given an element x in nf and a prime ideal pr in the format output by `idealprimedec`, computes the valuation v at pr of the element x. The valuation of 0 is `+oo`.

```
? nf = nfinit(x^2 + 1);
? P = idealprimedec(nf, 2)[1];
? nfeltval(nf, x+1, P)
%3 = 1
```

This particular valuation can also be obtained using `idealval(`nf`,x,`pr`)`, since x is then converted to a principal ideal.

If the y argument is present, sets $y = x\tau^v$, where τ is a fixed "anti-uniformizer" for pr: its valuation at pr is -1; its valuation is 0 at other prime ideals dividing pr`.p` and nonnegative at all other primes. In other words y is the part of x coprime to pr. If x is an algebraic integer, so is y.

```
? nfeltval(nf, x+1, P, &y); y
%4 = [0, 1]~
```

For instance if $x = \prod_i x_i^{e_i}$ is known to be coprime to pr, where the x_i are algebraic integers and $e_i \in \mathbf{Z}$ then, if $v_i = \texttt{nfeltval}(nf, x_i, pr, \&y_i)$, we still have $x = \prod_i y_i^{e_i}$, where the y_i are still algebraic integers but now all of them are coprime to pr. They can then be mapped to the residue field of pr more efficiently than if the product had been expanded beforehand: we can reduce mod pr after each ring operation.

The library syntax is `GEN gpnfvalrem(GEN nf, GEN x, GEN pr, GEN *y = NULL)`. Also available is `long nfvalrem(GEN nf, GEN x, GEN pr, GEN *y = NULL)`, which returns `LONG_MAX` if $x = 0$ and the valuation as a `long` integer.

3.8.105 nffactor(nf, T). Factorization of the univariate polynomial T over the number field nf given by `nfinit`; T has coefficients in nf (i.e. either scalar, polmod, polynomial or column vector). The factors are sorted by increasing degree.

The main variable of nf must be of *lower* priority than that of T, see Section 2.5.3. However if the polynomial defining the number field occurs explicitly in the coefficients of T as modulus of a `t_POLMOD` or as a `t_POL` coefficient, its main variable must be *the same* as the main variable of T. For example,

```
? nf = nfinit(y^2 + 1);
? nffactor(nf, x^2 + y); \\ OK
? nffactor(nf, x^2 + Mod(y, y^2+1)); \\  OK
? nffactor(nf, x^2 + Mod(z, z^2+1)); \\  WRONG
```

It is possible to input a defining polynomial for nf instead, but this is in general less efficient since parts of an `nf` structure will then be computed internally. This is useful in two situations: when you do not need the `nf` elsewhere, or when you cannot initialize an `nf` due to integer factorization difficulties when attempting to compute the field discriminant and maximal order.

Caveat. `nfinit([T, listP])` allows to compute in polynomial time a conditional nf structure, which sets `nf.zk` to an order which is not guaranteed to be maximal at all primes. Always either use `nfcertify` first (which may not run in polynomial time) or make sure to input `nf.pol` instead of the conditional nf: `nffactor` is able to recover in polynomial time in this case, instead of potentially missing a factor.

The library syntax is `GEN nffactor(GEN nf, GEN T)`.

3.8.106 nffactorback$(nf, f, \{e\})$. Gives back the nf element corresponding to a factorization. The integer 1 corresponds to the empty factorization.

If e is present, e and f must be vectors of the same length (e being integral), and the corresponding factorization is the product of the $f[i]^{e[i]}$.

If not, and f is vector, it is understood as in the preceding case with e a vector of 1s: we return the product of the $f[i]$. Finally, f can be a regular factorization matrix.

```
? nf = nfinit(y^2+1);
? nffactorback(nf, [3, y+1, [1,2]~], [1, 2, 3])
%2 = [12, -66]~
? 3 * (I+1)^2 * (1+2*I)^3
%3 = 12 - 66*I
```

The library syntax is `GEN nffactorback(GEN nf, GEN f, GEN e = NULL)`.

3.8.107 nffactormod(nf, Q, pr). This routine is obsolete, use `nfmodpr` and `factorff`.

Factors the univariate polynomial Q modulo the prime ideal pr in the number field nf. The coefficients of Q belong to the number field (scalar, polmod, polynomial, even column vector) and the main variable of nf must be of lower priority than that of Q (see Section 2.5.3). The prime ideal pr is either in `idealprimedec` or (preferred) `modprinit` format. The coefficients of the polynomial factors are lifted to elements of nf:

```
? K = nfinit(y^2+1);
? P = idealprimedec(K, 3)[1];
? nffactormod(K, x^2 + y*x + 18*y+1, P)
%3 =
[x + (2*y + 1) 1]

[x + (2*y + 2) 1]
? P = nfmodprinit(K, P);  \\ convert to nfmodprinit format
? nffactormod(K, x^2 + y*x + 18*y+1)
%5 =
[x + (2*y + 1) 1]

[x + (2*y + 2) 1]
```

Same result, of course, here about 10% faster due to the precomputation.

The library syntax is `GEN nffactormod(GEN nf, GEN Q, GEN pr)`.

3.8.108 nfgaloisapply(nf, aut, x). Let nf be a number field as output by `nfinit`, and let aut be a Galois automorphism of nf expressed by its image on the field generator (such automorphisms can be found using `nfgaloisconj`). The function computes the action of the automorphism aut on the object x in the number field; x can be a number field element, or an ideal (possibly extended). Because of possible confusion with elements and ideals, other vector or matrix arguments are forbidden.

```
? nf = nfinit(x^2+1);
? L = nfgaloisconj(nf)
%2 = [-x, x]~
? aut = L[1]; /* the non-trivial automorphism */
? nfgaloisapply(nf, aut, x)
%4 = Mod(-x, x^2 + 1)
? P = idealprimedec(nf,5); /* prime ideals above 5 */
? nfgaloisapply(nf, aut, P[2]) == P[1]
%6 = 0 \\ !!!!
? idealval(nf, nfgaloisapply(nf, aut, P[2]), P[1])
%7 = 1
```

The surprising failure of the equality test (`%7`) is due to the fact that although the corresponding prime ideals are equal, their representations are not. (A prime ideal is specified by a uniformizer, and there is no guarantee that applying automorphisms yields the same elements as a direct `idealprimedec` call.)

The automorphism can also be given as a column vector, representing the image of `Mod(x, nf.pol)` as an algebraic number. This last representation is more efficient and should be preferred if a given automorphism must be used in many such calls.

```
? nf = nfinit(x^3 - 37*x^2 + 74*x - 37);
? aut = nfgaloisconj(nf)[2]; \\  an automorphism in basistoalg form
%2 = -31/11*x^2 + 1109/11*x - 925/11
? AUT = nfalgtobasis(nf, aut); \\  same in algtobasis form
%3 = [16, -6, 5]~
? v = [1, 2, 3]~; nfgaloisapply(nf, aut, v) == nfgaloisapply(nf, AUT, v)
%4 = 1 \\  same result...
? for (i=1,10^5, nfgaloisapply(nf, aut, v))
time = 463 ms.
? for (i=1,10^5, nfgaloisapply(nf, AUT, v))
time = 343 ms.  \\  but the latter is faster
```

The library syntax is `GEN galoisapply(GEN nf, GEN aut, GEN x)`.

3.8.109 nfgaloisconj$(nf, \{flag = 0\}, \{d\})$. *nf* being a number field as output by `nfinit`, computes the conjugates of a root r of the non-constant polynomial $x = nf[1]$ expressed as polynomials in r. This also makes sense when the number field is not Galois since some conjugates may lie in the field. *nf* can simply be a polynomial.

If no flags or $flag = 0$, use a combination of flag 4 and 1 and the result is always complete. There is no point whatsoever in using the other flags.

If $flag = 1$, use `nfroots`: a little slow, but guaranteed to work in polynomial time.

If $flag = 4$, use `galoisinit`: very fast, but only applies to (most) Galois fields. If the field is Galois with weakly super-solvable Galois group (see `galoisinit`), return the complete list of automorphisms, else only the identity element. If present, d is assumed to be a multiple of the least common denominator of the conjugates expressed as polynomial in a root of *pol*.

This routine can only compute $\mathbf{Q}$-automorphisms, but it may be used to get K-automorphism for any base field K as follows:

```
rnfgaloisconj(nfK, R) = \\ K-automorphisms of L = K[X] / (R)
{
  my(polabs, N,al,S, ala,k, vR);
  R *= Mod(1, nfK.pol); \\ convert coeffs to polmod elts of K
  vR = variable(R);
  al = Mod(variable(nfK.pol),nfK.pol);
  [polabs,ala,k] = rnfequation(nfK, R, 1);
  Rt = if(k==0,R,subst(R,vR,vR-al*k));
  N = nfgaloisconj(polabs) % Rt; \\ Q-automorphisms of L
  S = select(s->subst(Rt, vR, Mod(s,Rt)) == 0, N);
  if (k==0, S, apply(s->subst(s,vR,vR+k*al)-k*al,S));
}
K  = nfinit(y^2 + 7);
rnfgaloisconj(K, x^4 - y*x^3 - 3*x^2 + y*x + 1)  \\ K-automorphisms of L
```

The library syntax is `GEN galoisconj0(GEN nf, long flag, GEN d = NULL, long prec)`. Use directly `GEN galoisconj(GEN nf, GEN d)`, corresponding to $flag = 0$, the others only have historical interest.

3.8.110 nfgrunwaldwang$(nf, Lpr, Ld, pl, \{v =' x\})$. Given nf a number field in nf or bnf format, a `t_VEC` Lpr of primes of nf and a `t_VEC` Ld of positive integers of the same length, a `t_VECSMALL` pl of length r_1 the number of real places of nf, computes a polynomial with coefficients in nf defining a cyclic extension of nf of minimal degree satisfying certain local conditions:

- at the prime `Lpr[i]`, the extension has local degree a multiple of `Ld[i]`;
- at the i-th real place of nf, it is complex if $pl[i] = -1$ (no condition if $pl[i] = 0$).

The extension has degree the LCM of the local degrees. Currently, the degree is restricted to be a prime power for the search, and to be prime for the construction because of the `rnfkummer` restrictions.

When nf is $\mathbf{Q}$, prime integers are accepted instead of `prid` structures. However, their primality is not checked and the behaviour is undefined if you provide a composite number.

Warning. If the number field nf does not contain the n-th roots of unity where n is the degree of the extension to be computed, triggers the computation of the bnf of $nf(\zeta_n)$, which may be costly.

```
? nf = nfinit(y^2-5);
? pr = idealprimedec(nf,13)[1];
? pol = nfgrunwaldwang(nf, [pr], [2], [0,-1], 'x)
%3 = x^2 + Mod(3/2*y + 13/2, y^2 - 5)
```

The library syntax is `GEN nfgrunwaldwang(GEN nf, GEN Lpr, GEN Ld, GEN pl, long v = -1)` where `v` is a variable number.

3.8.111 nfhilbert$(nf, a, b, \{pr\})$. If pr is omitted, compute the global quadratic Hilbert symbol (a, b) in nf, that is 1 if $x^2 - ay^2 - bz^2$ has a non trivial solution (x, y, z) in nf, and -1 otherwise. Otherwise compute the local symbol modulo the prime ideal pr, as output by `idealprimedec`.

The library syntax is `long nfhilbert0(GEN nf, GEN a, GEN b, GEN pr = NULL)`.

Also available is `long nfhilbert(GEN bnf, GEN a, GEN b)` (global quadratic Hilbert symbol).

3.8.112 nfhnf$(nf, x, \{flag = 0\})$. Given a pseudo-matrix (A, I), finds a pseudo-basis (B, J) in Hermite normal form of the module it generates. If $flag$ is non-zero, also return the transformation matrix U such that $AU = [0|B]$.

The library syntax is `GEN nfhnf0(GEN nf, GEN x, long flag)`. Also available:

`GEN nfhnf(GEN nf, GEN x)` ($flag = 0$).

`GEN rnfsimplifybasis(GEN bnf, GEN x)` simplifies the pseudo-basis given by $x = (A, I)$. The ideals in the list I are integral, primitive and either trivial (equal to the full ring of integer) or non-principal.

3.8.113 nfhnfmod$(nf, x, detx)$. Given a pseudo-matrix (A, I) and an ideal $detx$ which is contained in (read integral multiple of) the determinant of (A, I), finds a pseudo-basis in Hermite normal form of the module generated by (A, I). This avoids coefficient explosion. $detx$ can be computed using the function `nfdetint`.

The library syntax is `GEN nfhnfmod(GEN nf, GEN x, GEN detx)`.

3.8.114 nfinit$(pol, \{flag = 0\})$. *pol* being a non-constant, preferably monic, irreducible polynomial in $\mathbf{Z}[X]$, initializes a *number field* structure (`nf`) attached to the field K defined by *pol*. As such, it's a technical object passed as the first argument to most `nf`*xxx* functions, but it contains some information which may be directly useful. Access to this information via *member functions* is preferred since the specific data organization given below may change in the future. Currently, `nf` is a row vector with 9 components:

$nf[1]$ contains the polynomial *pol* (*nf*`.pol`).

$nf[2]$ contains $[r1, r2]$ (*nf*`.sign`, *nf*`.r1`, *nf*`.r2`), the number of real and complex places of K.

$nf[3]$ contains the discriminant $d(K)$ (*nf*`.disc`) of K.

$nf[4]$ contains the index of $nf[1]$ (*nf*`.index`), i.e. $[\mathbf{Z}_K : \mathbf{Z}[\theta]]$, where θ is any root of $nf[1]$.

$nf[5]$ is a vector containing 7 matrices M, G, *roundG*, T, MD, TI, MDI useful for certain computations in the number field K.

• M is the $(r1+r2)\times n$ matrix whose columns represent the numerical values of the conjugates of the elements of the integral basis.

• G is an $n \times n$ matrix such that $T2 = {}^tGG$, where $T2$ is the quadratic form $T_2(x) = \sum |\sigma(x)|^2$, σ running over the embeddings of K into $\mathbf{C}$.

• *roundG* is a rescaled copy of G, rounded to nearest integers.

• T is the $n \times n$ matrix whose coefficients are $\mathrm{Tr}(\omega_i\omega_j)$ where the ω_i are the elements of the integral basis. Note also that $\det(T)$ is equal to the discriminant of the field K. Also, when understood as an ideal, the matrix T^{-1} generates the codifferent ideal.

• The columns of MD (*nf*`.diff`) express a $\mathbf{Z}$-basis of the different of K on the integral basis.

• TI is equal to the primitive part of T^{-1}, which has integral coefficients.

• Finally, MDI is a two-element representation (for faster ideal product) of $d(K)$ times the codifferent ideal (*nf*`.disc*`*nf*`.codiff`, which is an integral ideal). MDI is only used in `idealinv`.

$nf[6]$ is the vector containing the $r1+r2$ roots (*nf*`.roots`) of $nf[1]$ corresponding to the $r1+r2$ embeddings of the number field into $\mathbf{C}$ (the first $r1$ components are real, the next $r2$ have positive imaginary part).

$nf[7]$ is an integral basis for $\mathbf{Z}_K$ (*nf*`.zk`) expressed on the powers of θ. Its first element is guaranteed to be 1. This basis is LLL-reduced with respect to T_2 (strictly speaking, it is a permutation of such a basis, due to the condition that the first element be 1).

$nf[8]$ is the $n \times n$ integral matrix expressing the power basis in terms of the integral basis, and finally

$nf[9]$ is the $n \times n^2$ matrix giving the multiplication table of the integral basis.

If a non monic polynomial is input, `nfinit` will transform it into a monic one, then reduce it (see *flag* = 3). It is allowed, though not very useful given the existence of `nfnewprec`, to input a *nf* or a *bnf* instead of a polynomial. It is also allowed to input a *rnf*, in which case an `nf` structure attached to the absolute defining polynomial `polabs` is returned (*flag*is then ignored).

```
? nf = nfinit(x^3 - 12); \\ initialize number field Q[X] / (X^3 - 12)
? nf.pol   \\ defining polynomial
```

```
%2 = x^3 - 12
? nf.disc  \\ field discriminant
%3 = -972
? nf.index \\ index of power basis order in maximal order
%4 = 2
? nf.zk    \\ integer basis, lifted to Q[X]
%5 = [1, x, 1/2*x^2]
? nf.sign  \\ signature
%6 = [1, 1]
? factor(abs(nf.disc ))  \\ determines ramified primes
%7 =
[2 2]

[3 5]
? idealfactor(nf, 2)
%8 =
[[2, [0, 0, -1]~, 3, 1, [0, 1, 0]~] 3]  \\  p_2^3
```

Huge discriminants, helping nfdisc.

In case *pol* has a huge discriminant which is difficult to factor, it is hard to compute from scratch the maximal order. The special input format $[pol, B]$ is also accepted where *pol* is a polynomial as above and B has one of the following forms

• an integer basis, as would be computed by `nfbasis`: a vector of polynomials with first element 1. This is useful if the maximal order is known in advance.

• an argument `listP` which specifies a list of primes (see `nfbasis`). Instead of the maximal order, `nfinit` then computes an order which is maximal at these particular primes as well as the primes contained in the private prime table (see `addprimes`). The result is unconditionaly correct when the discriminant `nf.disc` factors completely over this set of primes. The function `nfcertify` automates this:

```
? pol = polcompositum(x^5 - 101, polcyclo(7))[1];
? nf = nfinit( [pol, 10^3] );
? nfcertify(nf)
%3 = []
```

A priori, `nf.zk` defines an order which is only known to be maximal at all primes $\leq 10^3$ (no prime $\leq 10^3$ divides `nf.index`). The certification step proves the correctness of the computation.

If *flag* = 2: *pol* is changed into another polynomial P defining the same number field, which is as simple as can easily be found using the `polredbest` algorithm, and all the subsequent computations are done using this new polynomial. In particular, the first component of the result is the modified polynomial.

If *flag* = 3, apply `polredbest` as in case 2, but outputs $[nf, \texttt{Mod}(a, P)]$, where nf is as before and $\texttt{Mod}(a, P) = \texttt{Mod}(x, pol)$ gives the change of variables. This is implicit when *pol* is not monic: first a linear change of variables is performed, to get a monic polynomial, then `polredbest`.

The library syntax is `GEN nfinit0(GEN pol, long flag, long prec)`. Also available are `GEN nfinit(GEN x, long prec)` (*flag* = 0), `GEN nfinitred(GEN x, long prec)` (*flag* = 2), `GEN nfinitred2(GEN x, long prec)` (*flag* = 3). Instead of the above hardcoded numerical flags in `nfinit0`, one should rather use

`GEN nfinitall(GEN x, long flag, long prec)`, where *flag* is an or-ed combination of

• `nf_RED`: find a simpler defining polynomial,

• `nf_ORIG`: if `nf_RED` set, also return the change of variable,

• `nf_ROUND2`: *Deprecated*. Slow down the routine by using an obsolete normalization algorithm (do not use this one!),

• `nf_PARTIALFACT`: *Deprecated*. Lazy factorization of the polynomial discriminant. Result is conditional unless `nfcertify` can certify it.

3.8.115 nfisideal(nf, x). Returns 1 if x is an ideal in the number field nf, 0 otherwise.

The library syntax is `long isideal(GEN nf, GEN x)`.

3.8.116 nfisincl(x, y). Tests whether the number field K defined by the polynomial x is conjugate to a subfield of the field L defined by y (where x and y must be in $\mathbf{Q}[X]$). If they are not, the output is the number 0. If they are, the output is a vector of polynomials, each polynomial a representing an embedding of K into L, i.e. being such that $y \mid x \circ a$.

If y is a number field (nf), a much faster algorithm is used (factoring x over y using `nffactor`). Before version 2.0.14, this wasn't guaranteed to return all the embeddings, hence was triggered by a special flag. This is no more the case.

The library syntax is `GEN nfisincl(GEN x, GEN y)`.

3.8.117 nfisisom(x, y). As `nfisincl`, but tests for isomorphism. If either x or y is a number field, a much faster algorithm will be used.

The library syntax is `GEN nfisisom(GEN x, GEN y)`.

3.8.118 nfislocalpower(nf, pr, a, n). Let nf be a number field structure attached to K, let $a \in K$ and let pr be a *prid* attched to the maximal ideal v. Return 1 if a is an n-th power in the completed local field K_v, and 0 otherwise.

```
? K = nfinit(y^2+1);
? P = idealprimedec(K,2)[1]; \\ the ramified prime above 2
? nfislocalpower(K,P,-1, 2) \\ -1 is a square
%3 = 1
? nfislocalpower(K,P,-1, 4) \\ ... but not a 4-th power
%4 = 0
? nfislocalpower(K,P,2, 2)  \\ 2 is not a square
%5 = 0
? Q = idealprimedec(K,5)[1]; \\ a prime above 5
? nfislocalpower(K,Q, [0, 32]~, 30)  \\ 32*I is locally a 30-th power
%7 = 1
```

The library syntax is `long nfislocalpower(GEN nf, GEN pr, GEN a, GEN n)`.

3.8.119 nfkermodpr(nf, x, pr). This function is obsolete, use `nfmodpr`.

Kernel of the matrix a in $\mathbf{Z}_K/pr$, where pr is in **modpr** format (see `nfmodprinit`).

The library syntax is `GEN nfkermodpr(GEN nf, GEN x, GEN pr)`. This function is normally useless in library mode. Project your inputs to the residue field using `nfM_to_FqM`, then work there.

3.8.120 nfmodpr(nf, x, pr). Map x to the residue field modulo pr, to a `t_FFELT`. The argument pr is either a maximal ideal in `idealprimedec` format or, preferably, a `modpr` structure from `nfmodprinit`. The function `nfmodprlift` allows to lift back to $\mathbf{Z}_K$.

Note that the function applies to number field elements and not to vector / matrices / polynomials of such. Use `apply` to convert recursive structures.

```
? K = nfinit(y^3-250);
? P = idealprimedec(K, 5)[2]
? modP = nfmodprinit(K,P);
? K.zk
%4 = [1, 1/5*y, 1/25*y^2]
? apply(t->nfmodpr(K,t,modP), K.zk)
%5 = [1, y, 2*y + 1]
```

The library syntax is `GEN nfmodpr(GEN nf, GEN x, GEN pr)`.

3.8.121 nfmodprinit(nf, pr). Transforms the prime ideal pr into `modpr` format necessary for all operations modulo pr in the number field nf. The functions `nfmodpr` and `nfmodprlift` allow to project to and lift from the residue field.

The library syntax is `GEN nfmodprinit(GEN nf, GEN pr)`.

3.8.122 nfmodprlift(nf, x, pr). Lift the `t_FFELT` x (from `nfmodpr`) to the residue field modulo pr. Vectors and matrices are also supported. For polynomials, use `apply` and the present function.

The argument `pr` is either a maximal ideal in `idealprimedec` format or, preferably, a `modpr` structure from `nfmodprinit`. There are no compatibility checks to try and decide whether x is attached the same residue field as defined by `pr`: the result is undefined if not.

The function `nfmodpr` allows to reduce to the residue field.

```
? K = nfinit(y^3-250);
? P = idealprimedec(K, 5)[2]
? modP = nfmodprinit(K,P);
? K.zk
%4 = [1, 1/5*y, 1/25*y^2]
? apply(t->nfmodpr(K,t,modP), K.zk)
%5 = [1, y, 2*y + 1]
? nfmodprlift(K, %, modP)
%6 = [1, 1/5*y, 2/5*y + 1]
? nfeltval(K, %[3] - K.zk[3], P)
%7 = 1
```

The library syntax is `GEN nfmodprlift(GEN nf, GEN x, GEN pr)`.

3.8.123 nfnewprec(nf). Transforms the number field nf into the corresponding data using current (usually larger) precision. This function works as expected if nf is in fact a *bnf* or a *bnr* (update structure to current precision) but may be quite slow: many generators of principal ideals have to be computed; note that in this latter case, the *bnf* must contain fundamental units.

The library syntax is `GEN nfnewprec(GEN nf, long prec)`. See also `GEN bnfnewprec(GEN bnf, long prec)` and `GEN bnrnewprec(GEN bnr, long prec)`.

3.8.124 nfroots$(\{nf\}, x)$. Roots of the polynomial x in the number field nf given by `nfinit` without multiplicity (in **Q** if nf is omitted). x has coefficients in the number field (scalar, polmod, polynomial, column vector). The main variable of nf must be of lower priority than that of x (see Section 2.5.3). However if the coefficients of the number field occur explicitly (as polmods) as coefficients of x, the variable of these polmods *must* be the same as the main variable of t (see `nffactor`).

It is possible to input a defining polynomial for nf instead, but this is in general less efficient since parts of an `nf` structure will then be computed internally. This is useful in two situations: when you do not need the `nf` elsewhere, or when you cannot initialize an `nf` due to integer factorization difficulties when attempting to compute the field discriminant and maximal order.

Caveat. `nfinit([T, listP])` allows to compute in polynomial time a conditional nf structure, which sets `nf.zk` to an order which is not guaranteed to be maximal at all primes. Always either use `nfcertify` first (which may not run in polynomial time) or make sure to input `nf.pol` instead of the conditional nf: `nfroots` is able to recover in polynomial time in this case, instead of potentially missing a factor.

The library syntax is `GEN nfroots(GEN nf = NULL, GEN x)`. See also `GEN nfrootsQ(GEN x)`, corresponding to `nf` = `NULL`.

3.8.125 nfrootsof1(nf). Returns a two-component vector $[w, z]$ where w is the number of roots of unity in the number field nf, and z is a primitive w-th root of unity.

```
? K = nfinit(polcyclo(11));
? nfrootsof1(K)
%2 = [22, [0, 0, 0, 0, 0, -1, 0, 0, 0, 0]~]
? z = nfbasistoalg(K, %[2])   \\ in algebraic form
%3 = Mod(-x^5, x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)
? [lift(z^11), lift(z^2)]     \\ proves that the order of z is 22
%4 = [-1, -x^9 - x^8 - x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x - 1]
```

This function guesses the number w as the gcd of the $\#k(v)^*$ for unramified v above odd primes, then computes the roots in nf of the w-th cyclotomic polynomial: the algorithm is polynomial time with respect to the field degree and the bitsize of the multiplication table in nf (both of them polynomially bounded in terms of the size of the discriminant). Fields of degree up to 100 or so should require less than one minute.

The library syntax is `GEN rootsof1(GEN nf)`. Also available is `GEN rootsof1_kannan(GEN nf)`, that computes all algebraic integers of T_2 norm equal to the field degree (all roots of 1, by Kronecker's theorem). This is in general a little faster than the default when there *are* roots of 1 in the field (say twice faster), but can be much slower (say, *days* slower), since the algorithm is a priori exponential in the field degree.

3.8.126 nfsnf$(nf, x, \{flag = 0\})$. Given a torsion $\mathbf{Z}_K$-module x attached to the square integral invertible pseudo-matrix (A, I, J), returns an ideal list $D = [d_1, \ldots, d_n]$ which is the Smith normal form of x. In other words, x is isomorphic to $\mathbf{Z}_K/d_1 \oplus \cdots \oplus \mathbf{Z}_K/d_n$ and d_i divides d_{i-1} for $i \geq 2$. If *flag* is non-zero return $[D, U, V]$, where UAV is the identity.

See Section 3.8.4.1 for the definition of integral pseudo-matrix; briefly, it is input as a 3-component row vector $[A, I, J]$ where $I = [b_1, \ldots, b_n]$ and $J = [a_1, \ldots, a_n]$ are two ideal lists, and A is a square $n \times n$ matrix with columns $(A_1, \ldots, A_n)$, seen as elements in K^n (with canonical basis $(e_1, \ldots, e_n)$). This data defines the $\mathbf{Z}_K$ module x given by

$$(b_1 e_1 \oplus \cdots \oplus b_n e_n)/(a_1 A_1 \oplus \cdots \oplus a_n A_n) \ ,$$

The integrality condition is $a_{i,j} \in b_i a_j^{-1}$ for all i, j. If it is not satisfied, then the d_i will not be integral. Note that every finitely generated torsion module is isomorphic to a module of this form and even with $b_i = Z_K$ for all i.

The library syntax is `GEN nfsnf0(GEN nf, GEN x, long flag)`. Also available:

`GEN nfsnf(GEN nf, GEN x)` (*flag* $= 0$).

3.8.127 nfsolvemodpr(nf, a, b, P). This function is obsolete, use `nfmodpr`.

Let P be a prime ideal in **modpr** format (see `nfmodprinit`), let a be a matrix, invertible over the residue field, and let b be a column vector or matrix. This function returns a solution of $a \cdot x = b$; the coefficients of x are lifted to nf elements.

```
? K = nfinit(y^2+1);
? P = idealprimedec(K, 3)[1];
? P = nfmodprinit(K, P);
? a = [y+1, y; y, 0]; b = [1, y]~
? nfsolvemodpr(K, a,b, P)
%5 = [1, 2]~
```

The library syntax is `GEN nfsolvemodpr(GEN nf, GEN a, GEN b, GEN P)`. This function is normally useless in library mode. Project your inputs to the residue field using `nfM_to_FqM`, then work there.

3.8.128 nfsplitting$(nf, \{d\})$. Defining polynomial over $\mathbf{Q}$ for the splitting field of nf; if d is given, it must be a multiple of the splitting field degree. Instead of `nf`, it is possible to input a defining (irreducible) polynomial T for `nf`, but in general this is less efficient.

```
? K = nfinit(x^3-2);
? nfsplitting(K)
%2 = x^6 + 108
?  nfsplitting(x^8-2)
%3 = x^16 + 272*x^8 + 64
```

Specifying the degree of the splitting field can make the computation faster.

```
? nfsplitting(x^17-123);
time = 3,607 ms.
? poldegree(%)
%2 = 272
? nfsplitting(x^17-123,272);
```

```
time = 150 ms.
? nfsplitting(x^17-123,273);
 *** nfsplitting: Warning: ignoring incorrect degree bound 273
time = 3,611 ms.
```

The complexity of the algorithm is polynomial in the degree d of the splitting field and the bitsize of T; if d is large the result will likely be unusable, e.g. `nfinit` will not be an option:

```
? nfsplitting(x^6-x-1)
[... degree 720 polynomial deleted ...]
time = 11,020 ms.
```

The library syntax is `GEN nfsplitting(GEN nf, GEN d = NULL)`.

3.8.129 nfsubfields$(pol, \{d = 0\})$. Finds all subfields of degree d of the number field defined by the (monic, integral) polynomial *pol* (all subfields if d is null or omitted). The result is a vector of subfields, each being given by $[g, h]$, where g is an absolute equation and h expresses one of the roots of g in terms of the root x of the polynomial defining *nf*. This routine uses J. Klüners's algorithm in the general case, and B. Allombert's `galoissubfields` when *nf* is Galois (with weakly supersolvable Galois group).

The library syntax is `GEN nfsubfields(GEN pol, long d)`.

3.8.130 polcompositum$(P, Q, \{flag = 0\})$. P and Q being squarefree polynomials in $\mathbf{Z}[X]$ in the same variable, outputs the simple factors of the étale $\mathbf{Q}$-algebra $A = \mathbf{Q}(X, Y)/(P(X), Q(Y))$. The factors are given by a list of polynomials R in $\mathbf{Z}[X]$, attached to the number field $\mathbf{Q}(X)/(R)$, and sorted by increasing degree (with respect to lexicographic ordering for factors of equal degrees). Returns an error if one of the polynomials is not squarefree.

Note that it is more efficient to reduce to the case where P and Q are irreducible first. The routine will not perform this for you, since it may be expensive, and the inputs are irreducible in most applications anyway. In this case, there will be a single factor R if and only if the number fields defined by P and Q are linearly disjoint (their intersection is $\mathbf{Q}$).

Assuming P is irreducible (of smaller degree than Q for efficiency), it is in general much faster to proceed as follows

```
nf = nfinit(P); L = nffactor(nf, Q)[,1];
vector(#L, i, rnfequation(nf, L[i]))
```

to obtain the same result. If you are only interested in the degrees of the simple factors, the `rnfequation` instruction can be replaced by a trivial `poldegree(P) * poldegree(L[i])`.

The binary digits of *flag* mean

1: outputs a vector of 4-component vectors $[R, a, b, k]$, where R ranges through the list of all possible compositums as above, and a (resp. b) expresses the root of P (resp. Q) as an element of $\mathbf{Q}(X)/(R)$. Finally, k is a small integer such that $b + ka = X$ modulo R.

2: assume that P and Q define number fields which are linearly disjoint: both polynomials are irreducible and the corresponding number fields have no common subfield besides $\mathbf{Q}$. This allows to save a costly factorization over $\mathbf{Q}$. In this case return the single simple factor instead of a vector with one element.

A compositum is often defined by a complicated polynomial, which it is advisable to reduce before further work. Here is an example involving the field $\mathbf{Q}(\zeta_5, 5^{1/5})$:

```
? L = polcompositum(x^5 - 5, polcyclo(5), 1); \\ list of [R, a, b, k]
? [R, a] = L[1];  \\ pick the single factor, extract R, a (ignore b, k)
? R               \\ defines the compositum
%3 = x^20 + 5*x^19 + 15*x^18 + 35*x^17 + 70*x^16 + 141*x^15 + 260*x^14\
+ 355*x^13 + 95*x^12 - 1460*x^11 - 3279*x^10 - 3660*x^9 - 2005*x^8    \
+ 705*x^7 + 9210*x^6 + 13506*x^5 + 7145*x^4 - 2740*x^3 + 1040*x^2     \
- 320*x + 256
? a^5 - 5         \\ a fifth root of 5
%4 = 0
? [T, X] = polredbest(R, 1);
? T     \\ simpler defining polynomial for Q[x]/(R)
%6 = x^20 + 25*x^10 + 5
? X     \\  root of R in Q[y]/(T(y))
%7 = Mod(-1/11*x^15 - 1/11*x^14 + 1/22*x^10 - 47/22*x^5 - 29/11*x^4 + 7/22,\
x^20 + 25*x^10 + 5)
? a = subst(a.pol, 'x, X)  \\ a in the new coordinates
%8 = Mod(1/11*x^14 + 29/11*x^4, x^20 + 25*x^10 + 5)
? a^5 - 5
%9 = 0
```

In the above example, $x^5 - 5$ and the 5-th cyclotomic polynomial are irreducible over $\mathbf{Q}$; they have coprime degrees so define linearly disjoint extensions and we could have started by

```
? [R,a] = polcompositum(x^5 - 5, polcyclo(5), 3); \\ [R, a, b, k]
```

The library syntax is `GEN polcompositum0(GEN P, GEN Q, long flag)`. Also available are `GEN compositum(GEN P, GEN Q)` (*flag* = 0) and `GEN compositum2(GEN P, GEN Q)` (*flag* = 1).

3.8.131 polgalois(T). Galois group of the non-constant polynomial $T \in \mathbf{Q}[X]$. In the present version 2.9.2, T must be irreducible and the degree d of T must be less than or equal to 7. If the `galdata` package has been installed, degrees 8, 9, 10 and 11 are also implemented. By definition, if $K = \mathbf{Q}[x]/(T)$, this computes the action of the Galois group of the Galois closure of K on the d distinct roots of T, up to conjugacy (corresponding to different root orderings).

The output is a 4-component vector $[n, s, k, name]$ with the following meaning: n is the cardinality of the group, s is its signature ($s = 1$ if the group is a subgroup of the alternating group A_d, $s = -1$ otherwise) and name is a character string containing name of the transitive group according to the GAP 4 transitive groups library by Alexander Hulpke.

k is more arbitrary and the choice made up to version 2.2.3 of PARI is rather unfortunate: for $d > 7$, k is the numbering of the group among all transitive subgroups of S_d, as given in "The transitive groups of degree up to eleven", G. Butler and J. McKay, *Communications in Algebra*, vol. 11, 1983, pp. 863–911 (group k is denoted T_k there). And for $d \leq 7$, it was ad hoc, so as to ensure that a given triple would denote a unique group. Specifically, for polynomials of degree $d \leq 7$, the groups are coded as follows, using standard notations

In degree 1: $S_1 = [1, 1, 1]$.

In degree 2: $S_2 = [2, -1, 1]$.

In degree 3: $A_3 = C_3 = [3,1,1]$, $S_3 = [6,-1,1]$.

In degree 4: $C_4 = [4,-1,1]$, $V_4 = [4,1,1]$, $D_4 = [8,-1,1]$, $A_4 = [12,1,1]$, $S_4 = [24,-1,1]$.

In degree 5: $C_5 = [5,1,1]$, $D_5 = [10,1,1]$, $M_{20} = [20,-1,1]$, $A_5 = [60,1,1]$, $S_5 = [120,-1,1]$.

In degree 6: $C_6 = [6,-1,1]$, $S_3 = [6,-1,2]$, $D_6 = [12,-1,1]$, $A_4 = [12,1,1]$, $G_{18} = [18,-1,1]$, $S_4^- = [24,-1,1]$, $A_4 \times C_2 = [24,-1,2]$, $S_4^+ = [24,1,1]$, $G_{36}^- = [36,-1,1]$, $G_{36}^+ = [36,1,1]$, $S_4 \times C_2 = [48,-1,1]$, $A_5 = PSL_2(5) = [60,1,1]$, $G_{72} = [72,-1,1]$, $S_5 = PGL_2(5) = [120,-1,1]$, $A_6 = [360,1,1]$, $S_6 = [720,-1,1]$.

In degree 7: $C_7 = [7,1,1]$, $D_7 = [14,-1,1]$, $M_{21} = [21,1,1]$, $M_{42} = [42,-1,1]$, $PSL_2(7) = PSL_3(2) = [168,1,1]$, $A_7 = [2520,1,1]$, $S_7 = [5040,-1,1]$.

This is deprecated and obsolete, but for reasons of backward compatibility, we cannot change this behavior yet. So you can use the default `new_galois_format` to switch to a consistent naming scheme, namely k is always the standard numbering of the group among all transitive subgroups of S_n. If this default is in effect, the above groups will be coded as:

In degree 1: $S_1 = [1,1,1]$.

In degree 2: $S_2 = [2,-1,1]$.

In degree 3: $A_3 = C_3 = [3,1,1]$, $S_3 = [6,-1,2]$.

In degree 4: $C_4 = [4,-1,1]$, $V_4 = [4,1,2]$, $D_4 = [8,-1,3]$, $A_4 = [12,1,4]$, $S_4 = [24,-1,5]$.

In degree 5: $C_5 = [5,1,1]$, $D_5 = [10,1,2]$, $M_{20} = [20,-1,3]$, $A_5 = [60,1,4]$, $S_5 = [120,-1,5]$.

In degree 6: $C_6 = [6,-1,1]$, $S_3 = [6,-1,2]$, $D_6 = [12,-1,3]$, $A_4 = [12,1,4]$, $G_{18} = [18,-1,5]$, $A_4 \times C_2 = [24,-1,6]$, $S_4^+ = [24,1,7]$, $S_4^- = [24,-1,8]$, $G_{36}^- = [36,-1,9]$, $G_{36}^+ = [36,1,10]$, $S_4 \times C_2 = [48,-1,11]$, $A_5 = PSL_2(5) = [60,1,12]$, $G_{72} = [72,-1,13]$, $S_5 = PGL_2(5) = [120,-1,14]$, $A_6 = [360,1,15]$, $S_6 = [720,-1,16]$.

In degree 7: $C_7 = [7,1,1]$, $D_7 = [14,-1,2]$, $M_{21} = [21,1,3]$, $M_{42} = [42,-1,4]$, $PSL_2(7) = PSL_3(2) = [168,1,5]$, $A_7 = [2520,1,6]$, $S_7 = [5040,-1,7]$.

Warning. The method used is that of resolvent polynomials and is sensitive to the current precision. The precision is updated internally but, in very rare cases, a wrong result may be returned if the initial precision was not sufficient.

The library syntax is `GEN polgalois(GEN T, long prec)`. To enable the new format in library mode, set the global variable `new_galois_format` to 1.

3.8.132 polred$(T, \{flag = 0\})$. This function is *deprecated*, use `polredbest` instead. Finds polynomials with reasonably small coefficients defining subfields of the number field defined by T. One of the polynomials always defines $\mathbf{Q}$ (hence is equal to $x - 1$), and another always defines the same number field as T if T is irreducible.

All T accepted by `nfinit` are also allowed here; in particular, the format `[T, listP]` is recommended, e.g. with `listP` $= 10^5$ or a vector containing all ramified primes. Otherwise, the maximal order of $\mathbf{Q}[x]/(T)$ must be computed.

The following binary digits of *flag* are significant:

1: Possibly use a suborder of the maximal order. The primes dividing the index of the order chosen are larger than `primelimit` or divide integers stored in the `addprimes` table. This flag is *deprecated*, the `[T, listP]` format is more flexible.

2: gives also elements. The result is a two-column matrix, the first column giving primitive elements defining these subfields, the second giving the corresponding minimal polynomials.

```
? M = polred(x^4 + 8, 2)
%1 =
[1 x - 1]

[1/2*x^2 x^2 + 2]

[1/4*x^3 x^4 + 2]

[x x^4 + 8]
? minpoly(Mod(M[2,1], x^4+8))
%2 = x^2 + 2
```

The library syntax is **polred**(`GEN T`) (*flag* $= 0$). Also available is `GEN polred2(GEN T)` (*flag* $=$ 2). The function `polred0` is deprecated, provided for backward compatibility.

3.8.133 polredabs$(T, \{flag = 0\})$. Returns a canonical defining polynomial P for the number field $\mathbf{Q}[X]/(T)$ defined by T, such that the sum of the squares of the modulus of the roots (i.e. the T_2-norm) is minimal. Different T defining isomorphic number fields will yield the same P. All T accepted by `nfinit` are also allowed here, e.g. non-monic polynomials, or pairs `[T, listP]` specifying that a non-maximal order may be used. For convenience, any number field structure (*nf*, *bnf*,...) can also be used instead of T.

```
? polredabs(x^2 + 16)
%1 = x^2 + 1
? K = bnfinit(x^2 + 16); polredabs(K)
%2 = x^2 + 1
```

Warning 1. Using a `t_POL` T requires computing and fully factoring the discriminant d_K of the maximal order which may be very hard. You can use the format `[T, listP]`, where `listP` encodes a list of known coprime divisors of disc(T) (see `??nfbasis`), to help the routine, thereby replacing this part of the algorithm by a polynomial time computation But this may only compute a suborder of the maximal order, when the divisors are not squarefree or do not include all primes dividing d_K. The routine attempts to certify the result independently of this order computation as per `nfcertify`: we try to prove that the computed order is maximal. If the certification fails, the routine then fully factors the integers returned by `nfcertify`. You can use `polredbest` or `polredabs(,16)` to avoid this factorization step; in both cases, the result is no longer canonical.

Warning 2. Apart from the factorization of the discriminant of T, this routine runs in polynomial time for a *fixed* degree. But the complexity is exponential in the degree: this routine may be exceedingly slow when the number field has many subfields, hence a lot of elements of small T_2-norm. If you do not need a canonical polynomial, the function `polredbest` is in general much faster (it runs in polynomial time), and tends to return polynomials with smaller discriminants.

The binary digits of *flag* mean

1: outputs a two-component row vector $[P, a]$, where P is the default output and `Mod(a, P)` is a root of the original T.

4: gives *all* polynomials of minimal T_2 norm; of the two polynomials $P(x)$ and $\pm P(-x)$, only one is given.

16: Possibly use a suborder of the maximal order, *without* attempting to certify the result as in Warning 1: we always return a polynomial and never 0. The result is a priori not canonical.

```
? T = x^16 - 136*x^14 + 6476*x^12 - 141912*x^10 + 1513334*x^8 \
      - 7453176*x^6 + 13950764*x^4 - 5596840*x^2 + 46225
? T1 = polredabs(T); T2 = polredbest(T);
? [ norml2(polroots(T1)), norml2(polroots(T2)) ]
%3 = [88.0000000, 120.000000]
? [ sizedigit(poldisc(T1)), sizedigit(poldisc(T2)) ]
%4 = [75, 67]
```

The library syntax is `GEN polredabs0(GEN T, long flag)`. Instead of the above hardcoded numerical flags, one should use an or-ed combination of

• `nf_PARTIALFACT`: possibly use a suborder of the maximal order, *without* attempting to certify the result.

• `nf_ORIG`: return $[P, a]$, where `Mod(a, P)` is a root of T.

• `nf_RAW`: return $[P, b]$, where `Mod(b, T)` is a root of P. The algebraic integer b is the raw result produced by the small vectors enumeration in the maximal order; P was computed as the characteristic polynomial of `Mod(b, T)`. `Mod(a, P)` as in `nf_ORIG` is obtained with `modreverse`.

• `nf_ADDZK`: if r is the result produced with some of the above flags (of the form P or $[P, c]$), return `[r,zk]`, where `zk` is a $\mathbf{Z}$-basis for the maximal order of $\mathbf{Q}[X]/(P)$.

• `nf_ALL`: return a vector of results of the above form, for all polynomials of minimal T_2-norm.

3.8.134 polredbest$(T, \{flag = 0\})$. Finds a polynomial with reasonably small coefficients defining the same number field as T. All T accepted by `nfinit` are also allowed here (e.g. non-monic polynomials, `nf`, `bnf`, `[T,Z_K_basis]`). Contrary to `polredabs`, this routine runs in polynomial time, but it offers no guarantee as to the minimality of its result.

This routine computes an LLL-reduced basis for the ring of integers of $\mathbf{Q}[X]/(T)$, then examines small linear combinations of the basis vectors, computing their characteristic polynomials. It returns the *separable* P polynomial of smallest discriminant (the one with lexicographically smallest `abs(Vec(P))` in case of ties). This is a good candidate for subsequent number field computations, since it guarantees that the denominators of algebraic integers, when expressed in the power basis, are reasonably small. With no claim of minimality, though.

It can happen that iterating this functions yields better and better polynomials, until it stabilizes:

```
? \p5
? P = X^12+8*X^8-50*X^6+16*X^4-3069*X^2+625;
? poldisc(P)*1.
%2 = 1.2622 E55
? P = polredbest(P);
? poldisc(P)*1.
%4 = 2.9012 E51
? P = polredbest(P);
? poldisc(P)*1.
%6 = 8.8704 E44
```

In this example, the initial polynomial P is the one returned by `polredabs`, and the last one is stable.

If *flag* = 1: outputs a two-component row vector $[P, a]$, where P is the default output and `Mod(a, P)` is a root of the original T.

```
? [P,a] = polredbest(x^4 + 8, 1)
%1 = [x^4 + 2, Mod(x^3, x^4 + 2)]
? charpoly(a)
%2 = x^4 + 8
```

In particular, the map $\mathbf{Q}[x]/(T) \to \mathbf{Q}[x]/(P)$, $x \mapsto$ `Mod(a,P)` defines an isomorphism of number fields, which can be computed as

```
subst(lift(Q), 'x, a)
```

if Q is a `t_POLMOD` modulo T; `b = modreverse(a)` returns a `t_POLMOD` giving the inverse of the above map (which should be useless since $\mathbf{Q}[x]/(P)$ is a priori a better representation for the number field and its elements).

The library syntax is `GEN polredbest(GEN T, long flag)`.

3.8.135 polredord(x). This function is obsolete, use polredbest.

The library syntax is `GEN polredord(GEN x)`.

3.8.136 poltschirnhaus(x). Applies a random Tschirnhausen transformation to the polynomial x, which is assumed to be non-constant and separable, so as to obtain a new equation for the étale algebra defined by x. This is for instance useful when computing resolvents, hence is used by the `polgalois` function.

The library syntax is `GEN tschirnhaus(GEN x)`.

3.8.137 rnfalgtobasis(rnf, x). Expresses x on the relative integral basis. Here, *rnf* is a relative number field extension L/K as output by `rnfinit`, and x an element of L in absolute form, i.e. expressed as a polynomial or polmod with polmod coefficients, *not* on the relative integral basis.

The library syntax is `GEN rnfalgtobasis(GEN rnf, GEN x)`.

3.8.138 rnfbasis(bnf, M). Let K the field represented by bnf, as output by `bnfinit`. M is a projective $\mathbf{Z}_K$-module of rank n ($M \otimes K$ is an n-dimensional K-vector space), given by a pseudo-basis of size n. The routine returns either a true $\mathbf{Z}_K$-basis of M (of size n) if it exists, or an $n+1$-element generating set of M if not.

It is allowed to use an irreducible polynomial P in $K[X]$ instead of M, in which case, M is defined as the ring of integers of $K[X]/(P)$, viewed as a $\mathbf{Z}_K$-module.

The library syntax is `GEN rnfbasis(GEN bnf, GEN M)`.

3.8.139 rnfbasistoalg(rnf, x). Computes the representation of x as a polmod with polmods coefficients. Here, rnf is a relative number field extension L/K as output by `rnfinit`, and x an element of L expressed on the relative integral basis.

The library syntax is `GEN rnfbasistoalg(GEN rnf, GEN x)`.

3.8.140 rnfcharpoly$(nf, T, a, \{var = ' x\})$. Characteristic polynomial of a over nf, where a belongs to the algebra defined by T over nf, i.e. $nf[X]/(T)$. Returns a polynomial in variable v (x by default).

```
? nf = nfinit(y^2+1);
? rnfcharpoly(nf, x^2+y*x+1, x+y)
%2 = x^2 + Mod(-y, y^2 + 1)*x + 1
```

The library syntax is `GEN rnfcharpoly(GEN nf, GEN T, GEN a, long var = -1)` where `var` is a variable number.

3.8.141 rnfconductor(bnf, pol). Given bnf as output by `bnfinit`, and pol a relative polynomial defining an Abelian extension, computes the class field theory conductor of this Abelian extension. The result is a 3-component vector $[conductor, bnr, subgroup]$, where $conductor$ is the conductor of the extension given as a 2-component row vector $[f_0, f_\infty]$, bnr is the attached `bnr` structure and $subgroup$ is a matrix in HNF defining the subgroup of the ray class group on `bnr.gen`.

The library syntax is `GEN rnfconductor(GEN bnf, GEN pol)`.

3.8.142 rnfdedekind$(nf, pol, \{pr\}, \{flag = 0\})$. Given a number field K coded by nf and a monic polynomial $P \in \mathbf{Z}_K[X]$, irreducible over K and thus defining a relative extension L of K, applies Dedekind's criterion to the order $\mathbf{Z}_K[X]/(P)$, at the prime ideal pr. It is possible to set pr to a vector of prime ideals (test maximality at all primes in the vector), or to omit altogether, in which case maximality at *all* primes is tested; in this situation $flag$ is automatically set to 1.

The default historic behavior ($flag$ is 0 or omitted and pr is a single prime ideal) is not so useful since `rnfpseudobasis` gives more information and is generally not that much slower. It returns a 3-component vector $[max, basis, v]$:

• $basis$ is a pseudo-basis of an enlarged order O produced by Dedekind's criterion, containing the original order $\mathbf{Z}_K[X]/(P)$ with index a power of pr. Possibly equal to the original order.

• max is a flag equal to 1 if the enlarged order O could be proven to be pr-maximal and to 0 otherwise; it may still be maximal in the latter case if pr is ramified in L,

• v is the valuation at pr of the order discriminant.

If *flag* is non-zero, on the other hand, we just return 1 if the order $\mathbf{Z}_K[X]/(P)$ is pr-maximal (resp. maximal at all relevant primes, as described above), and 0 if not. This is much faster than the default, since the enlarged order is not computed.

```
? nf = nfinit(y^2-3); P = x^3 - 2*y;
? pr3 = idealprimedec(nf,3)[1];
? rnfdedekind(nf, P, pr3)
%3 = [1, [[1, 0, 0; 0, 1, 0; 0, 0, 1], [1, 1, 1]], 8]
? rnfdedekind(nf, P, pr3, 1)
%4 = 1
```

In this example, `pr3` is the ramified ideal above 3, and the order generated by the cube roots of y is already `pr3`-maximal. The order-discriminant has valuation 8. On the other hand, the order is not maximal at the prime above 2:

```
? pr2 = idealprimedec(nf,2)[1];
? rnfdedekind(nf, P, pr2, 1)
%6 = 0
? rnfdedekind(nf, P, pr2)
%7 = [0, [[2, 0, 0; 0, 1, 0; 0, 0, 1], [[1, 0; 0, 1], [1, 0; 0, 1],
     [1, 1/2; 0, 1/2]]], 2]
```

The enlarged order is not proven to be `pr2`-maximal yet. In fact, it is; it is in fact the maximal order:

```
? B = rnfpseudobasis(nf, P)
%8 = [[1, 0, 0; 0, 1, 0; 0, 0, 1], [1, 1, [1, 1/2; 0, 1/2]],
     [162, 0; 0, 162], -1]
? idealval(nf,B[3], pr2)
%9 = 2
```

It is possible to use this routine with non-monic $P = \sum_{i \leq n} a_i X^i \in \mathbf{Z}_K[X]$ if *flag* $= 1$; in this case, we test maximality of Dedekind's order generated by

$$1, a_n\alpha, a_n\alpha^2 + a_{n-1}\alpha, \ldots, a_n\alpha^{n-1} + a_{n-1}\alpha^{n-2} + \cdots + a_1\alpha.$$

The routine will fail if P is 0 on the projective line over the residue field $\mathbf{Z}_K/\mathtt{pr}$ (FIXME).

The library syntax is `GEN rnfdedekind(GEN nf, GEN pol, GEN pr = NULL, long flag)`.

3.8.143 rnfdet(nf, M). Given a pseudo-matrix M over the maximal order of nf, computes its determinant.

The library syntax is `GEN rnfdet(GEN nf, GEN M)`.

3.8.144 rnfdisc(nf, pol). Given a number field nf as output by `nfinit` and a polynomial pol with coefficients in nf defining a relative extension L of nf, computes the relative discriminant of L. This is a two-element row vector $[D, d]$, where D is the relative ideal discriminant and d is the relative discriminant considered as an element of nf^*/nf^{*2}. The main variable of nf *must* be of lower priority than that of pol, see Section 2.5.3.

The library syntax is `GEN rnfdiscf(GEN nf, GEN pol)`.

3.8.145 rnfeltabstorel(rnf, x). Let *rnf* be a relative number field extension L/K as output by `rnfinit` and let x be an element of L expressed as a polynomial modulo the absolute equation *rnf*`.pol`, or in terms of the absolute $\mathbf{Z}$-basis for $\mathbf{Z}_L$ if *rnf* contains one (as in `rnfinit(nf,pol,1)`, or after a call to `nfinit(rnf)`). Computes x as an element of the relative extension L/K as a polmod with polmod coefficients.

```
? K = nfinit(y^2+1); L = rnfinit(K, x^2-y);
? L.polabs
%2 = x^4 + 1
? rnfeltabstorel(L, Mod(x, L.polabs))
%3 = Mod(x, x^2 + Mod(-y, y^2 + 1))
? rnfeltabstorel(L, 1/3)
%4 = 1/3
? rnfeltabstorel(L, Mod(x, x^2-y))
%5 = Mod(x, x^2 + Mod(-y, y^2 + 1))
? rnfeltabstorel(L, [0,0,0,1]~) \\ Z_L not initialized yet
 ***   at top-level: rnfeltabstorel(L,[0,
 ***                 ^--------------------
 *** rnfeltabstorel: incorrect type in rnfeltabstorel, apply nfinit(rnf).
? nfinit(L); \\ initialize now
? rnfeltabstorel(L, [0,0,0,1]~)
%6 = Mod(Mod(y, y^2 + 1)*x, x^2 + Mod(-y, y^2 + 1))
```

The library syntax is `GEN rnfeltabstorel(GEN rnf, GEN x)`.

3.8.146 rnfeltdown$(rnf, x, \{flag = 0\})$. *rnf* being a relative number field extension L/K as output by `rnfinit` and x being an element of L expressed as a polynomial or polmod with polmod coefficients (or as a `t_COL` on `nfinit(rnf).zk`), computes x as an element of K as a `t_POLMOD` if $flag = 0$ and as a `t_COL` otherwise. If x is not in K, a domain error occurs.

```
? K = nfinit(y^2+1); L = rnfinit(K, x^2-y);
? L.pol
%2 = x^4 + 1
? rnfeltdown(L, Mod(x^2, L.pol))
%3 = Mod(y, y^2 + 1)
? rnfeltdown(L, Mod(x^2, L.pol), 1)
%4 = [0, 1]~
? rnfeltdown(L, Mod(y, x^2-y))
%5 = Mod(y, y^2 + 1)
? rnfeltdown(L, Mod(y,K.pol))
%6 = Mod(y, y^2 + 1)
? rnfeltdown(L, Mod(x, L.pol))
 ***   at top-level: rnfeltdown(L,Mod(x,x
 ***                 ^--------------------
 *** rnfeltdown: domain error in rnfeltdown: element not in the base field
? rnfeltdown(L, Mod(y, x^2-y), 1) \\ as a t_COL
%7 = [0, 1]~
? rnfeltdown(L, [0,1,0,0]~) \\ not allowed without absolute nf struct
  *** rnfeltdown: incorrect type in rnfeltdown (t_COL).
? nfinit(L); \\ add absolute nf structure to L
```

```
? rnfeltdown(L, [0,1,0,0]~) \\ now OK
%8 = Mod(y, y^2 + 1)
```

If we had started with `L = rnfinit(K, x^2-y, 1)`, then the final would have worked directly.

The library syntax is `GEN rnfeltdown0(GEN rnf, GEN x, long flag)`. Also available is `GEN rnfeltdown(GEN rnf, GEN x)` (*flag* = 0).

3.8.147 rnfeltnorm(rnf, x). *rnf* being a relative number field extension L/K as output by `rnfinit` and x being an element of L, returns the relative norm $N_{L/K}(x)$ as an element of K.

```
? K = nfinit(y^2+1); L = rnfinit(K, x^2-y);
? rnfeltnorm(L, Mod(x, L.pol))
%2 = Mod(x, x^2 + Mod(-y, y^2 + 1))
? rnfeltnorm(L, 2)
%3 = 4
? rnfeltnorm(L, Mod(x, x^2-y))
```

The library syntax is `GEN rnfeltnorm(GEN rnf, GEN x)`.

3.8.148 rnfeltreltoabs(rnf, x). *rnf* being a relative number field extension L/K as output by `rnfinit` and x being an element of L expressed as a polynomial or polmod with polmod coefficients, computes x as an element of the absolute extension $L/\mathbf{Q}$ as a polynomial modulo the absolute equation *rnf*`.pol`.

```
? K = nfinit(y^2+1); L = rnfinit(K, x^2-y);
? L.pol
%2 = x^4 + 1
? rnfeltreltoabs(L, Mod(x, L.pol))
%3 = Mod(x, x^4 + 1)
? rnfeltreltoabs(L, Mod(y, x^2-y))
%4 = Mod(x^2, x^4 + 1)
? rnfeltreltoabs(L, Mod(y,K.pol))
%5 = Mod(x^2, x^4 + 1)
```

The library syntax is `GEN rnfeltreltoabs(GEN rnf, GEN x)`.

3.8.149 rnfelttrace(rnf, x). *rnf* being a relative number field extension L/K as output by `rnfinit` and x being an element of L, returns the relative trace $Tr_{L/K}(x)$ as an element of K.

```
? K = nfinit(y^2+1); L = rnfinit(K, x^2-y);
? rnfelttrace(L, Mod(x, L.pol))
%2 = 0
? rnfelttrace(L, 2)
%3 = 4
? rnfelttrace(L, Mod(x, x^2-y))
```

The library syntax is `GEN rnfelttrace(GEN rnf, GEN x)`.

3.8.150 rnfeltup($rnf, x, \{flag = 0\}$). *rnf* being a relative number field extension L/K as output by `rnfinit` and x being an element of K, computes x as an element of the absolute extension $L/\mathbf{Q}$. As a `t_POLMOD` modulo *rnf*`.pol` if *flag* $= 0$ and as a `t_COL` on the absolute field integer basis if *flag* $= 1$.

```
? K = nfinit(y^2+1); L = rnfinit(K, x^2-y);
? L.pol
%2 = x^4 + 1
? rnfeltup(L, Mod(y, K.pol))
%3 = Mod(x^2, x^4 + 1)
? rnfeltup(L, y)
%4 = Mod(x^2, x^4 + 1)
? rnfeltup(L, [1,2]~) \\ in terms of K.zk
%5 = Mod(2*x^2 + 1, x^4 + 1)
? rnfeltup(L, y, 1) \\ in terms of nfinit(L).zk
%6 = [0, 1, 0, 0]~
? rnfeltup(L, [1,2]~, 1)
%7 = [1, 2, 0, 0]~
```

The library syntax is `GEN rnfeltup0(GEN rnf, GEN x, long flag)`.

3.8.151 rnfequation($nf, pol, \{flag = 0\}$). Given a number field *nf* as output by `nfinit` (or simply a polynomial) and a polynomial *pol* with coefficients in *nf* defining a relative extension L of *nf*, computes an absolute equation of L over $\mathbf{Q}$.

The main variable of *nf* *must* be of lower priority than that of *pol* (see Section 2.5.3). Note that for efficiency, this does not check whether the relative equation is irreducible over *nf*, but only if it is squarefree. If it is reducible but squarefree, the result will be the absolute equation of the étale algebra defined by *pol*. If *pol* is not squarefree, raise an `e_DOMAIN` exception.

```
? rnfequation(y^2+1, x^2 - y)
%1 = x^4 + 1
? T = y^3-2; rnfequation(nfinit(T), (x^3-2)/(x-Mod(y,T)))
%2 = x^6 + 108  \\ Galois closure of Q(2^(1/3))
```

If *flag* is non-zero, outputs a 3-component row vector $[z, a, k]$, where

- z is the absolute equation of L over $\mathbf{Q}$, as in the default behavior,
- a expresses as a `t_POLMOD` modulo z a root α of the polynomial defining the base field *nf*,
- k is a small integer such that $\theta = \beta + k\alpha$ is a root of z, where β is a root of *pol*.

```
? T = y^3-2; pol = x^2 +x*y + y^2;
? [z,a,k] = rnfequation(T, pol, 1);
? z
%3 = x^6 + 108
? subst(T, y, a)
%4 = 0
? alpha= Mod(y, T);
? beta = Mod(x*Mod(1,T), pol);
? subst(z, x, beta + k*alpha)
%7 = 0
```

The library syntax is `GEN rnfequation0(GEN nf, GEN pol, long flag)`. Also available are `GEN rnfequation(GEN nf, GEN pol)` (*flag* = 0) and `GEN rnfequation2(GEN nf, GEN pol)` (*flag* = 1).

3.8.152 rnfhnfbasis(*bnf*, *x*). Given *bnf* as output by `bnfinit`, and either a polynomial x with coefficients in *bnf* defining a relative extension L of *bnf*, or a pseudo-basis x of such an extension, gives either a true *bnf*-basis of L in upper triangular Hermite normal form, if it exists, and returns 0 otherwise.

The library syntax is `GEN rnfhnfbasis(GEN bnf, GEN x)`.

3.8.153 rnfidealabstorel(*rnf*, *x*). Let *rnf* be a relative number field extension L/K as output by `rnfinit` and x be an ideal of the absolute extension $L/\mathbf{Q}$ given by a $\mathbf{Z}$-basis of elements of L. Returns the relative pseudo-matrix in HNF giving the ideal x considered as an ideal of the relative extension L/K, i.e. as a $\mathbf{Z}_K$-module.

The reason why the input does not use the customary HNF in terms of a fixed $\mathbf{Z}$-basis for $\mathbf{Z}_L$ is precisely that no such basis has been explicitly specified. On the other hand, if you already computed an (absolute) `nf` structure `Labs` attached to L, and m is in HNF, defining an (absolute) ideal with respect to the $\mathbf{Z}$-basis `Labs.zk`, then `Labs.zk * m` is a suitable $\mathbf{Z}$-basis for the ideal, and

```
rnfidealabstorel(rnf, Labs.zk * m)
```

converts m to a relative ideal.

```
? K = nfinit(y^2+1); L = rnfinit(K, x^2-y); Labs = nfinit(L);
? m = idealhnf(Labs, 17, x^3+2);
? B = rnfidealabstorel(L, Labs.zk * m)
%3 = [[1, 8; 0, 1], [[17, 4; 0, 1], 1]]  \\ pseudo-basis for m as Z_K-module
? A = rnfidealreltoabs(L, B)
%4 = [17, x^2 + 4, x + 8, x^3 + 8*x^2]   \\ Z-basis for m in Q[x]/(L.pol)
? mathnf(matalgtobasis(Labs, A))
%5 =
[17 8 4 2]

[ 0 1 0 0]

[ 0 0 1 0]

[ 0 0 0 1]
? % == m
%6 = 1
```

The library syntax is `GEN rnfidealabstorel(GEN rnf, GEN x)`.

3.8.154 rnfidealdown(*rnf*, *x*). Let *rnf* be a relative number field extension L/K as output by `rnfinit`, and x an ideal of L, given either in relative form or by a $\mathbf{Z}$-basis of elements of L (see Section 3.8.153). This function returns the ideal of K below x, i.e. the intersection of x with K.

The library syntax is `GEN rnfidealdown(GEN rnf, GEN x)`.

3.8.155 rnfidealfactor(rnf, x). Factors into prime ideal powers the ideal x in the attached absolute number field $L =$ `nfinit`(rnf). The output format is similar to the `factor` function, and the prime ideals are represented in the form output by the `idealprimedec` function for L.

```
? rnf = rnfinit(nfinit(y^2+1), x^2-y+1);
? rnfidealfactor(rnf, y+1)  \\ P_2^2
%2 =
[[2, [0,0,1,0]~, 4, 1, [0,0,0,2;0,0,-2,0;-1,-1,0,0;1,-1,0,0]] 2]

? rnfidealfactor(rnf, x) \\ P_2
%3 =
[[2, [0,0,1,0]~, 4, 1, [0,0,0,2;0,0,-2,0;-1,-1,0,0;1,-1,0,0]] 1]

? L = nfinit(rnf);
? id = idealhnf(L, idealhnf(L, 25, (x+1)^2));
? idealfactor(L, id) == rnfidealfactor(rnf, id)
%6 = 1
```

Note that ideals of the base field K must be explicitly lifted to L via `rnfidealup` before they can be factored.

The library syntax is `GEN rnfidealfactor(GEN rnf, GEN x)`.

3.8.156 rnfidealhnf(rnf, x). *rnf* being a relative number field extension L/K as output by `rnfinit` and x being a relative ideal (which can be, as in the absolute case, of many different types, including of course elements), computes the HNF pseudo-matrix attached to x, viewed as a $\mathbf{Z}_K$-module.

The library syntax is `GEN rnfidealhnf(GEN rnf, GEN x)`.

3.8.157 rnfidealmul(rnf, x, y). *rnf* being a relative number field extension L/K as output by `rnfinit` and x and y being ideals of the relative extension L/K given by pseudo-matrices, outputs the ideal product, again as a relative ideal.

The library syntax is `GEN rnfidealmul(GEN rnf, GEN x, GEN y)`.

3.8.158 rnfidealnormabs(rnf, x). Let *rnf* be a relative number field extension L/K as output by `rnfinit` and let x be a relative ideal (which can be, as in the absolute case, of many different types, including of course elements). This function computes the norm of the x considered as an ideal of the absolute extension $L/\mathbf{Q}$. This is identical to

```
idealnorm(rnf, rnfidealnormrel(rnf,x))
```

but faster.

The library syntax is `GEN rnfidealnormabs(GEN rnf, GEN x)`.

3.8.159 rnfidealnormrel(rnf, x). Let *rnf* be a relative number field extension L/K as output by `rnfinit` and let x be a relative ideal (which can be, as in the absolute case, of many different types, including of course elements). This function computes the relative norm of x as an ideal of K in HNF.

The library syntax is `GEN rnfidealnormrel(GEN rnf, GEN x)`.

3.8.160 rnfidealprimedec(rnf, pr). Let *rnf* be a relative number field extension L/K as output by `rnfinit`, and `pr` a maximal ideal of K (`prid`), this function completes the *rnf* with a *nf* structure attached to L (see Section 3.8.164) and returns the prime ideal decomposition of `pr` in L/K.

```
? K = nfinit(y^2+1); rnf = rnfinit(K, x^3+y+1);
? P = idealprimedec(K, 2)[1];
? S = rnfidealprimedec(rnf, P);
? #S
%4 = 1
```

The argument `pr` is also allowed to be a prime number p, in which case we return a pair of vectors `[SK,SL]`, where `SK` contains the primes of K above p and `SL`$[i]$ is the vector of primes of L above `SK`$[i]$.

```
? [SK,SL] = rnfidealprimedec(rnf, 5);
? [#SK, vector(#SL,i,#SL[i])]
%6 = [2, [2, 2]]
```

The library syntax is `GEN rnfidealprimedec(GEN rnf, GEN pr)`.

3.8.161 rnfidealreltoabs$(rnf, x, \{flag = 0\})$. Let *rnf* be a relative number field extension L/K as output by `rnfinit` and let x be a relative ideal, given as a $\mathbf{Z}_K$-module by a pseudo matrix $[A, I]$. This function returns the ideal x as an absolute ideal of $L/\mathbf{Q}$. If $flag = 0$, the result is given by a vector of `t_POLMOD`s modulo `rnf.pol` forming a $\mathbf{Z}$-basis; if $flag = 1$, it is given in HNF in terms of the fixed $\mathbf{Z}$-basis for $\mathbf{Z}_L$, see Section 3.8.164.

```
? K = nfinit(y^2+1); rnf = rnfinit(K, x^2-y);
? P = idealprimedec(K,2)[1];
? P = rnfidealup(rnf, P)
%3 = [2, x^2 + 1, 2*x, x^3 + x]
? Prel = rnfidealhnf(rnf, P)
%4 = [[1, 0; 0, 1], [[2, 1; 0, 1], [2, 1; 0, 1]]]
? rnfidealreltoabs(rnf,Prel)
%5 = [2, x^2 + 1, 2*x, x^3 + x]
? rnfidealreltoabs(rnf,Prel,1)
%6 =
[2 1 0 0]

[0 1 0 0]

[0 0 2 1]

[0 0 0 1]
```

The reason why we do not return by default ($flag = 0$) the customary HNF in terms of a fixed $\mathbf{Z}$-basis for $\mathbf{Z}_L$ is precisely because a *rnf* does not contain such a basis by default. Completing the structure so that it contains a *nf* structure for L is polynomial time but costly when the absolute degree is large, thus it is not done by default. Note that setting $flag = 1$ will complete the *rnf*.

The library syntax is `GEN rnfidealreltoabs0(GEN rnf, GEN x, long flag)`. Also available is `GEN rnfidealreltoabs(GEN rnf, GEN x)` ($flag = 0$).

3.8.162 rnfidealtwoelt(rnf, x). *rnf* being a relative number field extension L/K as output by `rnfinit` and x being an ideal of the relative extension L/K given by a pseudo-matrix, gives a vector of two generators of x over $\mathbf{Z}_L$ expressed as polmods with polmod coefficients.

The library syntax is `GEN rnfidealtwoelement(GEN rnf, GEN x)`.

3.8.163 rnfidealup$(rnf, x, \{flag = 0\})$. Let *rnf* be a relative number field extension L/K as output by `rnfinit` and let x be an ideal of K. This function returns the ideal $x\mathbf{Z}_L$ as an absolute ideal of $L/\mathbf{Q}$, in the form of a $\mathbf{Z}$-basis. If $flag = 0$, the result is given by a vector of polynomials (modulo `rnf.pol`); if $flag = 1$, it is given in HNF in terms of the fixed $\mathbf{Z}$-basis for $\mathbf{Z}_L$, see Section 3.8.164.

```
? K = nfinit(y^2+1); rnf = rnfinit(K, x^2-y);
? P = idealprimedec(K,2)[1];
? rnfidealup(rnf, P)
%3 = [2, x^2 + 1, 2*x, x^3 + x]
? rnfidealup(rnf, P,1)
%4 =
[2 1 0 0]

[0 1 0 0]

[0 0 2 1]

[0 0 0 1]
```

The reason why we do not return by default ($flag = 0$) the customary HNF in terms of a fixed $\mathbf{Z}$-basis for $\mathbf{Z}_L$ is precisely because a *rnf* does not contain such a basis by default. Completing the structure so that it contains a *nf* structure for L is polynomial time but costly when the absolute degree is large, thus it is not done by default. Note that setting $flag = 1$ will complete the *rnf*.

The library syntax is `GEN rnfidealup0(GEN rnf, GEN x, long flag)`. Also available is `GEN rnfidealup(GEN rnf, GEN x)` ($flag = 0$).

3.8.164 rnfinit$(nf, pol, \{flag = 0\})$. *nf* being a number field in `nfinit` format considered as base field, and *pol* a polynomial defining a relative extension over *nf*, this computes data to work in the relative extension. The main variable of *pol* must be of higher priority (see Section 2.5.3) than that of *nf*, and the coefficients of *pol* must be in *nf*.

The result is a row vector, whose components are technical. In the following description, we let K be the base field defined by *nf* and L/K the extension attached to the *rnf*. Furthermore, we let $m = [K : \mathbf{Q}]$ the degree of the base field, $n = [L : K]$ the relative degree, r_1 and r_2 the number of real and complex places of K. Access to this information via *member functions* is preferred since the specific data organization specified below will change in the future.

If $flag = 1$, add an *nf* structure attached to L to *rnf*. This is likely to be very expensive if the absolute degree mn is large, but fixes an integer basis for $\mathbf{Z}_L$ as a $\mathbf{Z}$-module and allows to input and output elements of L in absolute form: as `t_COL` for elements, as `t_MAT` in HNF for ideals, as `prid` for prime ideals. Without such a call, elements of L are represented as `t_POLMOD`, etc. Note that a subsequent `nfinit`(*rnf*) will also explicitly add such a component, and so will the following functions `rnfidealmul`, `rnfidealtwoelt`, `rnfidealprimedec`, `rnfidealup` (with flag 1) and `rnfidealreltoabs` (with flag 1). The absolute *nf* structure attached to L can be recovered using `nfinit(rnf)`.

rnf[1](`rnf.pol`) contains the relative polynomial *pol*.

rnf[2] contains the integer basis $[A, d]$ of K, as (integral) elements of $L/\mathbf{Q}$. More precisely, A is a vector of polynomial with integer coefficients, d is a denominator, and the integer basis is given by A/d.

rnf[3] (`rnf.disc`) is a two-component row vector $[\mathfrak{d}(L/K), s]$ where $\mathfrak{d}(L/K)$ is the relative ideal discriminant of L/K and s is the discriminant of L/K viewed as an element of $K^*/(K^*)^2$, in other words it is the output of `rnfdisc`.

rnf[4](`rnf.index`) is the ideal index $\mathfrak{f}$, i.e. such that $d(pol)\mathbf{Z}_K = \mathfrak{f}^2\mathfrak{d}(L/K)$.

rnf[5] is currently unused.

rnf[6] is currently unused.

rnf[7] (`rnf.zk`) is the pseudo-basis (A, I) for the maximal order $\mathbf{Z}_L$ as a $\mathbf{Z}_K$-module: A is the relative integral pseudo basis expressed as polynomials (in the variable of *pol*) with polmod coefficients in *nf*, and the second component I is the ideal list of the pseudobasis in HNF.

rnf[8] is the inverse matrix of the integral basis matrix, with coefficients polmods in *nf*.

rnf[9] is currently unused.

rnf[10] (`rnf.nf`) is *nf*.

rnf[11] is an extension of `rnfequation(K, pol, 1)`. Namely, a vector $[P, a, k,$ `K.pol`$,$ `pol`$]$ describing the *absolute* extension $L/\mathbf{Q}$: P is an absolute equation, more conveniently obtained as `rnf.polabs`; a expresses the generator $\alpha = y \bmod$ `K.pol` of the number field K as an element of L, i.e. a polynomial modulo the absolute equation P;

k is a small integer such that, if β is an abstract root of *pol* and α the generator of K given above, then $P(\beta + k\alpha) = 0$.

Caveat. Be careful if $k \neq 0$ when dealing simultaneously with absolute and relative quantities since $L = \mathbf{Q}(\beta + k\alpha) = K(\alpha)$, and the generator chosen for the absolute extension is not the same as for the relative one. If this happens, one can of course go on working, but we advise to change the relative polynomial so that its root becomes $\beta + k\alpha$. Typical GP instructions would be

```
[P,a,k] = rnfequation(K, pol, 1);
if (k, pol = subst(pol, x, x - k*Mod(y, K.pol)));
L = rnfinit(K, pol);
```

rnf[12] is by default unused and set equal to 0. This field is used to store further information about the field as it becomes available (which is rarely needed, hence would be too expensive to compute during the initial `rnfinit` call).

The library syntax is `GEN rnfinit0(GEN nf, GEN pol, long flag)`. Also available is `GEN rnfinit(GEN nf, GEN pol)` (*flag* $= 0$).

3.8.165 rnfisabelian(nf, T). T being a relative polynomial with coefficients in *nf*, return 1 if it defines an abelian extension, and 0 otherwise.

```
? K = nfinit(y^2 + 23);
? rnfisabelian(K, x^3 - 3*x - y)
%2 = 1
```

The library syntax is `long rnfisabelian(GEN nf, GEN T)`.

3.8.166 rnfisfree(bnf, x). Given *bnf* as output by `bnfinit`, and either a polynomial x with coefficients in *bnf* defining a relative extension L of *bnf*, or a pseudo-basis x of such an extension, returns true (1) if L/bnf is free, false (0) if not.

The library syntax is `long rnfisfree(GEN bnf, GEN x)`.

3.8.167 rnfislocalcyclo(rnf). Let *rnf* a a relative number field extension L/K as output by `rnfinit` whole degree $[L : K]$ is a power of a prime ℓ. Return 1 if the ℓ-extension is locally cyclotomic (locally contained in the cyclotomic $\mathbf{Z}_\ell$-extension of K_v at all places $v|\ell$), and 0 if not.

```
? K = nfinit(y^2 + y + 1);
? L = rnfinit(K, x^3 - y); /* = K(zeta_9), globally cyclotomic */
? rnfislocalcyclo(L)
%3 = 1
\\ we expect 3-adic continuity by Krasner's lemma
? vector(5, i, rnfislocalcyclo(rnfinit(K, x^3 - y + 3^i)))
%5 = [0, 1, 1, 1, 1]
```

The library syntax is `long rnfislocalcyclo(GEN rnf)`.

3.8.168 rnfisnorm$(T, a, \{flag = 0\})$. Similar to `bnfisnorm` but in the relative case. T is as output by `rnfisnorminit` applied to the extension L/K. This tries to decide whether the element a in K is the norm of some x in the extension L/K.

The output is a vector $[x, q]$, where $a = \mathrm{Norm}(x) * q$. The algorithm looks for a solution x which is an S-integer, with S a list of places of K containing at least the ramified primes, the generators of the class group of L, as well as those primes dividing a. If L/K is Galois, then this is enough; otherwise, *flag* is used to add more primes to S: all the places above the primes $p \leq$ *flag* (resp. $p|$*flag*) if *flag* > 0 (resp. *flag* < 0).

The answer is guaranteed (i.e. a is a norm iff $q = 1$) if the field is Galois, or, under GRH, if S contains all primes less than $12 \log^2 |\mathrm{disc}(M)|$, where M is the normal closure of L/K.

If `rnfisnorminit` has determined (or was told) that L/K is Galois, and *flag* $\neq 0$, a Warning is issued (so that you can set *flag* $= 1$ to check whether L/K is known to be Galois, according to T). Example:

```
bnf = bnfinit(y^3 + y^2 - 2*y - 1);
p = x^2 + Mod(y^2 + 2*y + 1, bnf.pol);
T = rnfisnorminit(bnf, p);
rnfisnorm(T, 17)
```

checks whether 17 is a norm in the Galois extension $\mathbf{Q}(\beta)/\mathbf{Q}(\alpha)$, where $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$ and $\beta^2 + \alpha^2 + 2\alpha + 1 = 0$ (it is).

The library syntax is `GEN rnfisnorm(GEN T, GEN a, long flag)`.

3.8.169 rnfisnorminit$(pol, polrel, \{flag = 2\})$. Let K be defined by a root of *pol*, and L/K the extension defined by the polynomial *polrel*. As usual, *pol* can in fact be an *nf*, or *bnf*, etc; if *pol* has degree 1 (the base field is $\mathbf{Q}$), polrel is also allowed to be an *nf*, etc. Computes technical data needed by `rnfisnorm` to solve norm equations $Nx = a$, for x in L, and a in K.

If $flag = 0$, do not care whether L/K is Galois or not.

If $flag = 1$, L/K is assumed to be Galois (unchecked), which speeds up `rnfisnorm`.

If $flag = 2$, let the routine determine whether L/K is Galois.

The library syntax is `GEN rnfisnorminit(GEN pol, GEN polrel, long flag)`.

3.8.170 rnfkummer$(bnr, \{subgp\}, \{d = 0\})$. *bnr* being as output by `bnrinit`, finds a relative equation for the class field corresponding to the module in *bnr* and the given congruence subgroup (the full ray class field if *subgp* is omitted). If d is positive, outputs the list of all relative equations of degree d contained in the ray class field defined by *bnr*, with the *same* conductor as $(bnr, subgp)$.

Warning. This routine only works for subgroups of prime index. It uses Kummer theory, adjoining necessary roots of unity (it needs to compute a tough `bnfinit` here), and finds a generator via Hecke's characterization of ramification in Kummer extensions of prime degree. If your extension does not have prime degree, for the time being, you have to split it by hand as a tower / compositum of such extensions.

The library syntax is `GEN rnfkummer(GEN bnr, GEN subgp = NULL, long d, long prec)`.

3.8.171 rnflllgram$(nf, pol, order)$. Given a polynomial *pol* with coefficients in *nf* defining a relative extension L and a suborder *order* of L (of maximal rank), as output by `rnfpseudobasis`(nf, pol) or similar, gives $[[neworder], U]$, where *neworder* is a reduced order and U is the unimodular transformation matrix.

The library syntax is `GEN rnflllgram(GEN nf, GEN pol, GEN order, long prec)`.

3.8.172 rnfnormgroup(bnr, pol). *bnr* being a big ray class field as output by `bnrinit` and *pol* a relative polynomial defining an Abelian extension, computes the norm group (alias Artin or Takagi group) corresponding to the Abelian extension of *bnf* =`bnr.bnf` defined by *pol*, where the module corresponding to *bnr* is assumed to be a multiple of the conductor (i.e. *pol* defines a subextension of bnr). The result is the HNF defining the norm group on the given generators of `bnr.gen`. Note that neither the fact that *pol* defines an Abelian extension nor the fact that the module is a multiple of the conductor is checked. The result is undefined if the assumption is not correct, but the function will return the empty matrix `[;]` if it detects a problem; it may also not detect the problem and return a wrong result.

The library syntax is `GEN rnfnormgroup(GEN bnr, GEN pol)`.

3.8.173 rnfpolred(nf, pol). This function is obsolete: use `rnfpolredbest` instead. Relative version of `polred`. Given a monic polynomial *pol* with coefficients in *nf*, finds a list of relative polynomials defining some subfields, hopefully simpler and containing the original field. In the present version 2.9.2, this is slower and less efficient than `rnfpolredbest`.

Remark. this function is based on an incomplete reduction theory of lattices over number fields, implemented by `rnflllgram`, which deserves to be improved.

The library syntax is `GEN rnfpolred(GEN nf, GEN pol, long prec)`.

3.8.174 rnfpolredabs$(nf, pol, \{flag = 0\})$. This function is obsolete: use `rnfpolredbest` instead. Relative version of `polredabs`. Given a monic polynomial *pol* with coefficients in *nf*, finds a simpler relative polynomial defining the same field. The binary digits of *flag* mean

The binary digits of *flag* correspond to 1: add information to convert elements to the new representation, 2: absolute polynomial, instead of relative, 16: possibly use a suborder of the maximal order. More precisely:

0: default, return P

1: returns $[P, a]$ where P is the default output and a, a `t_POLMOD` modulo P, is a root of *pol*.

2: returns *Pabs*, an absolute, instead of a relative, polynomial. Same as but faster than

```
rnfequation(nf, rnfpolredabs(nf,pol))
```

3: returns $[Pabs, a, b]$, where *Pabs* is an absolute polynomial as above, a, b are `t_POLMOD` modulo *Pabs*, roots of `nf.pol` and *pol* respectively.

16: possibly use a suborder of the maximal order. This is slower than the default when the relative discriminant is smooth, and much faster otherwise. See Section 3.8.133.

Warning. In the present implementation, `rnfpolredabs` produces smaller polynomials than `rnfpolred` and is usually faster, but its complexity is still exponential in the absolute degree. The function `rnfpolredbest` runs in polynomial time, and tends to return polynomials with smaller discriminants.

The library syntax is `GEN rnfpolredabs(GEN nf, GEN pol, long flag)`.

3.8.175 rnfpolredbest$(nf, pol, \{flag = 0\})$. Relative version of `polredbest`. Given a monic polynomial *pol* with coefficients in *nf*, finds a simpler relative polynomial P defining the same field. As opposed to `rnfpolredabs` this function does not return a *smallest* (canonical) polynomial with respect to some measure, but it does run in polynomial time.

The binary digits of *flag* correspond to 1: add information to convert elements to the new representation, 2: absolute polynomial, instead of relative. More precisely:

0: default, return P

1: returns $[P, a]$ where P is the default output and a, a `t_POLMOD` modulo P, is a root of *pol*.

2: returns *Pabs*, an absolute, instead of a relative, polynomial. Same as but faster than

```
rnfequation(nf, rnfpolredbest(nf,pol))
```

3: returns $[Pabs, a, b]$, where *Pabs* is an absolute polynomial as above, a, b are `t_POLMOD` modulo *Pabs*, roots of `nf.pol` and *pol* respectively.

```
? K = nfinit(y^3-2); pol = x^2 +x*y + y^2;
? [P, a] = rnfpolredbest(K,pol,1);
? P
%3 = x^2 - x + Mod(y - 1, y^3 - 2)
```

```
? a
%4 = Mod(Mod(2*y^2+3*y+4,y^3-2)*x + Mod(-y^2-2*y-2,y^3-2),
         x^2 - x + Mod(y-1,y^3-2))
? subst(K.pol,y,a)
%5 = 0
? [Pabs, a, b] = rnfpolredbest(K,pol,3);
? Pabs
%7 = x^6 - 3*x^5 + 5*x^3 - 3*x + 1
? a
%8 = Mod(-x^2+x+1, x^6-3*x^5+5*x^3-3*x+1)
? b
%9 = Mod(2*x^5-5*x^4-3*x^3+10*x^2+5*x-5, x^6-3*x^5+5*x^3-3*x+1)
? subst(K.pol,y,a)
%10 = 0
? substvec(pol,[x,y],[a,b])
%11 = 0
```

The library syntax is `GEN rnfpolredbest(GEN nf, GEN pol, long flag)`.

3.8.176 rnfpseudobasis(nf, pol). Given a number field nf as output by `nfinit` and a polynomial pol with coefficients in nf defining a relative extension L of nf, computes a pseudo-basis (A, I) for the maximal order $\mathbf{Z}_L$ viewed as a $\mathbf{Z}_K$-module, and the relative discriminant of L. This is output as a four-element row vector $[A, I, D, d]$, where D is the relative ideal discriminant and d is the relative discriminant considered as an element of nf^*/nf^{*2}.

The library syntax is `GEN rnfpseudobasis(GEN nf, GEN pol)`.

3.8.177 rnfsteinitz(nf, x). Given a number field nf as output by `nfinit` and either a polynomial x with coefficients in nf defining a relative extension L of nf, or a pseudo-basis x of such an extension as output for example by `rnfpseudobasis`, computes another pseudo-basis (A, I) (not in HNF in general) such that all the ideals of I except perhaps the last one are equal to the ring of integers of nf, and outputs the four-component row vector $[A, I, D, d]$ as in `rnfpseudobasis`. The name of this function comes from the fact that the ideal class of the last ideal of I, which is well defined, is the Steinitz class of the $\mathbf{Z}_K$-module $\mathbf{Z}_L$ (its image in $SK_0(\mathbf{Z}_K)$).

The library syntax is `GEN rnfsteinitz(GEN nf, GEN x)`.

3.8.178 subgrouplist$(bnr, \{bound\}, \{flag = 0\})$. *bnr* being as output by `bnrinit` or a list of cyclic components of a finite Abelian group G, outputs the list of subgroups of G. Subgroups are given as HNF left divisors of the SNF matrix corresponding to G.

If $flag = 0$ (default) and *bnr* is as output by `bnrinit`, gives only the subgroups whose modulus is the conductor. Otherwise, the modulus is not taken into account.

If *bound* is present, and is a positive integer, restrict the output to subgroups of index less than *bound*. If *bound* is a vector containing a single positive integer B, then only subgroups of index exactly equal to B are computed. For instance

```
? subgrouplist([6,2])
%1 = [[6, 0; 0, 2], [2, 0; 0, 2], [6, 3; 0, 1], [2, 1; 0, 1], [3, 0; 0, 2],
[1, 0; 0, 2], [6, 0; 0, 1], [2, 0; 0, 1], [3, 0; 0, 1], [1, 0; 0, 1]]
? subgrouplist([6,2],3)     \\ index less than 3
```

```
%2 = [[2, 1; 0, 1], [1, 0; 0, 2], [2, 0; 0, 1], [3, 0; 0, 1], [1, 0; 0, 1]]
? subgrouplist([6,2],[3])  \\ index 3
%3 = [[3, 0; 0, 1]]
? bnr = bnrinit(bnfinit(x), [120,[1]], 1);
? L = subgrouplist(bnr, [8]);
```

In the last example, L corresponds to the 24 subfields of $\mathbf{Q}(\zeta_{120})$, of degree 8 and conductor 120∞ (by setting *flag*, we see there are a total of 43 subgroups of degree 8).

```
? vector(#L, i, galoissubcyclo(bnr, L[i]))
```

will produce their equations. (For a general base field, you would have to rely on `bnrstark`, or `rnfkummer`.)

The library syntax is `GEN subgrouplist0(GEN bnr, GEN bound = NULL, long flag)`.

3.9 Associative and central simple algebras.

This section collects functions related to associative algebras and central simple algebras over number fields. Let A be a finite-dimensional unitary associative algebra over a field K. We say that A is *central* if the center of A is K, and that A is *simple* if it has no nontrivial two-sided ideals.

We provide functions to manipulate associative algebras of finite dimension over $\mathbf{Q}$ or $\mathbf{F}_p$. We represent them by the left multiplication table on a basis over the prime subfield. The function `algtableinit` creates the object representing an associative algebra. We also provide functions to manipulate central simple algebras over number fields. We represent them either by the left multiplication table on a basis over the center, or by a cyclic algebra (see below). The function `alginit` creates the object representing a central simple algebra.

The set of elements of an algebra A that annihilate every simple left A-module is a two-sided ideal, called the *Jacobson radical* of A. An algebra is *semisimple* if its Jacobson radical is trivial. A semisimple algebra is isomorphic to a direct sum of simple algebras. The dimension of a central simple algebra A over K is always a square d^2, and the integer d is called the *degree* of the algebra A over K. A central simple algebra A over a field K is always isomorphic to $M_d(D)$ for some integer d and some central division algebra D of degree e : the integer e is called the *index* of A.

Let L/K be a cyclic extension of degree d, let σ be a generator of $\mathrm{Gal}(L/K)$ and let $b \in K^*$. Then the *cyclic algebra* $(L/K, \sigma, b)$ is the algebra $\bigoplus_{i=0}^{d-1} x^i L$ with $x^d = b$ and $\ell x = x\sigma(\ell)$ for all $\ell \in L$. The algebra $(L/K, \sigma, b)$ is a central simple K-algebra of degree d, and it is an L-vector space. Left multiplication is L-linear and induces a K-algebra homomorphism $(L/K, \sigma, b) \to M_d(L)$.

Let K be a nonarchimedean local field with uniformizer π, and let L/K be the unique unramified extension of degree d. Then every central simple algebra A of degree d over K is isomorphic to $(L/K, \mathrm{Frob}, \pi^h)$ for some integer h. The element $h/d \in (1/d)\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$ is called the *Hasse invariant* of A.

Let A be an algebra of finite dimension over $\mathbf{Q}$. An *order* in A is a finitely generated $\mathbf{Z}$-submodule $\mathcal{O}$ such that $\mathbf{Q}\mathcal{O} = A$, that is also a subring with unit. We define natural orders in central simple algebras defined by a cyclic algebra or by a multiplication table over the center. Let $A = (L/K, \sigma, b) = \bigoplus_{i=0}^{d-1} x^i L$ be a cyclic algebra over a number field K of degree n with ring of integers $\mathbf{Z}_K$. Let $\mathbf{Z}_L$ be the ring of integers of L, and assume that b is integral. Then the submodule $\mathcal{O} = \bigoplus_{i=0}^{d-1} x^i \mathbf{Z}_L$ is an order in A, called the *natural order*. Let $\omega_0, \ldots, \omega_{nd-1}$ be a $\mathbf{Z}$-basis of $\mathbf{Z}_L$. The *natural basis* of $\mathcal{O}$ is $b_0, \ldots, b_{nd^2-1}$ where $b_i = x^{i/(nd)}\omega_{(i \bmod nd)}$. Now

let A be a central simple algebra of degree d over a number field K of degree n with ring of integers $\mathbf{Z}_K$. Let $e_0, \dots, e_{d^2-1}$ be a basis of A over K and assume that the left multiplication table of A on (e_i) is integral. Then the submodule $\mathcal{O} = \bigoplus_{i=0}^{d^2-1} \mathbf{Z}_K e_i$ is an order in A, called the *natural order*. Let $\omega_0, \dots, \omega_{n-1}$ be a $\mathbf{Z}$-basis of $\mathbf{Z}_K$. The *natural basis* of $\mathcal{O}$ is $b_0, \dots, b_{nd^2-1}$ where $b_i = \omega_{(i \bmod n)} e_{i/n}$.

As with number fields, we represent elements of central simple algebras in two ways, called the *algebraic representation* and the *basis representation*, and you can convert betweeen the two with the functions `algalgtobasis` and `algbasistoalg`. In every central simple algebra object, we store a $\mathbf{Z}$-basis of an order $\mathcal{O}_0$, and the basis representation is simply a `t_COL` with coefficients in $\mathbf{Q}$ expressing the element in that basis. If no maximal order was computed, then $\mathcal{O}_0$ is the natural order. If a maximal order was computed, then $\mathcal{O}_0$ is a maximal order containing the natural order. For a cyclic algebra $A = (L/K, \sigma, b)$, the algebraic representation is a `t_COL` with coefficients in L representing the element in the decomposition $A = \bigoplus_{i=0}^{d-1} x^i L$. For a central simple algebra defined by a multiplication table over its center K on a basis (e_i), the algebraic representation is a `t_COL` with coefficients in K representing the element on the basis (e_i).

Warning. The coefficients in the decomposition $A = \bigoplus_{i=0}^{d-1} x^i L$ are not the same as those in the decomposition $A = \bigoplus_{i=0}^{d-1} L x^i$! The i-th coefficients are related by conjugating by x^i, which on L amounts to acting by σ^i.

Warning. For a central simple algebra over $\mathbf{Q}$ defined by a multiplication table, we cannot distinguish between the basis and the algebraic representations from the size of the vectors. The behaviour is then to always interpret the column vector as a basis representation if the coefficients are `t_INT` or `t_FRAC`, and as an algebraic representation if the coefficients are `t_POL` or `t_POLMOD`.

3.9.1 algabsdim(al). Given an algebra al output by `alginit` or by `algtableinit`, returns the dimension of al over its prime subfield ($\mathbf{Q}$ or $\mathbf{F}_p$).

```
? nf = nfinit(y^3-y+1);
? A = alginit(nf, [-1,-1]);
? algabsdim(A)
%3 = 12
```

The library syntax is `long algabsdim(GEN al)`.

3.9.2 algadd(al, x, y). Given two elements x and y in al, computes their sum $x+y$ in the algebra al.

```
? A = alginit(nfinit(y),[-1,1]);
? algadd(A,[1,0]~,[1,2]~)
%2 = [2, 2]~
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN algadd(GEN al, GEN x, GEN y)`.

3.9.3 algalgtobasis(al, x). Given an element x in the central simple algebra al output by `alginit`, transforms it to a column vector on the integral basis of al. This is the inverse function of `algbasistoalg`.

```
? A = alginit(nfinit(y^2-5),[2,y]);
? algalgtobasis(A,[y,1]~)
%2 = [0, 2, 0, -1, 2, 0, 0, 0]~
? algbasistoalg(A,algalgtobasis(A,[y,1]~))
%3 = [Mod(Mod(y, y^2 - 5), x^2 - 2), 1]~
```

The library syntax is `GEN algalgtobasis(GEN al, GEN x)`.

3.9.4 algaut(al). Given a cyclic algebra $al = (L/K, \sigma, b)$ output by `alginit`, returns the automorphism σ.

```
? nf = nfinit(y);
? p = idealprimedec(nf,7)[1];
? p2 = idealprimedec(nf,11)[1];
? A = alginit(nf,[3,[[p,p2],[1/3,2/3]],[0]]);
? algaut(A)
%5 = -1/3*x^2 + 1/3*x + 26/3
```

The library syntax is `GEN algaut(GEN al)`.

3.9.5 algb(al). Given a cyclic algebra $al = (L/K, \sigma, b)$ output by `alginit`, returns the element $b \in K$.

```
nf = nfinit(y);
? p = idealprimedec(nf,7)[1];
? p2 = idealprimedec(nf,11)[1];
? A = alginit(nf,[3,[[p,p2],[1/3,2/3]],[0]]);
? algb(A)
%5 = Mod(-77, y)
```

The library syntax is `GEN algb(GEN al)`.

3.9.6 algbasis(al). Given an central simple algebra al output by `alginit`, returns a $\mathbf{Z}$-basis of the order $\mathcal{O}_0$ stored in al with respect to the natural order in al. It is a maximal order if one has been computed.

```
A = alginit(nfinit(y), [-1,-1]);
? algbasis(A)
%2 =
[1 0 0 1/2]

[0 1 0 1/2]

[0 0 1 1/2]

[0 0 0 1/2]
```

The library syntax is `GEN algbasis(GEN al)`.

3.9.7 algbasistoalg(al, x). Given an element x in the central simple algebra al output by `alginit`, transforms it to its algebraic representation in al. This is the inverse function of `algalgtobasis`.

```
? A = alginit(nfinit(y^2-5),[2,y]);
? z = algbasistoalg(A,[0,1,0,0,2,-3,0,0]~);
? liftall(z)
%3 = [(-1/2*y - 2)*x + (-1/4*y + 5/4), -3/4*y + 7/4]~
? algalgtobasis(A,z)
%4 = [0, 1, 0, 0, 2, -3, 0, 0]~
```

The library syntax is `GEN algbasistoalg(GEN al, GEN x)`.

3.9.8 algcenter(al). If al is a table algebra output by `algtableinit`, returns a basis of the center of the algebra al over its prime field ($\mathbf{Q}$ or $\mathbf{F}_p$). If al is a central simple algebra output by `alginit`, returns the center of al, which is stored in al.

A simple example: the 2×2 upper triangular matrices over $\mathbf{Q}$, generated by I_2, $a = [0,1;0,0]$ and $b = [0,0;0,1]$, such that $a^2 = 0$, $ab = a$, $ba = 0$, $b^2 = b$: the diagonal matrices form the center.

```
? mt = [matid(3),[0,0,0;1,0,1;0,0,0],[0,0,0;0,0,0;1,0,1]];
? A = algtableinit(mt);
? algcenter(A) \\ = (I_2)
%3 =
[1]

[0]

[0]
```

An example in the central simple case:

```
? nf = nfinit(y^3-y+1);
? A = alginit(nf, [-1,-1]);
? algcenter(A).pol
%3 = y^3 - y + 1
```

The library syntax is `GEN algcenter(GEN al)`.

3.9.9 algcentralproj$(al, z, \{maps = 0\})$. Given a table algebra al output by `algtableinit` and a `t_VEC` $z = [z_1, \ldots, z_n]$ of orthogonal central idempotents, returns a `t_VEC` $[al_1, \ldots, al_n]$ of algebras such that $al_i = z_i\, al$. If $maps = 1$, each al_i is a `t_VEC` $[quo, proj, lift]$ where *quo* is the quotient algebra, *proj* is a `t_MAT` representing the projection onto this quotient and *lift* is a `t_MAT` representing a lift.

A simple example: $\mathbf{F}_2 \oplus \mathbf{F}_4$, generated by $1 = (1,1)$, $e = (1,0)$ and x such that $x^2 + x + 1 = 0$. We have $e^2 = e$, $x^2 = x + 1$ and $ex = 0$.

```
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,2);
? e = [0,1,0]~;
? e2 = algsub(A,[1,0,0]~,e);
? [a,a2] = algcentralproj(A,[e,e2]);
? algdim(a)
%6 = 1
? algdim(a2)
```

```
%7 = 2
```

The library syntax is `GEN alg_centralproj(GEN al, GEN z, long maps)`.

3.9.10 algchar(al). Given an algebra al output by `alginit` or `algtableinit`, returns the characteristic of al.

```
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,13);
? algchar(A)
%3 = 13
```

The library syntax is `GEN algchar(GEN al)`.

3.9.11 algcharpoly$(al, b, \{v =' x\})$. Given an element b in al, returns its characteristic polynomial as a polynomial in the variable v. If al is a table algebra output by `algtableinit`, returns the absolute characteristic polynomial of b, which is an element of $\mathbf{F}_p[v]$ or $\mathbf{Q}[v]$; if al is a central simple algebra output by `alginit`, returns the reduced characteristic polynomial of b, which is an element of $K[v]$ where K is the center of al.

```
? al = alginit(nfinit(y), [-1,-1]); \\ (-1,-1)_Q
? algcharpoly(al, [0,1]~)
%2 = x^2 + 1
```

Also accepts a square matrix with coefficients in al.

The library syntax is `GEN algcharpoly(GEN al, GEN b, long v = -1)` where `v` is a variable number.

3.9.12 algdecomposition(al). al being a table algebra output by `algtableinit`, returns $[J, [al_1, \ldots, al_n]]$ where J is a basis of the Jacobson radical of al and $al_1, \ldots, al_n$ are the simple factors of the semisimple algebra al/J.

The library syntax is `GEN alg_decomposition(GEN al)`.

3.9.13 algdegree(al). Given a central simple algebra al output by `alginit`, returns the degree of al.

```
? nf = nfinit(y^3-y+1);
? A = alginit(nf, [-1,-1]);
? algdegree(A)
%3 = 2
```

The library syntax is `long algdegree(GEN al)`.

3.9.14 algdim(al). Given a central simple algebra al output by `alginit`, returns the dimension of al over its center. Given a table algebra al output by `algtableinit`, returns the dimension of al over its prime subfield ($\mathbf{Q}$ or $\mathbf{F}_p$).

```
? nf = nfinit(y^3-y+1);
? A = alginit(nf, [-1,-1]);
? algdim(A)
%3 = 4
```

The library syntax is `long algdim(GEN al)`.

3.9.15 algdisc(al). Given a central simple algebra al output by `alginit`, computes the discriminant of the order $\mathcal{O}_0$ stored in al, that is the determinant of the trace form $\mathrm{Tr} : \mathcal{O}_0 \times \mathcal{O}_0 \to \mathbf{Z}$.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-3,1-y]);
? [PR,h] = alghassef(A);
%3 = [[[2, [2, 0]~, 1, 2, 1], [3, [3, 0]~, 1, 2, 1]], Vecsmall([0, 1])]
? n = algdegree(A);
? D = algabsdim(A);
? h = vector(#h, i, n - gcd(n,h[i]));
? n^D * nf.disc^(n^2) * idealnorm(nf, idealfactorback(nf,PR,h))^n
%4 = 12960000
? algdisc(A)
%5 = 12960000
```

The library syntax is `GEN algdisc(GEN al)`.

3.9.16 algdivl(al, x, y). Given two elements x and y in al, computes their left quotient $x\backslash y$ in the algebra al: an element z such that $xz = y$ (such an element is not unique when x is a zerodivisor). If x is invertible, this is the same as $x^{-1}y$. Assumes that y is left divisible by x (i.e. that z exists). Also accepts matrices with coefficients in al.

The library syntax is `GEN algdivl(GEN al, GEN x, GEN y)`.

3.9.17 algdivr(al, x, y). Given two elements x and y in al, return xy^{-1}. Also accepts matrices with coefficients in al.

The library syntax is `GEN algdivr(GEN al, GEN x, GEN y)`.

3.9.18 alggroup($gal, \{p = 0\}$). Initialize the group algebra $K[G]$ over $K = \mathbf{Q}$ (p omitted) or $\mathbf{F}_p$ where G is the underlying group of the `galoisinit` structure gal. The input gal is also allowed to be a `t_VEC` of permutations that is closed under products.

Example:

```
? K = nfsplitting(x^3-x+1);
? gal = galoisinit(K);
? al = alggroup(gal);
? algissemisimple(al)
%4 = 1
? G = [Vecsmall([1,2,3]), Vecsmall([1,3,2])];
? al2 = alggroup(G, 2);
? algissemisimple(al2)
%8 = 0
```

The library syntax is `GEN alggroup(GEN gal, GEN p = NULL)`.

3.9.19 alghasse(al, pl). Given a central simple algebra al output by `alginit` and a prime ideal or an integer between 1 and $r_1 + r_2$, returns a `t_FRAC` h : the local Hasse invariant of al at the place specified by pl.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,y]);
? alghasse(A, 1)
%3 = 1/2
? alghasse(A, 2)
%4 = 0
? alghasse(A, idealprimedec(nf,2)[1])
%5 = 1/2
? alghasse(A, idealprimedec(nf,5)[1])
%6 = 0
```

The library syntax is `GEN alghasse(GEN al, GEN pl)`.

3.9.20 alghassef(al). Given a central simple algebra al output by `alginit`, returns a `t_VEC` $[\mathtt{PR}, h_f]$ describing the local Hasse invariants at the finite places of the center: `PR` is a `t_VEC` of primes and h_f is a `t_VECSMALL` of integers modulo the degree d of al.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,2*y-1]);
? [PR,hf] = alghassef(A);
? PR
%4 = [[19, [10, 2]~, 1, 1, [-8, 2; 2, -10]], [2, [2, 0]~, 1, 2, 1]]
? hf
%5 = Vecsmall([1, 0])
```

The library syntax is `GEN alghassef(GEN al)`.

3.9.21 alghassei(al). Given a central simple algebra al output by `alginit`, returns a `t_VECSMALL` h_i of r_1 integers modulo the degree d of al, where r_1 is the number of real places of the center: the local Hasse invariants of al at infinite places.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,y]);
? alghassei(A)
%3 = Vecsmall([1, 0])
```

The library syntax is `GEN alghassei(GEN al)`.

3.9.22 algindex$(al, \{pl\})$. Return the index of the central simple algebra A over K (as output by alginit), that is the degree e of the unique central division algebra D over K such that A is isomorphic to some matrix algebra $M_d(D)$. If pl is set, it should be a prime ideal of K or an integer between 1 and $r_1 + r_2$, and in that case return the local index at the place pl instead.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,y]);
? algindex(A, 1)
%3 = 2
? algindex(A, 2)
%4 = 1
? algindex(A, idealprimedec(nf,2)[1])
%5 = 2
? algindex(A, idealprimedec(nf,5)[1])
%6 = 1
? algindex(A)
%7 = 2
```

The library syntax is `long algindex(GEN al, GEN pl = NULL)`.

3.9.23 alginit$(B, C, \{v\}, \{flag = 1\})$. Initialize the central simple algebra defined by data B, C and variable v, as follows.

• (multiplication table) B is the base number field K in `nfinit` form, C is a "multiplication table" over K. As a K-vector space, the algebra is generated by a basis $(e_1 = 1, \dots, e_n)$; the table is given as a `t_VEC` of n matrices in $M_n(K)$, giving the left multiplication by the basis elements e_i, in the given basis. Assumes that $e_1 = 1$, that the multiplication table is integral, and that $K[e_1, \dots, e_n]$ describes a central simple algebra over K.

```
{ m_i = [0,-1,0, 0;
         1, 0,0, 0;
         0, 0,0,-1;
         0, 0,1, 0];
  m_j = [0, 0,-1,0;
         0, 0, 0,1;
         1, 0, 0,0;
         0,-1, 0,0];
  m_k = [0, 0, 0, 0;
         0, 0,-1, 0;
         0, 1, 0, 0;
         1, 0, 0,-1];
  A = alginit(nfinit(y), [matid(4), m_i,m_j,m_k],  0); }
```

represents (in a complicated way) the quaternion algebra $(-1,-1)_{\mathbf{Q}}$. See below for a simpler solution.

• (cyclic algebra) B is an `rnf` structure attached to a cyclic number field extension L/K of degree d, C is a `t_VEC [sigma,b]` with 2 components: `sigma` is a `t_POLMOD` representing an automorphism generating $\mathrm{Gal}(L/K)$, b is an element in K^*. This represents the cyclic algebra $(L/K, \sigma, b)$. Currently the element b has to be integral.

```
? Q = nfinit(y); T = polcyclo(5, 'x); F = rnfinit(Q, T);
```

```
? A = alginit(F, [Mod(x^2,T), 3]);
```

defines the cyclic algebra $(L/\mathbf{Q}, \sigma, 3)$, where $L = \mathbf{Q}(\zeta_5)$ and $\sigma : \zeta \mapsto \zeta^2$ generates $\mathrm{Gal}(L/\mathbf{Q})$.

• (quaternion algebra, special case of the above) B is an `nf` structure attached to a number field K, $C = [a, b]$ is a vector containing two elements of K^* with a not a square in K, returns the quaternion algebra $(a, b)_K$. The variable v (`'x` by default) must have higher priority than the variable of K`.pol` and is used to represent elements in the splitting field $L = K[x]/(x^2 - a)$.

```
? Q = nfinit(y); A = alginit(Q, [-1,-1]);  \\ (-1,-1)_Q
```

• (algebra/K defined by local Hasse invariants) B is an `nf` structure attached to a number field K, $C = [d, [\mathtt{PR}, h_f], h_i]$ is a triple containing an integer $d > 1$, a pair $[\mathtt{PR}, h_f]$ describing the Hasse invariants at finite places, and h_i the Hasse invariants at archimedean (real) places. A local Hasse invariant belongs to $(1/d)\mathbf{Z}/\mathbf{Z} \subset \mathbf{Q}/\mathbf{Z}$, and is given either as a `t_FRAC` (lift to $(1/d)\mathbf{Z}$), a `t_INT` or `t_INTMOD` modulo d (lift to $\mathbf{Z}/d\mathbf{Z}$); a whole vector of local invariants can also be given as a `t_VECSMALL`, whose entries are handled as `t_INT`s. `PR` is a list of prime ideals (`prid` structures), and h_f is a vector of the same length giving the local invariants at those maximal ideals. The invariants at infinite real places are indexed by the real roots K`.roots`: if the Archimedean place v is attached to the j-th root, the value of h_v is given by $h_i[j]$, must be 0 or 1/2 (or $d/2$ modulo d), and can be nonzero only if d is even.

By class field theory, provided the local invariants h_v sum to 0, up to Brauer equivalence, there is a unique central simple algebra over K with given local invariants and trivial invariant elsewhere. In particular, up to isomorphism, there is a unique such algebra A of degree d.

We realize A as a cyclic algebra through class field theory. The variable v (`'x` by default) must have higher priority than the variable of K`.pol` and is used to represent elements in the (cyclic) splitting field extension L/K for A.

```
? nf = nfinit(y^2+1);
? PR = idealprimedec(nf,5); #PR
%2 = 2
? hi = [];
? hf = [PR, [1/3,-1/3]];
? A = alginit(nf, [3,hf,hi]);
? algsplittingfield(A).pol
%6 = x^3 - 21*x + 7
```

• (matrix algebra, toy example) B is an `nf` structure attached to a number field K, $C = d$ is a positive integer. Returns a cyclic algebra isomorphic to the matrix algebra $M_d(K)$.

In all cases, this function computes a maximal order for the algebra by default, which may require a lot of time. Setting *flag* $= 0$ prevents this computation.

The pari object representing such an algebra A is a `t_VEC` with the following data:

• A splitting field L of A of the same degree over K as A, in `rnfinit` format, accessed with `algsplittingfield`.

• The same splitting field L in `nfinit` format.

• The Hasse invariants at the real places of K, accessed with `alghassei`.

• The Hasse invariants of A at the finite primes of K that ramify in the natural order of A, accessed with `alghassef`.

• A basis of an order $\mathcal{O}_0$ expressed on the basis of the natural order, accessed with `algbasis`.

• A basis of the natural order expressed on the basis of $\mathcal{O}_0$, accessed with `alginvbasis`.

• The left multiplication table of $\mathcal{O}_0$ on the previous basis, accessed with `algmultable`.

• The characteristic of A (always 0), accessed with `algchar`.

• The absolute traces of the elements of the basis of $\mathcal{O}_0$.

• If A was constructed as a cyclic algebra $(L/K, \sigma, b)$ of degree d, a `t_VEC` $[\sigma, \sigma^2, \ldots, \sigma^{d-1}]$. The function `algaut` returns σ.

• If A was constructed as a cyclic algebra $(L/K, \sigma, b)$, the element b, accessed with `algb`.

• If A was constructed with its multiplication table mt over K, the `t_VEC` of `t_MAT` mt, accessed with `algrelmultable`.

• If A was constructed with its multiplication table mt over K, a `t_VEC` with three components: a `t_COL` representing an element of A generating the splitting field L as a maximal subfield of A, a `t_MAT` representing an L-basis $\mathcal{B}$ of A expressed on the $\mathbf{Z}$-basis of $\mathcal{O}_0$, and a `t_MAT` representing the $\mathbf{Z}$-basis of $\mathcal{O}_0$ expressed on $\mathcal{B}$. This data is accessed with `algsplittingdata`.

The library syntax is `GEN alginit(GEN B, GEN C, long v = -1, long flag)` where `v` is a variable number.

3.9.24 alginv(al, x). Given an element x in al, computes its inverse x^{-1} in the algebra al. Assumes that x is invertible.

```
? A = alginit(nfinit(y), [-1,-1]);
? alginv(A,[1,1,0,0]~)
%2 = [1/2, 1/2, 0, 0]~
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN alginv(GEN al, GEN x)`.

3.9.25 alginvbasis(al). Given an central simple algebra al output by `alginit`, returns a $\mathbf{Z}$-basis of the natural order in al with respect to the order $\mathcal{O}_0$ stored in al.

```
A = alginit(nfinit(y), [-1,-1]);
? alginvbasis(A)
%2 =
[1 0 0 -1]

[0 1 0 -1]

[0 0 1 -1]

[0 0 0  2]
```

The library syntax is `GEN alginvbasis(GEN al)`.

3.9.26 algisassociative$(mt, p = 0)$. Returns 1 if the multiplication table `mt` is suitable for `algtableinit(mt,p)`, 0 otherwise. More precisely, `mt` should be a `t_VEC` of n matrices in $M_n(K)$, giving the left multiplications by the basis elements $e_1, \dots, e_n$ (structure constants). We check whether the first basis element e_1 is 1 and $e_i(e_j e_k) = (e_i e_j)e_k$ for all i, j, k.

```
? mt = [matid(3),[0,0,0;1,0,1;0,0,0],[0,0,0;0,0,0;1,0,1]];
? algisassociative(mt)
%2 = 1
```

May be used to check a posteriori an algebra: we also allow `mt` as output by `algtableinit` (p is ignored in this case).

The library syntax is `GEN algisassociative(GEN mt, GEN p)`.

3.9.27 algiscommutative(al). al being a table algebra output by `algtableinit` or a central simple algebra output by `alginit`, tests whether the algebra al is commutative.

```
? mt = [matid(3),[0,0,0;1,0,1;0,0,0],[0,0,0;0,0,0;1,0,1]];
? A = algtableinit(mt);
? algiscommutative(A)
%3 = 0
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,2);
? algiscommutative(A)
%6 = 1
```

The library syntax is `GEN algiscommutative(GEN al)`.

3.9.28 algisdivision$(al, \{pl\})$. Given a central simple algebra al output by `alginit`, test whether al is a division algebra. If pl is set, it should be a prime ideal of K or an integer between 1 and $r_1 + r_2$, and in that case test whether al is locally a division algebra at the place pl instead.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,y]);
? algisdivision(A, 1)
%3 = 1
? algisdivision(A, 2)
%4 = 0
? algisdivision(A, idealprimedec(nf,2)[1])
%5 = 1
? algisdivision(A, idealprimedec(nf,5)[1])
%6 = 0
? algisdivision(A)
%7 = 1
```

The library syntax is `GEN algisdivision(GEN al, GEN pl = NULL)`.

3.9.29 algisdivl$(al, x, y, \{\&z\})$. Given two elements x and y in al, tests whether y is left divisible by x, that is whether there exists z in al such that $xz = y$, and sets z to this element if it exists.

```
? A = alginit(nfinit(y), [-1,1]);
? algisdivl(A,[x+2,-x-2]~,[x,1]~)
%2 = 0
? algisdivl(A,[x+2,-x-2]~,[-x,x]~,&z)
%3 = 1
? z
%4 = [Mod(-2/5*x - 1/5, x^2 + 1), 0]~
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN algisdivl(GEN al, GEN x, GEN y, GEN *z = NULL)`.

3.9.30 algisinv$(al, x, \{\&ix\})$. Given an element x in al, tests whether x is invertible, and sets ix to the inverse of x.

```
? A = alginit(nfinit(y), [-1,1]);
? algisinv(A,[-1,1]~)
%2 = 0
? algisinv(A,[1,2]~,&ix)
%3 = 1
? ix
%4 = [Mod(Mod(-1/3, y), x^2 + 1), Mod(Mod(2/3, y), x^2 + 1)]~
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN algisinv(GEN al, GEN x, GEN *ix = NULL)`.

3.9.31 algisramified$(al, \{pl\})$. Given a central simple algebra al output by `alginit`, test whether al is ramified, i.e. not isomorphic to a matrix algebra over its center. If pl is set, it should be a prime ideal of K or an integer between 1 and $r_1 + r_2$, and in that case test whether al is locally ramified at the place pl instead.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,y]);
? algisramified(A, 1)
%3 = 1
? algisramified(A, 2)
%4 = 0
? algisramified(A, idealprimedec(nf,2)[1])
%5 = 1
? algisramified(A, idealprimedec(nf,5)[1])
%6 = 0
? algisramified(A)
%7 = 1
```

The library syntax is `GEN algisramified(GEN al, GEN pl = NULL)`.

3.9.32 algissemisimple(al). al being a table algebra output by `algtableinit` or a central simple algebra output by `alginit`, tests whether the algebra al is semisimple.

```
? mt = [matid(3),[0,0,0;1,0,1;0,0,0],[0,0,0;0,0,0;1,0,1]];
? A = algtableinit(mt);
? algissemisimple(A)
%3 = 0
? m_i=[0,-1,0,0;1,0,0,0;0,0,0,-1;0,0,1,0]; \\ quaternion algebra (-1,-1)
? m_j=[0,0,-1,0;0,0,0,1;1,0,0,0;0,-1,0,0];
? m_k=[0,0,0,-1;0,0,-1,0;0,1,0,0;1,0,0,0];
? mt = [matid(4), m_i, m_j, m_k];
? A = algtableinit(mt);
? algissemisimple(A)
%9 = 1
```

The library syntax is `GEN algissemisimple(GEN al)`.

3.9.33 algissimple($al, \{ss = 0\}$). al being a table algebra output by `algtableinit` or a central simple algebra output by `alginit`, tests whether the algebra al is simple. If $ss = 1$, assumes that the algebra al is semisimple without testing it.

```
? mt = [matid(3),[0,0,0;1,0,1;0,0,0],[0,0,0;0,0,0;1,0,1]];
? A = algtableinit(mt); \\ matrices [*,*; 0,*]
? algissimple(A)
%3 = 0
? algissimple(A,1) \\ incorrectly assume that A is semisimple
%4 = 1
? m_i=[0,-1,0,0;1,0,0,0;0,0,0,-1;0,0,1,0];
? m_j=[0,0,-1,0;0,0,0,1;1,0,0,0;0,-1,0,0];
? m_k=[0,0,0,-1;0,0,b,0;0,1,0,0;1,0,0,0];
? mt = [matid(4), m_i, m_j, m_k];
? A = algtableinit(mt); \\ quaternion algebra (-1,-1)
? algissimple(A)
%10 = 1
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,2); \\ direct sum F_4+F_2
? algissimple(A)
%13 = 0
```

The library syntax is `GEN algissimple(GEN al, long ss)`.

3.9.34 algissplit$(al, \{pl\})$. Given a central simple algebra al output by `alginit`, test whether al is split, i.e. isomorphic to a matrix algebra over its center. If pl is set, it should be a prime ideal of K or an integer between 1 and $r_1 + r_2$, and in that case test whether al is locally split at the place pl instead.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,y]);
? algissplit(A, 1)
%3 = 0
? algissplit(A, 2)
%4 = 1
? algissplit(A, idealprimedec(nf,2)[1])
%5 = 0
? algissplit(A, idealprimedec(nf,5)[1])
%6 = 1
? algissplit(A)
%7 = 0
```

The library syntax is `GEN algissplit(GEN al, GEN pl = NULL)`.

3.9.35 alglathnf(al, m). Given an algebra al and a square invertible matrix m with size the dimension of al, returns the lattice generated by the columns of m.

```
? al = alginit(nfinit(y^2+7), [-1,-1]);
? a = [1,1,-1/2,1,1/3,-1,1,1]~;
? mt = algleftmultable(al,a);
? lat = alglathnf(al,mt);
? lat[2]
%5 = 1/6
```

The library syntax is `GEN alglathnf(GEN al, GEN m)`.

3.9.36 algleftmultable(al, x). Given an element x in al, computes its left multiplication table. If x is given in basis form, returns its multiplication table on the integral basis; if x is given in algebraic form, returns its multiplication table on the basis corresponding to the algebraic form of elements of al. In every case, if x is a `t_COL` of length n, then the output is a $n \times n$ `t_MAT`. Also accepts a square matrix with coefficients in al.

```
? A = alginit(nfinit(y), [-1,-1]);
? algleftmultable(A,[0,1,0,0]~)
%2 =
[0 -1  1  0]

[1  0  1  1]

[0  0  1  1]

[0  0 -2 -1]
```

The library syntax is `GEN algleftmultable(GEN al, GEN x)`.

3.9.37 algmul(al, x, y). Given two elements x and y in al, computes their product $x * y$ in the algebra al.

```
? A = alginit(nfinit(y), [-1,-1]);
? algmul(A,[1,1,0,0]~,[0,0,2,1]~)
%2 = [2, 3, 5, -4]~
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN algmul(GEN al, GEN x, GEN y)`.

3.9.38 algmultable(al). Returns a multiplication table of al over its prime subfield ($\mathbf{Q}$ or $\mathbf{F}_p$), as a `t_VEC` of `t_MAT`: the left multiplication tables of basis elements. If al was output by `algtableinit`, returns the multiplication table used to define al. If al was output by `alginit`, returns the multiplication table of the order $\mathcal{O}_0$ stored in al.

```
? A = alginit(nfinit(y), [-1,-1]);
? M = algmultable(A);
? #M
%3 = 4
? M[1]  \\ multiplication by e_1 = 1
%4 =
[1 0 0 0]

[0 1 0 0]

[0 0 1 0]

[0 0 0 1]

? M[2]
%5 =
[0 -1  1  0]

[1  0  1  1]

[0  0  1  1]

[0  0 -2 -1]
```

The library syntax is `GEN algmultable(GEN al)`.

3.9.39 algneg(al, x). Given an element x in al, computes its opposite $-x$ in the algebra al.

```
? A = alginit(nfinit(y), [-1,-1]);
? algneg(A,[1,1,0,0]~)
%2 = [-1, -1, 0, 0]~
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN algneg(GEN al, GEN x)`.

3.9.40 algnorm(al, x). Given an element x in al, computes its norm. If al is a table algebra output by `algtableinit`, returns the absolute norm of x, which is an element of $\mathbf{F}_p$ of $\mathbf{Q}$; if al is a central simple algebra output by `alginit`, returns the reduced norm of x, which is an element of the center of al.

```
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,19);
? algnorm(A,[0,-2,3]~)
%3 = 18
```

Also accepts a square matrix with coefficients in al.

The library syntax is `GEN algnorm(GEN al, GEN x)`.

3.9.41 algpoleval(al, T, b). Given an element b in al and a polynomial T in $K[X]$, computes $T(b)$ in al.

The library syntax is `GEN algpoleval(GEN al, GEN T, GEN b)`.

3.9.42 algpow(al, x, n). Given an element x in al and an integer n, computes the power x^n in the algebra al.

```
? A = alginit(nfinit(y), [-1,-1]);
? algpow(A,[1,1,0,0]~,7)
%2 = [8, -8, 0, 0]~
```

Also accepts a square matrix with coefficients in al.

The library syntax is `GEN algpow(GEN al, GEN x, GEN n)`.

3.9.43 algprimesubalg(al). al being the output of `algtableinit` representing a semisimple algebra of positive characteristic, returns a basis of the prime subalgebra of al. The prime subalgebra of al is the subalgebra fixed by the Frobenius automorphism of the center of al. It is abstractly isomorphic to a product of copies of $\mathbf{F}_p$.

```
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,2);
? algprimesubalg(A)
%3 =
[1 0]

[0 1]

[0 0]
```

The library syntax is `GEN algprimesubalg(GEN al)`.

3.9.44 algquotient$(al, I, \{flag = 0\})$. al being a table algebra output by `algtableinit` and I being a basis of a two-sided ideal of al represented by a matrix, returns the quotient al/I. When $flag = 1$, returns a `t_VEC` $[al/I, proj, lift]$ where *proj* and *lift* are matrices respectively representing the projection map and a section of it.

```
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,2);
? AQ = algquotient(A,[0;1;0]);
? algdim(AQ)
%4 = 2
```

The library syntax is `GEN alg_quotient(GEN al, GEN I, long flag)`.

3.9.45 algradical(al). al being a table algebra output by `algtableinit`, returns a basis of the Jacobson radical of the algebra al over its prime field ($\mathbf{Q}$ or $\mathbf{F}_p$).

Here is an example with $A = \mathbf{Q}[x]/(x^2)$, generated by $(1, x)$:

```
? mt = [matid(2),[0,0;1,0]];
? A = algtableinit(mt);
? algradical(A) \\ = (x)
%3 =
[0]

[1]
```

Another one with 2×2 upper triangular matrices over $\mathbf{Q}$, generated by I_2, $a = [0, 1; 0, 0]$ and $b = [0, 0; 0, 1]$, such that $a^2 = 0$, $ab = a$, $ba = 0$, $b^2 = b$:

```
? mt = [matid(3),[0,0,0;1,0,1;0,0,0],[0,0,0;0,0,0;1,0,1]];
? A = algtableinit(mt);
? algradical(A) \\ = (a)
%6 =
[0]

[1]

[0]
```

The library syntax is `GEN algradical(GEN al)`.

3.9.46 algramifiedplaces(al). Given a central simple algebra al output by `alginit`, return a `t_VEC` containing the list of places of the center of al that are ramified in al. Each place is described as an integer between 1 and r_1 or as a prime ideal.

```
? nf = nfinit(y^2-5);
? A = alginit(nf, [-1,y]);
? algramifiedplaces(A)
%3 = [1, [2, [2, 0]~, 1, 2, 1]]
```

The library syntax is `GEN algramifiedplaces(GEN al)`.

3.9.47 algrandom(al, b). Given an algebra al and an integer b, returns a random element in al with coefficients in $[-b, b]$.

The library syntax is `GEN algrandom(GEN al, GEN b)`.

3.9.48 algrelmultable(al). Given a central simple algebra al output by `alginit` defined by a multiplication table over its center (a number field), returns this multiplication table.

```
? nf = nfinit(y^3-5); a = y; b = y^2;
? {m_i = [0,a,0,0;
          1,0,0,0;
          0,0,0,a;
          0,0,1,0];}
? {m_j = [0, 0,b, 0;
          0, 0,0,-b;
          1, 0,0, 0;
          0,-1,0, 0];}
? {m_k = [0, 0,0,-a*b;
          0, 0,b,   0;
          0,-a,0,   0;
          1, 0,0,   0];}
? mt = [matid(4), m_i, m_j, m_k];
? A = alginit(nf,mt,'x);
? M = algrelmultable(A);
? M[2] == m_i
%8 = 1
? M[3] == m_j
%9 = 1
? M[4] == m_k
%10 = 1
```

The library syntax is `GEN algrelmultable(GEN al)`.

3.9.49 algsimpledec($al, \{flag = 0\}$). al being the output of `algtableinit` representing a semisimple algebra, returns a `t_VEC` $[al_1, al_2, \ldots, al_n]$ such that al is isomorphic to the direct sum of the simple algebras al_i. When $flag = 1$, each component is instead a `t_VEC` $[al_i, proj_i, lift_i]$ where $proj_i$ and $lift_i$ are matrices respectively representing the projection map on the i-th factor and a section of it. The factors are sorted by increasing dimension, then increasing dimension of the center. This ensures that the ordering of the isomorphism classes of the factors is deterministic over finite fields, but not necessarily over $\mathbf{Q}$.

Warning. The images of the $lift_i$ are not guaranteed to form a direct sum.

The library syntax is `GEN algsimpledec(GEN al, long flag)`.

3.9.50 algsplittingdata(al). Given a central simple algebra al output by `alginit` defined by a multiplication table over its center K (a number field), returns data stored to compute a splitting of al over an extension. This data is a `t_VEC [t,Lbas,Lbasinv]` with 3 components:

• an element t of al such that $L = K(t)$ is a maximal subfield of al;

• a matrix `Lbas` expressing a L-basis of al (given an L-vector space structure by multiplication on the right) on the integral basis of al;

• a matrix `Lbasinv` expressing the integral basis of al on the previous L-basis.

```
? nf = nfinit(y^3-5); a = y; b = y^2;
? {m_i = [0,a,0,0;
```

```
          1,0,0,0;
          0,0,0,a;
          0,0,1,0];}
? {m_j = [0, 0,b, 0;
          0, 0,0,-b;
          1, 0,0, 0;
          0,-1,0, 0];}
? {m_k = [0, 0,0,-a*b;
          0, 0,b,   0;
          0,-a,0,   0;
          1, 0,0,   0];}
? mt = [matid(4), m_i, m_j, m_k];
? A = alginit(nf,mt,'x);
? [t,Lb,Lbi] = algsplittingdata(A);
? t
%8 = [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0]~;
? matsize(Lb)
%9 = [12, 2]
? matsize(Lbi)
%10 = [2, 12]
```

The library syntax is `GEN algsplittingdata(GEN al)`.

3.9.51 algsplittingfield(al). Given a central simple algebra al output by `alginit`, returns an `rnf` structure: the splitting field of al that is stored in al, as a relative extension of the center.

```
nf = nfinit(y^3-5);
a = y; b = y^2;
{m_i = [0,a,0,0;
       1,0,0,0;
       0,0,0,a;
       0,0,1,0];}
{m_j = [0, 0,b, 0;
       0, 0,0,-b;
       1, 0,0, 0;
       0,-1,0, 0];}
{m_k = [0, 0,0,-a*b;
       0, 0,b,   0;
       0,-a,0,   0;
       1, 0,0,   0];}
mt = [matid(4), m_i, m_j, m_k];
A = alginit(nf,mt,'x);
algsplittingfield(A).pol
%8 = x^2 - y
```

The library syntax is `GEN algsplittingfield(GEN al)`.

3.9.52 algsplittingmatrix(al, x). A central simple algebra al output by `alginit` contains data describing an isomorphism $\phi : A \otimes_K L \to M_d(L)$, where d is the degree of the algebra and L is an extension of L with $[L : K] = d$. Returns the matrix $\phi(x)$.

```
? A = alginit(nfinit(y), [-1,-1]);
? algsplittingmatrix(A,[0,0,0,2]~)
%2 =
[Mod(x + 1, x^2 + 1) Mod(Mod(1, y)*x + Mod(-1, y), x^2 + 1)]

[Mod(x + 1, x^2 + 1)                    Mod(-x + 1, x^2 + 1)]
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN algsplittingmatrix(GEN al, GEN x)`.

3.9.53 algsqr(al, x). Given an element x in al, computes its square x^2 in the algebra al.

```
? A = alginit(nfinit(y), [-1,-1]);
? algsqr(A,[1,0,2,0]~)
%2 = [-3, 0, 4, 0]~
```

Also accepts a square matrix with coefficients in al.

The library syntax is `GEN algsqr(GEN al, GEN x)`.

3.9.54 algsub(al, x, y). Given two elements x and y in al, computes their difference $x - y$ in the algebra al.

```
? A = alginit(nfinit(y), [-1,-1]);
? algsub(A,[1,1,0,0]~,[1,0,1,0]~)
%2 = [0, 1, -1, 0]~
```

Also accepts matrices with coefficients in al.

The library syntax is `GEN algsub(GEN al, GEN x, GEN y)`.

3.9.55 algsubalg(al, B). al being a table algebra output by `algtableinit` and B being a basis of a subalgebra of al represented by a matrix, returns an algebra isomorphic to B.

```
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,2);
? B = algsubalg(A,[1,0; 0,0; 0,1]);
? algdim(A)
%4 = 3
? algdim(B)
%5 = 2
```

The library syntax is `GEN algsubalg(GEN al, GEN B)`.

3.9.56 algtableinit$(mt, \{p = 0\})$. Initialize the associative algebra over $K = \mathbf{Q}$ (p omitted) or $\mathbf{F}_p$ defined by the multiplication table mt. As a K-vector space, the algebra is generated by a basis $(e_1 = 1, e_2, \ldots, e_n)$; the table is given as a `t_VEC` of n matrices in $M_n(K)$, giving the left multiplication by the basis elements e_i, in the given basis. Assumes that $e_1 = 1$, that $Ke_1 \oplus \ldots \oplus Ke_n]$ describes an associative algebra over K, and in the case $K = \mathbf{Q}$ that the multiplication table is integral. If the algebra is already known to be central and simple, then the case $K = \mathbf{F}_p$ is useless, and one should use `alginit` directly.

The point of this function is to input a finite dimensional K-algebra, so as to later compute its radical, then to split the quotient algebra as a product of simple algebras over K.

The pari object representing such an algebra A is a `t_VEC` with the following data:

- The characteristic of A, accessed with `algchar`.
- The multiplication table of A, accessed with `algmultable`.
- The traces of the elements of the basis.

A simple example: the 2×2 upper triangular matrices over $\mathbf{Q}$, generated by I_2, $a = [0, 1; 0, 0]$ and $b = [0, 0; 0, 1]$, such that $a^2 = 0$, $ab = a$, $ba = 0$, $b^2 = b$:

```
? mt = [matid(3),[0,0,0;1,0,1;0,0,0],[0,0,0;0,0,0;1,0,1]];
? A = algtableinit(mt);
? algradical(A) \\ = (a)
%6 =
[0]

[1]

[0]
? algcenter(A) \\ = (I_2)
%7 =
[1]

[0]

[0]
```

The library syntax is `GEN algtableinit(GEN mt, GEN p = NULL)`.

3.9.57 algtensor$(al1, al2, \{maxord = 1\})$. Given two algebras *al1* and *al2*, computes their tensor product. For table algebras output by `algtableinit`, the flag *maxord* is ignored. For central simple algebras output by `alginit`, computes a maximal order by default. Prevent this computation by setting $maxord = 0$.

Currently only implemented for cyclic algebras of coprime degree over the same center K, and the tensor product is over K.

The library syntax is `GEN algtensor(GEN al1, GEN al2, long maxord)`.

3.9.58 algtrace(al, x). Given an element x in al, computes its trace. If al is a table algebra output by `algtableinit`, returns the absolute trace of x, which is an element of $\mathbf{F}_p$ or $\mathbf{Q}$; if al is the output of `alginit`, returns the reduced trace of x, which is an element of the center of al.

```
? A = alginit(nfinit(y), [-1,-1]);
? algtrace(A,[5,0,0,1]~)
%2 = 11
```

Also accepts a square matrix with coefficients in al.

The library syntax is `GEN algtrace(GEN al, GEN x)`.

3.9.59 algtype(al). Given an algebra al output by `alginit` or by `algtableinit`, returns an integer indicating the type of algebra:

- 0: not a valid algebra.
- 1: table algebra output by `algtableinit`.
- 2: central simple algebra output by `alginit` and represented by a multiplication table over its center.
- 3: central simple algebra output by `alginit` and represented by a cyclic algebra.

```
? algtype([])
%1 = 0
? mt = [matid(3), [0,0,0; 1,1,0; 0,0,0], [0,0,1; 0,0,0; 1,0,1]];
? A = algtableinit(mt,2);
? algtype(A)
%4 = 1
? nf = nfinit(y^3-5);
?  a = y; b = y^2;
?  {m_i = [0,a,0,0;
           1,0,0,0;
           0,0,0,a;
           0,0,1,0];}
?  {m_j = [0, 0,b, 0;
           0, 0,0,-b;
           1, 0,0, 0;
           0,-1,0, 0];}
?  {m_k = [0, 0,0,-a*b;
           0, 0,b,   0;
           0,-a,0,   0;
           1, 0,0,   0];}
?  mt = [matid(4), m_i, m_j, m_k];
?  A = alginit(nf,mt,'x);
? algtype(A)
%12 = 2
? A = alginit(nfinit(y), [-1,-1]);
? algtype(A)
%14 = 3
```

The library syntax is `long algtype(GEN al)`.

3.10 Polynomials and power series.

We group here all functions which are specific to polynomials or power series. Many other functions which can be applied on these objects are described in the other sections. Also, some of the functions described here can be applied to other types.

3.10.1 O($p\hat{}e$). If p is an integer greater than 2, returns a p-adic 0 of precision e. In all other cases, returns a power series zero with precision given by ev, where v is the X-adic valuation of p with respect to its main variable.

The library syntax is `GEN ggrando()`. `GEN zeropadic(GEN p, long e)` for a p-adic and `GEN zeroser(long v, long e)` for a power series zero in variable v.

3.10.2 bezoutres($A, B, \{v\}$). Deprecated alias for `polresultantext`

The library syntax is `GEN polresultantext0(GEN A, GEN B, long v = -1)` where `v` is a variable number.

3.10.3 deriv($x, \{v\}$). Derivative of x with respect to the main variable if v is omitted, and with respect to v otherwise. The derivative of a scalar type is zero, and the derivative of a vector or matrix is done componentwise. One can use x' as a shortcut if the derivative is with respect to the main variable of x.

By definition, the main variable of a `t_POLMOD` is the main variable among the coefficients from its two polynomial components (representative and modulus); in other words, assuming a polmod represents an element of $R[X]/(T(X))$, the variable X is a mute variable and the derivative is taken with respect to the main variable used in the base ring R.

The library syntax is `GEN deriv(GEN x, long v = -1)` where `v` is a variable number.

3.10.4 diffop($x, v, d, \{n = 1\}$). Let v be a vector of variables, and d a vector of the same length, return the image of x by the n-power (1 if n is not given) of the differential operator D that assumes the value `d[i]` on the variable `v[i]`. The value of D on a scalar type is zero, and D applies componentwise to a vector or matrix. When applied to a `t_POLMOD`, if no value is provided for the variable of the modulus, such value is derived using the implicit function theorem.

Some examples: This function can be used to differentiate formal expressions: If $E = \exp(X^2)$ then we have $E' = 2 * X * E$. We can derivate $X * exp(X^2)$ as follow:

```
? diffop(E*X,[X,E],[1,2*X*E])
%1 = (2*X^2 + 1)*E
```

Let `Sin` and `Cos` be two function such that $\texttt{Sin}^2 + \texttt{Cos}^2 = 1$ and $\texttt{Cos}' = -\texttt{Sin}$. We can differentiate `Sin/Cos` as follow, PARI inferring the value of $\texttt{Sin}'$ from the equation:

```
? diffop(Mod('Sin/'Cos,'Sin^2+'Cos^2-1),['Cos],[-'Sin])
%1 = Mod(1/Cos^2, Sin^2 + (Cos^2 - 1))
```

Compute the Bell polynomials (both complete and partial) via the Faa di Bruno formula:

```
Bell(k,n=-1)=
{
  my(var(i)=eval(Str("X",i)));
  my(x,v,dv);
```

```
  v=vector(k,i,if(i==1,'E,var(i-1)));
  dv=vector(k,i,if(i==1,'X*var(1)*'E,var(i)));
  x=diffop('E,v,dv,k)/'E;
  if(n<0,subst(x,'X,1),polcoeff(x,n,'X))
}
```

The library syntax is `GEN diffop0(GEN x, GEN v, GEN d, long n)`.

For $n = 1$, the function `GEN diffop(GEN x, GEN v, GEN d)` is also available.

3.10.5 eval(x). Replaces in x the formal variables by the values that have been assigned to them after the creation of x. This is mainly useful in GP, and not in library mode. Do not confuse this with substitution (see `subst`).

If x is a character string, `eval(`x`)` executes x as a GP command, as if directly input from the keyboard, and returns its output.

```
? x1 = "one"; x2 = "two";
? n = 1; eval(Str("x", n))
%2 = "one"
? f = "exp"; v = 1;
? eval(Str(f, "(", v, ")"))
%4 = 2.7182818284590452353602874713526624978
```

Note that the first construct could be implemented in a simpler way by using a vector `x = ["one","two"]; x[n]`, and the second by using a closure `f = exp; f(v)`. The final example is more interesting:

```
? genmat(u,v) = matrix(u,v,i,j, eval( Str("x",i,j) ));
? genmat(2,3)   \\ generic 2 x 3 matrix
%2 =
[x11 x12 x13]

[x21 x22 x23]
```

A syntax error in the evaluation expression raises an `e_SYNTAX` exception, which can be trapped as usual:

```
? 1a
 ***   syntax error, unexpected variable name, expecting $end or ';': 1a
 ***                                                                  ^-
? E(expr) =
  {
    iferr(eval(expr),
          e, print("syntax error"),
          errname(e) == "e_SYNTAX");
  }
? E("1+1")
%1 = 2
? E("1a")
syntax error
```

The library syntax is **geval**(`GEN x`).

3.10.6 factorpadic(pol, p, r). p-adic factorization of the polynomial *pol* to precision r, the result being a two-column matrix as in `factor`. Note that this is not the same as a factorization over $\mathbf{Z}/p^r\mathbf{Z}$ (polynomials over that ring do not form a unique factorization domain, anyway), but approximations in $\mathbf{Q}/p^r\mathbf{Z}$ of the true factorization in $\mathbf{Q}_p[X]$.

```
? factorpadic(x^2 + 9, 3,5)
%1 =
[(1 + O(3^5))*x^2 + O(3^5)*x + (3^2 + O(3^5)) 1]
? factorpadic(x^2 + 1, 5,3)
%2 =
[  (1 + O(5^3))*x + (2 + 5 + 2*5^2 + O(5^3)) 1]

[(1 + O(5^3))*x + (3 + 3*5 + 2*5^2 + O(5^3)) 1]
```

The factors are normalized so that their leading coefficient is a power of p. The method used is a modified version of the round 4 algorithm of Zassenhaus.

If *pol* has inexact `t_PADIC` coefficients, this is not always well-defined; in this case, the polynomial is first made integral by dividing out the p-adic content, then lifted to $\mathbf{Z}$ using `truncate` coefficientwise. Hence we actually factor exactly a polynomial which is only p-adically close to the input. To avoid pitfalls, we advise to only factor polynomials with exact rational coefficients.

The library syntax is **factorpadic**`(GEN f,GEN p, long r)` . The function `factorpadic0` is deprecated, provided for backward compatibility.

3.10.7 intformal$(x, \{v\})$. formal integration of x with respect to the variable v (wrt. the main variable if v is omitted). Since PARI cannot represent logarithmic or arctangent terms, any such term in the result will yield an error:

```
? intformal(x^2)
%1 = 1/3*x^3
? intformal(x^2, y)
%2 = y*x^2
? intformal(1/x)
  ***   at top-level: intformal(1/x)
  ***                 ^--------------
  *** intformal: domain error in intformal: residue(series, pole) != 0
```

The argument x can be of any type. When x is a rational function, we assume that the base ring is an integral domain of characteristic zero.

By definition, the main variable of a `t_POLMOD` is the main variable among the coefficients from its two polynomial components (representative and modulus); in other words, assuming a polmod represents an element of $R[X]/(T(X))$, the variable X is a mute variable and the integral is taken with respect to the main variable used in the base ring R. In particular, it is meaningless to integrate with respect to the main variable of `x.mod`:

```
? intformal(Mod(1,x^2+1), 'x)
*** intformal: incorrect priority in intformal: variable x = x
```

The library syntax is `GEN integ(GEN x, long v = -1)` where `v` is a variable number.

3.10.8 padicappr(pol, a). Vector of p-adic roots of the polynomial *pol* congruent to the p-adic number a modulo p, and with the same p-adic precision as a. The number a can be an ordinary p-adic number (type `t_PADIC`, i.e. an element of $\mathbf{Z}_p$) or can be an integral element of a finite extension of $\mathbf{Q}_p$, given as a `t_POLMOD` at least one of whose coefficients is a `t_PADIC`. In this case, the result is the vector of roots belonging to the same extension of $\mathbf{Q}_p$ as a.

The library syntax is `GEN padicappr(GEN pol, GEN a)`. Also available is `GEN Zp_appr(GEN f, GEN a)` when a is a `t_PADIC`.

3.10.9 padicfields$(p, N, \{flag = 0\})$. Returns a vector of polynomials generating all the extensions of degree N of the field $\mathbf{Q}_p$ of p-adic rational numbers; N is allowed to be a 2-component vector $[n, d]$, in which case we return the extensions of degree n and discriminant p^d.

The list is minimal in the sense that two different polynomials generate non-isomorphic extensions; in particular, the number of polynomials is the number of classes of non-isomorphic extensions. If P is a polynomial in this list, α is any root of P and $K = \mathbf{Q}_p(\alpha)$, then α is the sum of a uniformizer and a (lift of a) generator of the residue field of K; in particular, the powers of α generate the ring of p-adic integers of K.

If *flag* $= 1$, replace each polynomial P by a vector $[P, e, f, d, c]$ where e is the ramification index, f the residual degree, d the valuation of the discriminant, and c the number of conjugate fields. If *flag* $= 2$, only return the *number* of extensions in a fixed algebraic closure (Krasner's formula), which is much faster.

The library syntax is `GEN padicfields0(GEN p, GEN N, long flag)`. Also available is `GEN padicfields(GEN p, long n, long d, long flag)`, which computes extensions of $\mathbf{Q}_p$ of degree n and discriminant p^d.

3.10.10 polchebyshev$(n, \{flag = 1\}, \{a =' x\})$. Returns the n^{th} Chebyshev polynomial of the first kind T_n (*flag* $= 1$) or the second kind U_n (*flag* $= 2$), evaluated at a (`'x` by default). Both series of polynomials satisfy the 3-term relation

$$P_{n+1} = 2xP_n - P_{n-1},$$

and are determined by the initial conditions $U_0 = T_0 = 1$, $T_1 = x$, $U_1 = 2x$. In fact $T_n' = nU_{n-1}$ and, for all complex numbers z, we have $T_n(\cos z) = \cos(nz)$ and $U_{n-1}(\cos z) = \sin(nz)/\sin z$. If $n \geq 0$, then these polynomials have degree n. For $n < 0$, T_n is equal to T_{-n} and U_n is equal to $-U_{-2-n}$. In particular, $U_{-1} = 0$.

The library syntax is `GEN polchebyshev_eval(long n, long flag, GEN a = NULL)`. Also available are `GEN polchebyshev(long n, long flag, long v)`, `GEN polchebyshev1(long n, long v)` and `GEN polchebyshev2(long n, long v)` for T_n and U_n respectively.

3.10.11 polclass$(D, \{inv = 0\}, \{x =' x\})$. Return a polynomial in $\mathbf{Z}[x]$ generating the Hilbert class field for the imaginary quadratic discriminant D. If inv is 0 (the default), use the modular j-function and return the classical Hilbert polynomial, otherwise use a class invariant. The following invariants correspond to the different values of inv, where f denotes Weber's function `weber`, and $w_{p,q}$ the double eta quotient given by $w_{p,q} = \frac{\eta(x/p)\eta(x/q)}{\eta(x)\eta(x/pq)}$

The invariants $w_{p,q}$ are not allowed unless they satisfy the following technical conditions ensuring they do generate the Hilbert class field and not a strict subfield:

• if $p \neq q$, we need them both non-inert, prime to the conductor of $\mathbf{Z}[\sqrt{D}]$. Let P, Q be prime ideals above p and q; if both are unramified, we further require that $P^{\pm 1}Q^{\pm 1}$ be all distinct in the class group of $\mathbf{Z}[\sqrt{D}]$; if both are ramified, we require that $PQ \neq 1$ in the class group.

• if $p = q$, we want it split and prime to the conductor and the prime ideal above it must have order $\neq 1, 2, 4$ in the class group.

Invariants are allowed under the additional conditions on D listed below.

• $0 : j$

• $1 : f$, $D = 1 \bmod 8$ and $D = 1, 2 \bmod 3$;

• $2 : f^2$, $D = 1 \bmod 8$ and $D = 1, 2 \bmod 3$;

• $3 : f^3$, $D = 1 \bmod 8$;

• $4 : f^4$, $D = 1 \bmod 8$ and $D = 1, 2 \bmod 3$;

• $5 : \gamma_2 = j^{1/3}$, $D = 1, 2 \bmod 3$;

• $6 : w_{2,3}$, $D = 1 \bmod 8$ and $D = 1, 2 \bmod 3$;

• $8 : f^8$, $D = 1 \bmod 8$ and $D = 1, 2 \bmod 3$;

• $9 : w_{3,3}$, $D = 1 \bmod 2$ and $D = 1, 2 \bmod 3$;

• 10: $w_{2,5}$, $D \neq 60 \bmod 80$ and $D = 1, 2 \bmod 3$;

• 14: $w_{2,7}$, $D = 1 \bmod 8$;

• 15: $w_{3,5}$, $D = 1, 2 \bmod 3$;

• 21: $w_{3,7}$, $D = 1 \bmod 2$ and 21 does not divide D

• 23: $w_{2,3}^2$, $D = 1, 2 \bmod 3$;

• 24: $w_{2,5}^2$, $D = 1, 2 \bmod 3$;

• 26: $w_{2,13}$, $D \neq 156 \bmod 208$;

• 27: $w_{2,7}^2$, $D \neq 28 \bmod 112$;

• 28: $w_{3,3}^2$, $D = 1, 2 \bmod 3$;

• 35: $w_{5,7}$, $D = 1, 2 \bmod 3$;

• 39: $w_{3,13}$, $D = 1 \bmod 2$ and $D = 1, 2 \bmod 3$;

The algorithm for computing the polynomial does not use the floating point approach, which would evaluate a precise modular function in a precise complex argument. Instead, it relies on a faster Chinese remainder based approach modulo small primes, in which the class invariant is only defined algebraically by the modular polynomial relating the modular function to j. So in fact, any of the several roots of the modular polynomial may actually be the class invariant, and more precise assertions cannot be made.

For instance, while `polclass(D)` returns the minimal polynomial of $j(\tau)$ with τ (any) quadratic integer for the discriminant D, the polynomial returned by `polclass(D, 5)` can be the minimal polynomial of any of $\gamma_2(\tau)$, $\zeta_3\gamma_2(\tau)$ or $\zeta_3^2\gamma_2(\tau)$, the three roots of the modular polynomial $j = \gamma_2^3$, in which j has been specialised to $j(\tau)$.

The modular polynomial is given by $j = \frac{(f^{24}-16)^3}{f^{24}}$ for Weber's function f.

For the double eta quotients of level $N = pq$, all functions are covered such that the modular curve $X_0^+(N)$, the function field of which is generated by the functions invariant under $\Gamma^0(N)$ and the Fricke–Atkin–Lehner involution, is of genus 0 with function field generated by (a power of) the double eta quotient w. This ensures that the full Hilbert class field (and not a proper subfield) is generated by class invariants from these double eta quotients. Then the modular polynomial is of degree 2 in j, and of degree $\psi(N) = (p+1)(q+1)$ in w.

```
? polclass(-163)
%1 = x + 262537412640768000
? polclass(-51, , 'z)
%2 = z^2 + 5541101568*z + 6262062317568
? polclass(-151,1)
x^7 - x^6 + x^5 + 3*x^3 - x^2 + 3*x + 1
```

The library syntax is `GEN polclass(GEN D, long inv, long x = -1)` where `x` is a variable number.

3.10.12 polcoeff$(x, n, \{v\})$. Coefficient of degree n of the polynomial x, with respect to the main variable if v is omitted, with respect to v otherwise. If n is greater than the degree, the result is zero.

Naturally applies to scalars (polynomial of degree 0), as well as to rational functions whose denominator is a monomial. It also applies to power series: if n is less than the valuation, the result is zero. If it is greater than the largest significant degree, then an error message is issued.

For greater flexibility, x can be a vector or matrix type and the function then returns `component(x,n)`.

The library syntax is `GEN polcoeff0(GEN x, long n, long v = -1)` where `v` is a variable number.

3.10.13 polcyclo$(n, \{a =' x\})$. n-th cyclotomic polynomial, evaluated at a (`'x` by default). The integer n must be positive.

Algorithm used: reduce to the case where n is squarefree; to compute the cyclotomic polynomial, use $\Phi_{np}(x) = \Phi_n(x^p)/\Phi(x)$; to compute it evaluated, use $\Phi_n(x) = \prod_{d|n}(x^d - 1)^{\mu(n/d)}$. In the evaluated case, the algorithm assumes that $a^d - 1$ is either 0 or invertible, for all $d \mid n$. If this is not the case (the base ring has zero divisors), use `subst(polcyclo(n),x,a)`.

The library syntax is `GEN polcyclo_eval(long n, GEN a = NULL)`. The variant `GEN polcyclo(long n, long v)` returns the n-th cyclotomic polynomial in variable v.

3.10.14 polcyclofactors(f). Returns a vector of polynomials, whose product is the product of distinct cyclotomic polynomials dividing f.

```
? f = x^10+5*x^8-x^7+8*x^6-4*x^5+8*x^4-3*x^3+7*x^2+3;
? v = polcyclofactors(f)
%2 = [x^2 + 1, x^2 + x + 1, x^4 - x^3 + x^2 - x + 1]
? apply(poliscycloprod, v)
%3 = [1, 1, 1]
? apply(poliscyclo, v)
%4 = [4, 3, 10]
```

In general, the polynomials are products of cyclotomic polynomials and not themselves irreducible:

```
? g = x^8+2*x^7+6*x^6+9*x^5+12*x^4+11*x^3+10*x^2+6*x+3;
? polcyclofactors(g)
%2 = [x^6 + 2*x^5 + 3*x^4 + 3*x^3 + 3*x^2 + 2*x + 1]
? factor(%[1])
%3 =
[            x^2 + x + 1  1]

[x^4 + x^3 + x^2 + x + 1  1]
```

The library syntax is `GEN polcyclofactors(GEN f)`.

3.10.15 poldegree$(x, \{v\})$. Degree of the polynomial x in the main variable if v is omitted, in the variable v otherwise.

The degree of 0 is `-oo`. The degree of a non-zero scalar is 0. Finally, when x is a non-zero polynomial or rational function, returns the ordinary degree of x. Raise an error otherwise.

The library syntax is `GEN gppoldegree(GEN x, long v = -1)` where `v` is a variable number. Also available is `long poldegree(GEN x, long v)`, which returns `-LONG_MAX` if $x = 0$ and the degree as a `long` integer.

3.10.16 poldisc$(pol, \{v\})$. Discriminant of the polynomial *pol* in the main variable if v is omitted, in v otherwise. Uses a modular algorithm over $\mathbf{Z}$ or $\mathbf{Q}$, and the subresultant algorithm otherwise.

```
? T = x^4 + 2*x+1;
? poldisc(T)
%2 = -176
? poldisc(T^2)
%3 = 0
```

For convenience, the function also applies to types `t_QUAD` and `t_QFI`/`t_QFR`:

```
? z = 3*quadgen(8) + 4;
? poldisc(z)
%2 = 8
? q = Qfb(1,2,3);
? poldisc(q)
%4 = -8
```

The library syntax is `GEN poldisc0(GEN pol, long v = -1)` where `v` is a variable number.

3.10.17 poldiscreduced(f). Reduced discriminant vector of the (integral, monic) polynomial f. This is the vector of elementary divisors of $\mathbf{Z}[\alpha]/f'(\alpha)\mathbf{Z}[\alpha]$, where α is a root of the polynomial f. The components of the result are all positive, and their product is equal to the absolute value of the discriminant of f.

The library syntax is `GEN reduceddiscsmith(GEN f)`.

3.10.18 polgraeffe(f). Returns the Graeffe transform g of f, such that $g(x^2) = f(x)f(-x)$.

The library syntax is `GEN polgraeffe(GEN f)`.

3.10.19 polhensellift(A, B, p, e). Given a prime p, an integral polynomial A whose leading coefficient is a p-unit, a vector B of integral polynomials that are monic and pairwise relatively prime modulo p, and whose product is congruent to $A/\mathrm{lc}(A)$ modulo p, lift the elements of B to polynomials whose product is congruent to A modulo p^e.

More generally, if T is an integral polynomial irreducible mod p, and B is a factorization of A over the finite field $\mathbf{F}_p[t]/(T)$, you can lift it to $\mathbf{Z}_p[t]/(T, p^e)$ by replacing the p argument with $[p, T]$:

```
? { T = t^3 - 2; p = 7; A = x^2 + t + 1;
    B = [x + (3*t^2 + t + 1), x + (4*t^2 + 6*t + 6)];
    r = polhensellift(A, B, [p, T], 6) }
%1 = [x + (20191*t^2 + 50604*t + 75783), x + (97458*t^2 + 67045*t + 41866)]
? liftall( r[1] * r[2] * Mod(Mod(1,p^6),T) )
%2 = x^2 + (t + 1)
```

The library syntax is `GEN polhensellift(GEN A, GEN B, GEN p, long e)`.

3.10.20 polhermite$(n, \{a =' x\})$. n^{th} Hermite polynomial H_n evaluated at a (`'x` by default), i.e.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The library syntax is `GEN polhermite_eval(long n, GEN a = NULL)`. The variant `GEN polhermite(long n, long v)` returns the n-th Hermite polynomial in variable v.

3.10.21 polinterpolate$(X, \{Y\}, \{t =' x\}, \{\&e\})$. Given the data vectors X and Y of the same length n (X containing the x-coordinates, and Y the corresponding y-coordinates), this function finds the interpolating polynomial P of minimal degree passing through these points and evaluates it at t. If Y is omitted, the polynomial P interpolates the $(i, X[i])$. If present, e will contain an error estimate on the returned value.

The library syntax is `GEN polint(GEN X, GEN Y = NULL, GEN t = NULL, GEN *e = NULL)`.

3.10.22 poliscyclo(f). Returns 0 if f is not a cyclotomic polynomial, and $n > 0$ if $f = \Phi_n$, the n-th cyclotomic polynomial.

```
? poliscyclo(x^4-x^2+1)
%1 = 12
? polcyclo(12)
%2 = x^4 - x^2 + 1
? poliscyclo(x^4-x^2-1)
%3 = 0
```

The library syntax is `long poliscyclo(GEN f)`.

3.10.23 poliscycloprod(f). Returns 1 if f is a product of cyclotomic polynomial, and 0 otherwise.

```
? f = x^6+x^5-x^3+x+1;
? poliscycloprod(f)
%2 = 1
? factor(f)
%3 =
[  x^2 + x + 1 1]

[x^4 - x^2 + 1 1]
? [ poliscyclo(T) | T <- %[,1] ]
%4 = [3, 12]
? polcyclo(3) * polcyclo(12)
%5 = x^6 + x^5 - x^3 + x + 1
```

The library syntax is `long poliscycloprod(GEN f)`.

3.10.24 polisirreducible(pol). *pol* being a polynomial (univariate in the present version 2.9.2), returns 1 if *pol* is non-constant and irreducible, 0 otherwise. Irreducibility is checked over the smallest base field over which *pol* seems to be defined.

The library syntax is `long isirreducible(GEN pol)`.

3.10.25 pollead($x, \{v\}$). Leading coefficient of the polynomial or power series x. This is computed with respect to the main variable of x if v is omitted, with respect to the variable v otherwise.

The library syntax is `GEN pollead(GEN x, long v = -1)` where `v` is a variable number.

3.10.26 pollegendre($n, \{a =' x\}$). n^{th} Legendre polynomial evaluated at a (`'x` by default).

The library syntax is `GEN pollegendre_eval(long n, GEN a = NULL)`. To obtain the n-th Legendre polynomial in variable v, use `GEN pollegendre(long n, long v)`.

3.10.27 polmodular($L, \{inv = 0\}, \{x =' x\}, \{y =' y\}, \{derivs = 0\}$). Return the modular polynomial of prime level L in variables x and y for the modular function specified by `inv`. If `inv` is 0 (the default), use the modular j function, if `inv` is 1 use the Weber-f function, and if `inv` is 5 use $\gamma_2 = \sqrt{[3]j}$. See `polclass` for the full list of invariants. If x is given as `Mod(j, p)` or an element j of a finite field (as a `t_FFELT`), then return the modular polynomial of level L evaluated at j. If j is from a finite field and `derivs` is non-zero, then return a triple where the last two elements are the first and second derivatives of the modular polynomial evaluated at j.

```
? polmodular(3)
%1 = x^4 + (-y^3 + 2232*y^2 - 1069956*y + 36864000)*x^3 + ...
? polmodular(7, 1, , 'J)
%2 = x^8 - J^7*x^7 + 7*J^4*x^4 - 8*J*x + J^8
? polmodular(7, 5, 7*ffgen(19)^0, 'j)
%3 = j^8 + 4*j^7 + 4*j^6 + 8*j^5 + j^4 + 12*j^2 + 18*j + 18
? polmodular(7, 5, Mod(7,19), 'j)
%4 = Mod(1, 19)*j^8 + Mod(4, 19)*j^7 + Mod(4, 19)*j^6 + ...

? u = ffgen(5)^0; T = polmodular(3,0,,'j)*u;
? polmodular(3, 0, u,'j,1)
%6 = [j^4 + 3*j^2 + 4*j + 1, 3*j^2 + 2*j + 4, 3*j^3 + 4*j^2 + 4*j + 2]
```

```
? subst(T,x,u)
%7 = j^4 + 3*j^2 + 4*j + 1
? subst(T',x,u)
%8 = 3*j^2 + 2*j + 4
? subst(T'',x,u)
%9 = 3*j^3 + 4*j^2 + 4*j + 2
```

The library syntax is `GEN polmodular(long L, long inv, GEN x = NULL, long y = -1, long derivs)` where `y` is a variable number.

3.10.28 polrecip(pol). Reciprocal polynomial of pol, i.e. the coefficients are in reverse order. pol must be a polynomial.

The library syntax is `GEN polrecip(GEN pol)`.

3.10.29 polresultant($x, y, \{v\}, \{flag = 0\}$). Resultant of the two polynomials x and y with exact entries, with respect to the main variables of x and y if v is omitted, with respect to the variable v otherwise. The algorithm assumes the base ring is a domain. If you also need the u and v such that $x * u + y * v = \mathrm{Res}(x, y)$, use the `polresultantext` function.

If $flag = 0$ (default), uses the algorithm best suited to the inputs, either the subresultant algorithm (Lazard/Ducos variant, generic case), a modular algorithm (inputs in $\mathbf{Q}[X]$) or Sylvester's matrix (inexact inputs).

If $flag = 1$, uses the determinant of Sylvester's matrix instead; this should always be slower than the default.

The library syntax is `GEN polresultant0(GEN x, GEN y, long v = -1, long flag)` where `v` is a variable number.

3.10.30 polresultantext($A, B, \{v\}$). Finds polynomials U and V such that $A * U + B * V = R$, where R is the resultant of U and V with respect to the main variables of A and B if v is omitted, and with respect to v otherwise. Returns the row vector $[U, V, R]$. The algorithm used (subresultant) assumes that the base ring is a domain.

```
? A = x*y; B = (x+y)^2;
? [U,V,R] = polresultantext(A, B)
%2 = [-y*x - 2*y^2, y^2, y^4]
? A*U + B*V
%3 = y^4
? [U,V,R] = polresultantext(A, B, y)
%4 = [-2*x^2 - y*x, x^2, x^4]
? A*U+B*V
%5 = x^4
```

The library syntax is `GEN polresultantext0(GEN A, GEN B, long v = -1)` where `v` is a variable number. Also available is `GEN polresultantext(GEN x, GEN y)`.

3.10.31 polroots(x). Complex roots of the polynomial x, given as a column vector where each root is repeated according to its multiplicity. The precision is given as for transcendental functions: in GP it is kept in the variable `realprecision` and is transparent to the user, but it must be explicitly given as a second argument in library mode.

The algorithm used is a modification of A. Schönhage's root-finding algorithm, due to and originally implemented by X. Gourdon. Barring bugs, it is guaranteed to converge and to give the roots to the required accuracy.

The library syntax is `GEN roots(GEN x, long prec)`.

3.10.32 polrootsmod$(pol, p, \{flag = 0\})$. Row vector of roots modulo p of the polynomial *pol*. Multiple roots are *not* repeated.

```
? polrootsmod(x^2-1,2)
%1 = [Mod(1, 2)]~
```

If p is very small, you may set $flag = 1$, which uses a naive search.

The library syntax is `GEN rootmod0(GEN pol, GEN p, long flag)`.

3.10.33 polrootspadic(x, p, r). Vector of p-adic roots of the polynomial *pol*, given to p-adic precision r p is assumed to be a prime. Multiple roots are *not* repeated. Note that this is not the same as the roots in $\mathbf{Z}/p^r\mathbf{Z}$, rather it gives approximations in $\mathbf{Z}/p^r\mathbf{Z}$ of the true roots living in $\mathbf{Q}_p$.

```
? polrootspadic(x^3 - x^2 + 64, 2, 5)
%1 = [2^3 + O(2^5), 2^3 + 2^4 + O(2^5), 1 + O(2^5)]~
```

If *pol* has inexact `t_PADIC` coefficients, this is not always well-defined; in this case, the polynomial is first made integral by dividing out the p-adic content, then lifted to $\mathbf{Z}$ using `truncate` coefficientwise. Hence the roots given are approximations of the roots of an exact polynomial which is p-adically close to the input. To avoid pitfalls, we advise to only factor polynomials with eact rational coefficients.

The library syntax is `GEN rootpadic(GEN x, GEN p, long r)`.

3.10.34 polrootsreal$(T, \{ab\})$. Real roots of the polynomial T with rational coefficients, multiple roots being included according to their multiplicity. The roots are given to a relative accuracy of `realprecision`. If argument ab is present, it must be a vector $[a, b]$ with two components (of type `t_INT`, `t_FRAC` or `t_INFINITY`) and we restrict to roots belonging to that closed interval.

```
? \p9
? polrootsreal(x^2-2)
%1 = [-1.41421356, 1.41421356]~
? polrootsreal(x^2-2, [1,+oo])
%2 = [1.41421356]~
? polrootsreal(x^2-2, [2,3])
%3 = []~
? polrootsreal((x-1)*(x-2), [2,3])
%4 = [2.00000000]~
```

The algorithm used is a modification of Uspensky's method (relying on Descartes's rule of sign), following Rouillier and Zimmerman's article "Efficient isolation of a polynomial real roots" (`http://hal.inria.fr/inria-00072518/`). Barring bugs, it is guaranteed to converge and to give the roots to the required accuracy.

Remark. If the polynomial T is of the form $Q(x^h)$ for some $h \geq 2$ and ab is omitted, the routine will apply the algorithm to Q (restricting to non-negative roots when h is even), then take h-th roots. On the other hand, if you want to specify ab, you should apply the routine to Q yourself and a suitable interval $[a', b']$ using approximate h-th roots adapted to your problem: the function will not perform this change of variables if ab is present.

The library syntax is `GEN realroots(GEN T, GEN ab = NULL, long prec)`.

3.10.35 polsturm$(T, \{ab\})$. Number of real roots of the real squarefree polynomial T. If the argument ab is present, it must be a vector $[a, b]$ with two real components (of type `t_INT`, `t_REAL`, `t_FRAC` or `t_INFINITY`) and we count roots belonging to that closed interval.

If possible, you should stick to exact inputs, that is avoid `t_REAL`s in T and the bounds a, b: the result is then guaranteed and we use a fast algorithm (Uspensky's method, relying on Descartes's rule of sign, see `polrootsreal`); otherwise, we use Sturm's algorithm and the result may be wrong due to round-off errors.

```
? T = (x-1)*(x-2)*(x-3);
? polsturm(T)
%2 = 3
? polsturm(T, [-oo,2])
%3 = 2
? polsturm(T, [1/2,+oo])
%4 = 3
? polsturm(T, [1, Pi])  \\ Pi inexact: not recommended !
%5 = 3
? polsturm(T*1., [0, 4])  \\ T*1. inexact: not recommended !
%6 = 3
? polsturm(T^2, [0, 4])  \\ not squarefree
 ***   at top-level: polsturm(T^2,[0,4])
 ***                 ^-------------------
 *** polsturm: domain error in polsturm: issquarefree(pol) = 0
? polsturm((T*1.)^2, [0, 4])  \\ not squarefree AND inexact
 ***   at top-level: polsturm((T*1.)^2,[0
 ***                 ^--------------------
 *** polsturm: precision too low in polsturm.
```

In the last example, the input polynomial is not squarefree but there is no way to ascertain it from the given floating point approximation: we get a precision error in this case.

The library syntax is `long RgX_sturmpart(GEN T, GEN ab)` or `long sturm(GEN T)` (for the case `ab = NULL`). The function `long sturmpart(GEN T, GEN a, GEN b)` is obsolete and deprecated.

3.10.36 polsubcyclo$(n, d, \{v =' x\})$. Gives polynomials (in variable v) defining the sub-Abelian extensions of degree d of the cyclotomic field $\mathbf{Q}(\zeta_n)$, where $d \mid \phi(n)$.

If there is exactly one such extension the output is a polynomial, else it is a vector of polynomials, possibly empty. To get a vector in all cases, use `concat([], polsubcyclo(n,d))`.

The function `galoissubcyclo` allows to specify exactly which sub-Abelian extension should be computed.

The library syntax is `GEN polsubcyclo(long n, long d, long v = -1)` where `v` is a variable number.

3.10.37 polsylvestermatrix(x, y). Forms the Sylvester matrix corresponding to the two polynomials x and y, where the coefficients of the polynomials are put in the columns of the matrix (which is the natural direction for solving equations afterwards). The use of this matrix can be essential when dealing with polynomials with inexact entries, since polynomial Euclidean division doesn't make much sense in this case.

The library syntax is `GEN sylvestermatrix(GEN x, GEN y)`.

3.10.38 polsym(x, n). Creates the column vector of the symmetric powers of the roots of the polynomial x up to power n, using Newton's formula.

The library syntax is `GEN polsym(GEN x, long n)`.

3.10.39 poltchebi$(n, \{v =' x\})$. Deprecated alias for `polchebyshev`

The library syntax is `GEN polchebyshev1(long n,  long v = -1)` where `v` is a variable number.

3.10.40 polzagier(n, m). Creates Zagier's polynomial $P_n^{(m)}$ used in the functions `sumalt` and `sumpos` (with *flag* $= 1$), see "Convergence acceleration of alternating series", Cohen et al., *Experiment. Math.*, vol. 9, 2000, pp. 3–12.

If $m < 0$ or $m \geq n$, $P_n^{(m)} = 0$. We have $P_n := P_n^{(0)}$ is $T_n(2x-1)$, where T_n is the Legendre polynomial of the second kind. For $n > m > 0$, $P_n^{(m)}$ is the m-th difference with step 2 of the sequence $n^{m+1}P_n$; in this case, it satisfies

$$2P_n^{(m)}(sin^2 t) = \frac{d^{m+1}}{dt^{m+1}}(\sin(2t)^m \sin(2(n-m)t)).$$

The library syntax is `GEN polzag(long n, long m)`.

3.10.41 serconvol(x, y). Convolution (or Hadamard product) of the two power series x and y; in other words if $x = \sum a_k * X^k$ and $y = \sum b_k * X^k$ then `serconvol`$(x, y) = \sum a_k * b_k * X^k$.

The library syntax is `GEN convol(GEN x, GEN y)`.

3.10.42 serlaplace(x). x must be a power series with non-negative exponents or a polynomial. If $x = \sum (a_k/k!) * X^k$ then the result is $\sum a_k * X^k$.

The library syntax is `GEN laplace(GEN x)`.

3.10.43 serreverse(s). Reverse power series of s, i.e. the series t such that $t(s) = x$; s must be a power series whose valuation is exactly equal to one.

```
? \ps 8
? t = serreverse(tan(x))
%2 = x - 1/3*x^3 + 1/5*x^5 - 1/7*x^7 + O(x^8)
? tan(t)
%3 = x + O(x^8)
```

The library syntax is `GEN serreverse(GEN s)`.

3.10.44 subst(x, y, z). Replace the simple variable y by the argument z in the "polynomial" expression x. Every type is allowed for x, but if it is not a genuine polynomial (or power series, or rational function), the substitution will be done as if the scalar components were polynomials of degree zero. In particular, beware that:

```
? subst(1, x, [1,2; 3,4])
%1 =
[1 0]

[0 1]

? subst(1, x, Mat([0,1]))
  ***   at top-level: subst(1,x,Mat([0,1])
  ***                 ^--------------------
  *** subst: forbidden substitution by a non square matrix.
```

If x is a power series, z must be either a polynomial, a power series, or a rational function. Finally, if x is a vector, matrix or list, the substitution is applied to each individual entry.

Use the function `substvec` to replace several variables at once, or the function `substpol` to replace a polynomial expression.

The library syntax is `GEN gsubst(GEN x, long y, GEN z)` where `y` is a variable number.

3.10.45 substpol(x, y, z). Replace the "variable" y by the argument z in the "polynomial" expression x. Every type is allowed for x, but the same behavior as `subst` above apply.

The difference with `subst` is that y is allowed to be any polynomial here. The substitution is done moding out all components of x (recursively) by $y - t$, where t is a new free variable of lowest priority. Then substituting t by z in the resulting expression. For instance

```
? substpol(x^4 + x^2 + 1, x^2, y)
%1 = y^2 + y + 1
? substpol(x^4 + x^2 + 1, x^3, y)
%2 = x^2 + y*x + 1
? substpol(x^4 + x^2 + 1, (x+1)^2, y)
%3 = (-4*y - 6)*x + (y^2 + 3*y - 3)
```

The library syntax is `GEN gsubstpol(GEN x, GEN y, GEN z)`. Further, `GEN gdeflate(GEN T, long v, long d)` attempts to write $T(x)$ in the form $t(x^d)$, where $x =$`pol_x`(v), and returns `NULL` if the substitution fails (for instance in the example `%2` above).

3.10.46 substvec(x, v, w). v being a vector of monomials of degree 1 (variables), w a vector of expressions of the same length, replace in the expression x all occurrences of v_i by w_i. The substitutions are done simultaneously; more precisely, the v_i are first replaced by new variables in x, then these are replaced by the w_i:

```
? substvec([x,y], [x,y], [y,x])
%1 = [y, x]
? substvec([x,y], [x,y], [y,x+y])
%2 = [y, x + y]     \\ not [y, 2*y]
```

The library syntax is `GEN gsubstvec(GEN x, GEN v, GEN w)`.

3.10.47 sumformal$(f, \{v\})$. formal sum of the polynomial expression f with respect to the main variable if v is omitted, with respect to the variable v otherwise; it is assumed that the base ring has characteristic zero. In other words, considering f as a polynomial function in the variable v, returns F, a polynomial in v vanishing at 0, such that $F(b) - F(a) = sum_{v=a+1}^{b} f(v)$:

```
? sumformal(n)  \\ 1 + ... + n
%1 = 1/2*n^2 + 1/2*n
? f(n) = n^3+n^2+1;
? F = sumformal(f(n))  \\ f(1) + ... + f(n)
%3 = 1/4*n^4 + 5/6*n^3 + 3/4*n^2 + 7/6*n
? sum(n = 1, 2000, f(n)) == subst(F, n, 2000)
%4 = 1
? sum(n = 1001, 2000, f(n)) == subst(F, n, 2000) - subst(F, n, 1000)
%5 = 1
? sumformal(x^2 + x*y + y^2, y)
%6 = y*x^2 + (1/2*y^2 + 1/2*y)*x + (1/3*y^3 + 1/2*y^2 + 1/6*y)
? x^2 * y + x * sumformal(y) + sumformal(y^2) == %
%7 = 1
```

The library syntax is `GEN sumformal(GEN f, long v = -1)` where `v` is a variable number.

3.10.48 taylor$(x, t, \{d = seriesprecision\})$. Taylor expansion around 0 of x with respect to the simple variable t. x can be of any reasonable type, for example a rational function. Contrary to `Ser`, which takes the valuation into account, this function adds $O(t^d)$ to all components of x.

```
? taylor(x/(1+y), y, 5)
%1 = (y^4 - y^3 + y^2 - y + 1)*x + O(y^5)
? Ser(x/(1+y), y, 5)
 ***   at top-level: Ser(x/(1+y),y,5)
 ***                 ^----------------
 *** Ser: main variable must have higher priority in gtoser.
```

The library syntax is `GEN tayl(GEN x, long t, long precdl)` where `t` is a variable number.

3.10.49 thue$(tnf, a, \{sol\})$. Returns all solutions of the equation $P(x, y) = a$ in integers x and y, where *tnf* was created with `thueinit`(P). If present, *sol* must contain the solutions of $\mathrm{Norm}(x) = a$ modulo units of positive norm in the number field defined by P (as computed by `bnfisintnorm`). If there are infinitely many solutions, an error is issued.

It is allowed to input directly the polynomial P instead of a *tnf*, in which case, the function first performs `thueinit(P,0)`. This is very wasteful if more than one value of a is required.

If *tnf* was computed without assuming GRH (flag 1 in `thueinit`), then the result is unconditional. Otherwise, it depends in principle of the truth of the GRH, but may still be unconditionally correct in some favorable cases. The result is conditional on the GRH if $a \neq \pm 1$ and, P has a single irreducible rational factor, whose attached tentative class number h and regulator R (as computed assuming the GRH) satisfy

- $h > 1$,
- $R/0.2 > 1.5$.

Here's how to solve the Thue equation $x^{13} - 5y^{13} = -4$:

```
? tnf = thueinit(x^13 - 5);
? thue(tnf, -4)
%1 = [[1, 1]]
```

In this case, one checks that `bnfinit(x^13 -5).no` is 1. Hence, the only solution is $(x, y) = (1, 1)$, and the result is unconditional. On the other hand:

```
? P = x^3-2*x^2+3*x-17; tnf = thueinit(P);
? thue(tnf, -15)
%2 = [[1, 1]]  \\ a priori conditional on the GRH.
? K = bnfinit(P); K.no
%3 = 3
? K.reg
%4 = 2.8682185139262873674706034475498755834
```

This time the result is conditional. All results computed using this particular *tnf* are likewise conditional, *except* for a right-hand side of ± 1. The above result is in fact correct, so we did not just disprove the GRH:

```
? tnf = thueinit(x^3-2*x^2+3*x-17, 1 /*unconditional*/);
? thue(tnf, -15)
%4 = [[1, 1]]
```

Note that reducible or non-monic polynomials are allowed:

```
? tnf = thueinit((2*x+1)^5 * (4*x^3-2*x^2+3*x-17), 1);
? thue(tnf, 128)
%2 = [[-1, 0], [1, 0]]
```

Reducible polynomials are in fact much easier to handle.

The library syntax is `GEN thue(GEN tnf, GEN a, GEN sol = NULL)`.

3.10.50 thueinit$(P, \{flag = 0\})$. Initializes the *tnf* corresponding to P, a non-constant univariate polynomial with integer coefficients. The result is meant to be used in conjunction with `thue` to solve Thue equations $P(X/Y)Y^{\deg P} = a$, where a is an integer. Accordingly, P must either have at least two distinct irreducible factors over $\mathbf{Q}$, or have one irreducible factor T with degree > 2 or two conjugate complex roots: under these (necessary and sufficient) conditions, the equation has finitely many integer solutions.

```
? S = thueinit(t^2+1);
? thue(S, 5)
%2 = [[-2, -1], [-2, 1], [-1, -2], [-1, 2], [1, -2], [1, 2], [2, -1], [2, 1]]
? S = thueinit(t+1);
 ***   at top-level: thueinit(t+1)
 ***                 ^-------------
 *** thueinit: domain error in thueinit: P = t + 1
```

The hardest case is when $\deg P > 2$ and P is irreducible with at least one real root. The routine then uses Bilu-Hanrot's algorithm.

If *flag* is non-zero, certify results unconditionally. Otherwise, assume GRH, this being much faster of course. In the latter case, the result may still be unconditionally correct, see `thue`. For instance in most cases where P is reducible (not a pure power of an irreducible), *or* conditional computed class groups are trivial *or* the right hand side is ± 1, then results are unconditional.

Note. The general philosophy is to disprove the existence of large solutions then to enumerate bounded solutions naively. The implementation will overflow when there exist huge solutions and the equation has degree > 2 (the quadratic imaginary case is special, since we can use `bnfisintnorm`):

```
? thue(t^3+2, 10^30)
 ***   at top-level: L=thue(t^3+2,10^30)
 ***                   ^-----------------
 *** thue: overflow in thue (SmallSols): y <= 8066520378961903602892 8.
? thue(x^2+2, 10^30)  \\ quadratic case much easier
%1 = [[-1000000000000000, 0], [1000000000000000, 0]]
```

Note. It is sometimes possible to circumvent the above, and in any case obtain an important speed-up, if you can write $P = Q(x^d)$ for some $d > 1$ and Q still satisfying the `thueinit` hypotheses. You can then solve the equation attached to Q then eliminate all solutions (x, y) such that either x or y is not a d-th power.

```
? thue(x^4+1, 10^40); \\ stopped after 10 hours
? filter(L,d) =
    my(x,y); [[x,y] | v<-L, ispower(v[1],d,&x)&&ispower(v[2],d,&y)];
? L = thue(x^2+1, 10^40);
? filter(L, 2)
%4 = [[0, 10000000000], [10000000000, 0]]
```

The last 2 commands use less than 20ms.

The library syntax is `GEN thueinit(GEN P, long flag, long prec)`.

3.11 Vectors, matrices, linear algebra and sets.

Note that most linear algebra functions operating on subspaces defined by generating sets (such as `mathnf`, `qflll`, etc.) take matrices as arguments. As usual, the generating vectors are taken to be the *columns* of the given matrix.

Since PARI does not have a strong typing system, scalars live in unspecified commutative base rings. It is very difficult to write robust linear algebra routines in such a general setting. We thus assume that the base ring is a domain and work over its field of fractions. If the base ring is *not* a domain, one gets an error as soon as a non-zero pivot turns out to be non-invertible. Some functions, e.g. `mathnf` or `mathnfmod`, specifically assume that the base ring is $\mathbf{Z}$.

3.11.1 algdep$(z, k, \{flag = 0\})$. z being real/complex, or p-adic, finds a polynomial (in the variable `'x`) of degree at most k, with integer coefficients, having z as approximate root. Note that the polynomial which is obtained is not necessarily the "correct" one. In fact it is not even guaranteed to be irreducible. One can check the closeness either by a polynomial evaluation (use `subst`), or by computing the roots of the polynomial given by `algdep` (use `polroots` or `polrootspadic`).

Internally, `lindep`$([1, z, \ldots, z^k], flag)$ is used. A non-zero value of *flag* may improve on the default behavior if the input number is known to a *huge* accuracy, and you suspect the last bits are incorrect: if *flag* > 0 the computation is done with an accuracy of *flag* decimal digits; to get meaningful results, the parameter *flag* should be smaller than the number of correct decimal digits in the input. But default values are usually sufficient, so try without *flag* first:

```
? \p200
? z = 2^(1/6)+3^(1/5);
? algdep(z, 30);      \\ right in 280ms
? algdep(z, 30, 100); \\ wrong in 169ms
? algdep(z, 30, 170); \\ right in 288ms
? algdep(z, 30, 200); \\ wrong in 320ms
? \p250
? z = 2^(1/6)+3^(1/5); \\ recompute to new, higher, accuracy !
? algdep(z, 30);      \\ right in 329ms
? algdep(z, 30, 200); \\ right in 324ms
? \p500
? algdep(2^(1/6)+3^(1/5), 30); \\ right in 677ms
? \p1000
? algdep(2^(1/6)+3^(1/5), 30); \\ right in 1.5s
```

The changes in `realprecision` only affect the quality of the initial approximation to $2^{1/6} + 3^{1/5}$, `algdep` itself uses exact operations. The size of its operands depend on the accuracy of the input of course: more accurate input means slower operations.

Proceeding by increments of 5 digits of accuracy, `algdep` with default flag produces its first correct result at 195 digits, and from then on a steady stream of correct results:

```
\\ assume T contains the correct result, for comparison
forstep(d=100, 250, 5, localprec(d);\
  print(d, " ", algdep(2^(1/6)+3^(1/5),30) == T))
```

The above example is the test case studied in a 2000 paper by Borwein and Lisonek: Applications of integer relation algorithms, *Discrete Math.*, **217**, p. 65–82. The version of PARI tested there was 1.39, which succeeded reliably from precision 265 on, in about 200 as much time as the current version.

The library syntax is `GEN algdep0(GEN z, long k, long flag)`. Also available is `GEN algdep(GEN z, long k)` (*flag* $= 0$).

3.11.2 charpoly$(A, \{v =' x\}, \{flag = 5\})$. characteristic polynomial of A with respect to the variable v, i.e. determinant of $v * I - A$ if A is a square matrix.

```
? charpoly([1,2;3,4]);
%1 = x^2 - 5*x - 2
? charpoly([1,2;3,4],, 't)
%2 = t^2 - 5*t - 2
```

If A is not a square matrix, the function returns the characteristic polynomial of the map "multiplication by A" if A is a scalar:

```
? charpoly(Mod(x+2, x^3-2))
%1 = x^3 - 6*x^2 + 12*x - 10
? charpoly(I)
%2 = x^2 + 1
? charpoly(quadgen(5))
%3 = x^2 - x - 1
? charpoly(ffgen(ffinit(2,4)))
%4 = Mod(1, 2)*x^4 + Mod(1, 2)*x^3 + Mod(1, 2)*x^2 + Mod(1, 2)*x + Mod(1, 2)
```

The value of *flag* is only significant for matrices, and we advise to stick to the default value. Let n be the dimension of A.

If *flag* $= 0$, same method (Le Verrier's) as for computing the adjoint matrix, i.e. using the traces of the powers of A. Assumes that $n!$ is invertible; uses $O(n^4)$ scalar operations.

If *flag* $= 1$, uses Lagrange interpolation which is usually the slowest method. Assumes that $n!$ is invertible; uses $O(n^4)$ scalar operations.

If *flag* $= 2$, uses the Hessenberg form. Assumes that the base ring is a field. Uses $O(n^3)$ scalar operations, but suffers from coefficient explosion unless the base field is finite or $\mathbf{R}$.

If *flag* $= 3$, uses Berkowitz's division free algorithm, valid over any ring (commutative, with unit). Uses $O(n^4)$ scalar operations.

If *flag* $= 4$, x must be integral. Uses a modular algorithm: Hessenberg form for various small primes, then Chinese remainders.

If *flag* $= 5$ (default), uses the "best" method given x. This means we use Berkowitz unless the base ring is $\mathbf{Z}$ (use *flag* $= 4$) or a field where coefficient explosion does not occur, e.g. a finite field or the reals (use *flag* $= 2$).

The library syntax is `GEN charpoly0(GEN A, long v = -1, long flag)` where `v` is a variable number. Also available are `GEN charpoly(GEN x, long v)` (*flag* $= 5$), `GEN caract(GEN A, long v)` (*flag* $= 1$), `GEN carhess(GEN A, long v)` (*flag* $= 2$), `GEN carberkowitz(GEN A, long v)` (*flag* $= 3$) and `GEN caradj(GEN A, long v, GEN *pt)`. In this last case, if *pt* is not `NULL`, `*pt` receives the address of the adjoint matrix of A (see `matadjoint`), so both can be obtained at once.

3.11.3 concat$(x, \{y\})$. Concatenation of x and y. If x or y is not a vector or matrix, it is considered as a one-dimensional vector. All types are allowed for x and y, but the sizes must be compatible. Note that matrices are concatenated horizontally, i.e. the number of rows stays the same. Using transpositions, one can concatenate them vertically, but it is often simpler to use `matconcat`.

```
? x = matid(2); y = 2*matid(2);
? concat(x,y)
%2 =
[1 0 2 0]

[0 1 0 2]
? concat(x~,y~)~
%3 =
[1 0]

[0 1]

[2 0]

[0 2]
? matconcat([x;y])
%4 =
[1 0]

[0 1]

[2 0]

[0 2]
```

To concatenate vectors sideways (i.e. to obtain a two-row or two-column matrix), use `Mat` instead, or `matconcat`:

```
? x = [1,2];
? y = [3,4];
? concat(x,y)
%3 = [1, 2, 3, 4]

? Mat([x,y]~)
%4 =
[1 2]

[3 4]
? matconcat([x;y])
%5 =
[1 2]

[3 4]
```

Concatenating a row vector to a matrix having the same number of columns will add the row to the matrix (top row if the vector is x, i.e. comes first, and bottom row otherwise).

The empty matrix `[;]` is considered to have a number of rows compatible with any operation, in particular concatenation. (Note that this is *not* the case for empty vectors `[ ]` or `[ ]~`.)

If y is omitted, x has to be a row vector or a list, in which case its elements are concatenated, from left to right, using the above rules.

```
? concat([1,2], [3,4])
%1 = [1, 2, 3, 4]
? a = [[1,2]~, [3,4]~]; concat(a)
%2 =
[1 3]

[2 4]

? concat([1,2; 3,4], [5,6]~)
%3 =
[1 2 5]

[3 4 6]
? concat([%, [7,8]~, [1,2,3,4]])
%5 =
[1 2 5 7]

[3 4 6 8]

[1 2 3 4]
```

The library syntax is `GEN gconcat(GEN x, GEN y = NULL)`. `GEN gconcat1(GEN x)` is a shortcut for `gconcat(x,NULL)`.

3.11.4 forqfvec$(v, q, b, expr)$. q being a square and symmetric integral matrix representing a positive definite quadratic form, evaluate `expr` for all vector v such that $q(v) \leq b$. The formal variable v runs through all such vectors in turn.

```
? forqfvec(v, [3,2;2,3], 3, print(v))
[0, 1]~
[1, 0]~
[-1, 1]~
```

The library syntax is `void forqfvec0(GEN v, GEN q = NULL, GEN b)`. The following function is also available: `void forqfvec(void *E, long (*fun)(void *, GEN, GEN, double), GEN q, GEN b)`: Evaluate `fun(E,w,v,m)` on all v such that $q(v) < b$, where v is a `t_VECSMALL` and $m = q(v)$ is a C double. The function `fun` must return 0, unless `forqfvec` should stop, in which case, it should return 1.

3.11.5 lindep$(v, \{flag = 0\})$. finds a small non-trivial integral linear combination between components of v. If none can be found return an empty vector.

If v is a vector with real/complex entries we use a floating point (variable precision) LLL algorithm. If $flag = 0$ the accuracy is chosen internally using a crude heuristic. If $flag > 0$ the computation is done with an accuracy of $flag$ decimal digits. To get meaningful results in the latter case, the parameter $flag$ should be smaller than the number of correct decimal digits in the input.

```
? lindep([sqrt(2), sqrt(3), sqrt(2)+sqrt(3)])
%1 = [-1, -1, 1]~
```

If v is p-adic, $flag$ is ignored and the algorithm LLL-reduces a suitable (dual) lattice.

```
? lindep([1, 2 + 3 + 3^2 + 3^3 + 3^4 + O(3^5)])
%2 = [1, -2]~
```

If v is a matrix (or a vector of column vectors, or a vector of row vectors), $flag$ is ignored and the function returns a non trivial kernel vector if one exists, else an empty vector.

```
? lindep([1,2,3;4,5,6;7,8,9])
%3 = [1, -2, 1]~
? lindep([[1,0], [2,0]])
%4 = [2, -1]~
? lindep([[1,0], [0,1]])
%5 = []~
```

If v contains polynomials or power series over some base field, finds a linear relation with coefficients in the field.

```
? lindep([x*y, x^2 + y, x^2*y + x*y^2, 1])
%4 = [y, y, -1, -y^2]~
```

For better control, it is preferable to use `t_POL` rather than `t_SER` in the input, otherwise one gets a linear combination which is t-adically small, but not necessarily 0. Indeed, power series are first converted to the minimal absolute accuracy occurring among the entries of v (which can cause some coefficients to be ignored), then truncated to polynomials:

```
? v = [t^2+O(t^4), 1+O(t^2)]; L=lindep(v)
%1 = [1, 0]~
? v*L
```

```
%2 = t^2+O(t^4)  \\ small but not 0
```

The library syntax is `GEN lindep0(GEN v, long flag)`. Also available are `GEN lindep(GEN v)` (real/complex entries, *flag* = 0), `GEN lindep2(GEN v, long flag)` (real/complex entries) `GEN padic_lindep(GEN v)` (p-adic entries) and `GEN Xadic_lindep(GEN v)` (polynomial entries). Finally `GEN deplin(GEN v)` returns a non-zero kernel vector for a `t_MAT` input.

3.11.6 matadjoint$(M, \{flag = 0\})$. adjoint matrix of M, i.e. a matrix N of cofactors of M, satisfying $M * N = \det(M) * \mathrm{Id}$. M must be a (non-necessarily invertible) square matrix of dimension n. If *flag* is 0 or omitted, we try to use Leverrier-Faddeev's algorithm, which assumes that $n!$ invertible. If it fails or *flag* = 1, compute $T =$ `charpoly`(M) independently first and return $(-1)^{n-1}(T(x) - T(0))/x$ evaluated at M.

```
? a = [1,2,3;3,4,5;6,7,8] * Mod(1,4);
%2 =
[Mod(1, 4) Mod(2, 4) Mod(3, 4)]

[Mod(3, 4) Mod(0, 4) Mod(1, 4)]

[Mod(2, 4) Mod(3, 4) Mod(0, 4)]
```

Both algorithms use $O(n^4)$ operations in the base ring, and are usually slower than computing the characteristic polynomial or the inverse of M directly.

The library syntax is `GEN matadjoint0(GEN M, long flag)`. Also available are `GEN adj(GEN x)` (*flag*=0) and `GEN adjsafe(GEN x)` (*flag*=1).

3.11.7 matcompanion(x). The left companion matrix to the non-zero polynomial x.

The library syntax is `GEN matcompanion(GEN x)`.

3.11.8 matconcat(v). Returns a `t_MAT` built from the entries of v, which may be a `t_VEC` (concatenate horizontally), a `t_COL` (concatenate vertically), or a `t_MAT` (concatenate vertically each column, and concatenate vertically the resulting matrices). The entries of v are always considered as matrices: they can themselves be `t_VEC` (seen as a row matrix), a `t_COL` seen as a column matrix), a `t_MAT`, or a scalar (seen as an 1×1 matrix).

```
? A=[1,2;3,4]; B=[5,6]~; C=[7,8]; D=9;
? matconcat([A, B]) \\ horizontal
%1 =
[1 2 5]

[3 4 6]
? matconcat([A, C]~) \\ vertical
%2 =
[1 2]

[3 4]

[7 8]
? matconcat([A, B; C, D]) \\ block matrix
%3 =
[1 2 5]

[3 4 6]

[7 8 9]
```

If the dimensions of the entries to concatenate do not match up, the above rules are extended as follows:

• each entry $v_{i,j}$ of v has a natural length and height: 1×1 for a scalar, $1 \times n$ for a `t_VEC` of length n, $n \times 1$ for a `t_COL`, $m \times n$ for an $m \times n$ `t_MAT`

• let H_i be the maximum over j of the lengths of the $v_{i,j}$, let L_j be the maximum over i of the heights of the $v_{i,j}$. The dimensions of the (i,j)-th block in the concatenated matrix are $H_i \times L_j$.

• a scalar $s = v_{i,j}$ is considered as s times an identity matrix of the block dimension $\min(H_i, L_j)$

• blocks are extended by 0 columns on the right and 0 rows at the bottom, as needed.

```
? matconcat([1, [2,3]~, [4,5,6]~]) \\ horizontal
%4 =
[1 2 4]

[0 3 5]

[0 0 6]
? matconcat([1, [2,3], [4,5,6]]~) \\ vertical
%5 =
[1 0 0]

[2 3 0]

[4 5 6]
? matconcat([B, C; A, D]) \\ block matrix
%6 =
[5 0 7 8]

[6 0 0 0]

[1 2 9 0]

[3 4 0 9]
? U=[1,2;3,4]; V=[1,2,3;4,5,6;7,8,9];
? matconcat(matdiagonal([U, V])) \\ block diagonal
%7 =
[1 2 0 0 0]

[3 4 0 0 0]

[0 0 1 2 3]

[0 0 4 5 6]

[0 0 7 8 9]
```

The library syntax is `GEN matconcat(GEN v)`.

3.11.9 matdet$(x, \{flag = 0\})$. Determinant of the square matrix x.

If *flag* $= 0$, uses an appropriate algorithm depending on the coefficients:

- integer entries: modular method due to Dixon, Pernet and Stein.
- real or p-adic entries: classical Gaussian elimination using maximal pivot.
- intmod entries: classical Gaussian elimination using first non-zero pivot.
- other cases: Gauss-Bareiss.

If *flag* $= 1$, uses classical Gaussian elimination with appropriate pivoting strategy (maximal pivot for real or p-adic coefficients). This is usually worse than the default.

The library syntax is `GEN det0(GEN x, long flag)`. Also available are `GEN det(GEN x)` (*flag* $= 0$), `GEN det2(GEN x)` (*flag* $= 1$) and `GEN ZM_det(GEN x)` for integer entries.

3.11.10 matdetint(B). Let B be an $m \times n$ matrix with integer coefficients. The *determinant* D of the lattice generated by the columns of B is the square root of $\det(B^T B)$ if B has maximal rank m, and 0 otherwise.

This function uses the Gauss-Bareiss algorithm to compute a positive *multiple* of D. When B is square, the function actually returns $D = |\det B|$.

This function is useful in conjunction with `mathnfmod`, which needs to know such a multiple. If the rank is maximal and the matrix non-square, you can obtain D exactly using

```
matdet( mathnfmod(B, matdetint(B)) )
```

Note that as soon as one of the dimensions gets large (m or n is larger than 20, say), it will often be much faster to use `mathnf(B, 1)` or `mathnf(B, 4)` directly.

The library syntax is `GEN detint(GEN B)`.

3.11.11 matdiagonal(x). x being a vector, creates the diagonal matrix whose diagonal entries are those of x.

```
? matdiagonal([1,2,3]);
%1 =
[1 0 0]

[0 2 0]

[0 0 3]
```

Block diagonal matrices are easily created using `matconcat`:

```
? U=[1,2;3,4]; V=[1,2,3;4,5,6;7,8,9];
? matconcat(matdiagonal([U, V]))
%1 =
[1 2 0 0 0]

[3 4 0 0 0]

[0 0 1 2 3]

[0 0 4 5 6]

[0 0 7 8 9]
```

The library syntax is `GEN diagonal(GEN x)`.

3.11.12 mateigen$(x, \{flag = 0\})$. Returns the (complex) eigenvectors of x as columns of a matrix. If $flag = 1$, return $[L, H]$, where L contains the eigenvalues and H the corresponding eigenvectors; multiple eigenvalues are repeated according to the eigenspace dimension (which may be less than the eigenvalue multiplicity in the characteristic polynomial).

This function first computes the characteristic polynomial of x and approximates its complex roots (λ_i), then tries to compute the eigenspaces as kernels of the $x - \lambda_i$. This algorithm is ill-conditioned and is likely to miss kernel vectors if some roots of the characteristic polynomial are close, in particular if it has multiple roots.

```
? A = [13,2; 10,14]; mateigen(A)
%1 =
[-1/2 2/5]

[   1   1]
? [L,H] = mateigen(A, 1);
? L
%3 = [9, 18]
? H
%4 =
[-1/2 2/5]

[   1   1]
```

For symmetric matrices, use `qfjacobi` instead; for Hermitian matrices, compute

```
A = real(x);
B = imag(x);
y = matconcat([A, -B; B, A]);
```

and apply `qfjacobi` to y.

The library syntax is `GEN mateigen(GEN x, long flag, long prec)`. Also available is `GEN eigen(GEN x, long prec)` $(flag = 0)$

3.11.13 matfrobenius$(M, \{flag\}, \{v =' x\})$. Returns the Frobenius form of the square matrix `M`. If $flag = 1$, returns only the elementary divisors as a vector of polynomials in the variable `v`. If $flag = 2$, returns a two-components vector [F,B] where `F` is the Frobenius form and `B` is the basis change so that $M = B^{-1}FB$.

The library syntax is `GEN matfrobenius(GEN M, long flag, long v = -1)` where `v` is a variable number.

3.11.14 mathess(x). Returns a matrix similar to the square matrix x, which is in upper Hessenberg form (zero entries below the first subdiagonal).

The library syntax is `GEN hess(GEN x)`.

3.11.15 mathilbert(n). x being a `long`, creates the Hilbert matrixof order x, i.e. the matrix whose coefficient (i,j) is $1/(i + j - 1)$.

The library syntax is `GEN mathilbert(long n)`.

3.11.16 mathnf$(M, \{flag = 0\})$. Let R be a Euclidean ring, equal to $\mathbf{Z}$ or to $K[X]$ for some field K. If M is a (not necessarily square) matrix with entries in R, this routine finds the *upper triangular* Hermite normal form of M. If the rank of M is equal to its number of rows, this is a square matrix. In general, the columns of the result form a basis of the R-module spanned by the columns of M.

The values $0, 1, 2, 3$ of *flag* have a binary meaning, analogous to the one in `matsnf`; in this case, binary digits of *flag* mean:

• 1 (complete output): if set, outputs $[H, U]$, where H is the Hermite normal form of M, and U is a transformation matrix such that $MU = [0|H]$. The matrix U belongs to $\mathrm{GL}(R)$. When M has a large kernel, the entries of U are in general huge.

• 2 (generic input): *Deprecated*. If set, assume that $R = K[X]$ is a polynomial ring; otherwise, assume that $R = \mathbf{Z}$. This flag is now useless since the routine always checks whether the matrix has integral entries.

For these 4 values, we use a naive algorithm, which behaves well in small dimension only. Larger values correspond to different algorithms, are restricted to *integer* matrices, and all output the unimodular matrix U. From now on all matrices have integral entries.

• *flag* $= 4$, returns $[H, U]$ as in "complete output" above, using a variant of LLL reduction along the way. The matrix U is provably small in the L_2 sense, and in general close to optimal; but the reduction is in general slow, although provably polynomial-time.

If *flag* $= 5$, uses Batut's algorithm and output $[H, U, P]$, such that H and U are as before and P is a permutation of the rows such that P applied to MU gives H. This is in general faster than *flag* $= 4$ but the matrix U is usually worse; it is heuristically smaller than with the default algorithm.

When the matrix is dense and the dimension is large (bigger than 100, say), *flag* $= 4$ will be fastest. When M has maximal rank, then

```
H = mathnfmod(M, matdetint(M))
```

will be even faster. You can then recover U as $M^{-1}H$.

```
? M = matrix(3,4,i,j,random([-5,5]))
%1 =
[ 0 2  3  0]

[-5 3 -5 -5]

[ 4 3 -5  4]

? [H,U] = mathnf(M, 1);
? U
%3 =
[-1 0 -1 0]

[ 0 5  3 2]

[ 0 3  1 1]

[ 1 0  0 0]

? H
%5 =
[19 9 7]
```

```
[ 0 9 1]

[ 0 0 1]

? M*U
%6 =
[0 19 9 7]

[0  0 9 1]

[0  0 0 1]
```

For convenience, M is allowed to be a `t_VEC`, which is then automatically converted to a `t_MAT`, as per the `Mat` function. For instance to solve the generalized extended gcd problem, one may use

```
? v = [116085838, 181081878, 314252913,10346840];
? [H,U] = mathnf(v, 1);
? U
%2 =
[ 103 -603    15  -88]

[-146   13 -1208  352]

[  58  220   678 -167]

[-362 -144   381 -101]
? v*U
%3 = [0, 0, 0, 1]
```

This also allows to input a matrix as a `t_VEC` of `t_COL`s of the same length (which `Mat` would concatenate to the `t_MAT` having those columns):

```
? v = [[1,0,4]~, [3,3,4]~, [0,-4,-5]~]; mathnf(v)
%1 =
[47 32 12]

[ 0  1  0]

[ 0  0  1]
```

The library syntax is `GEN mathnf0(GEN M, long flag)`. Also available are `GEN hnf(GEN M)` (*flag* $= 0$) and `GEN hnfall(GEN M)` (*flag* $= 1$). To reduce *huge* relation matrices (sparse with small entries, say dimension 400 or more), you can use the pair `hnfspec` / `hnfadd`. Since this is quite technical and the calling interface may change, they are not documented yet. Look at the code in `basemath/hnf_snf.c`.

3.11.17 mathnfmod(x, d). If x is a (not necessarily square) matrix of maximal rank with integer entries, and d is a multiple of the (non-zero) determinant of the lattice spanned by the columns of x, finds the *upper triangular* Hermite normal form of x.

If the rank of x is equal to its number of rows, the result is a square matrix. In general, the columns of the result form a basis of the lattice spanned by the columns of x. Even when d is known, this is in general slower than `mathnf` but uses much less memory.

The library syntax is `GEN hnfmod(GEN x, GEN d)`.

3.11.18 mathnfmodid(x, d). Outputs the (upper triangular) Hermite normal form of x concatenated with the diagonal matrix with diagonal d. Assumes that x has integer entries. Variant: if d is an integer instead of a vector, concatenate d times the identity matrix.

```
? m=[0,7;-1,0;-1,-1]
%1 =
[ 0  7]

[-1  0]

[-1 -1]
? mathnfmodid(m, [6,2,2])
%2 =
[2 1 1]

[0 1 0]

[0 0 1]
? mathnfmodid(m, 10)
%3 =
[10 7 3]

[ 0 1 0]

[ 0 0 1]
```

The library syntax is `GEN hnfmodid(GEN x, GEN d)`.

3.11.19 mathouseholder(Q, v). applies a sequence Q of Householder transforms, as returned by `matqr`$(M, 1)$ to the vector or matrix v.

The library syntax is `GEN mathouseholder(GEN Q, GEN v)`.

3.11.20 matid(n). Creates the $n \times n$ identity matrix.

The library syntax is `GEN matid(long n)`.

3.11.21 matimage$(x, \{flag = 0\})$. Gives a basis for the image of the matrix x as columns of a matrix. A priori the matrix can have entries of any type. If $flag = 0$, use standard Gauss pivot. If $flag = 1$, use `matsupplement` (much slower: keep the default flag!).

The library syntax is `GEN matimage0(GEN x, long flag)`. Also available is `GEN image(GEN x)` ($flag = 0$).

3.11.22 matimagecompl(x). Gives the vector of the column indices which are not extracted by the function `matimage`, as a permutation (`t_VECSMALL`). Hence the number of components of `matimagecompl(x)` plus the number of columns of `matimage(x)` is equal to the number of columns of the matrix x.

The library syntax is `GEN imagecompl(GEN x)`.

3.11.23 matindexrank(x). x being a matrix of rank r, returns a vector with two `t_VECSMALL` components y and z of length r giving a list of rows and columns respectively (starting from 1) such that the extracted matrix obtained from these two vectors using `vecextract`(x, y, z) is invertible.

The library syntax is `GEN indexrank(GEN x)`.

3.11.24 matintersect(x, y). x and y being two matrices with the same number of rows each of whose columns are independent, finds a basis of the $\mathbf{Q}$-vector space equal to the intersection of the spaces spanned by the columns of x and y respectively. The faster function `idealintersect` can be used to intersect fractional ideals (projective $\mathbf{Z}_K$ modules of rank 1); the slower but much more general function `nfhnf` can be used to intersect general $\mathbf{Z}_K$-modules.

The library syntax is `GEN intersect(GEN x, GEN y)`.

3.11.25 matinverseimage(x, y). Given a matrix x and a column vector or matrix y, returns a preimage z of y by x if one exists (i.e such that $xz = y$), an empty vector or matrix otherwise. The complete inverse image is $z + \mathrm{Ker}x$, where a basis of the kernel of x may be obtained by `matker`.

```
? M = [1,2;2,4];
? matinverseimage(M, [1,2]~)
%2 = [1, 0]~
? matinverseimage(M, [3,4]~)
%3 = []~     \\ no solution
? matinverseimage(M, [1,3,6;2,6,12])
%4 =
[1 3 6]

[0 0 0]
? matinverseimage(M, [1,2;3,4])
%5 = [;]     \\ no solution
? K = matker(M)
%6 =
[-2]

[1]
```

The library syntax is `GEN inverseimage(GEN x, GEN y)`.

3.11.26 matisdiagonal(x). Returns true (1) if x is a diagonal matrix, false (0) if not.

The library syntax is `GEN isdiagonal(GEN x)`.

3.11.27 matker$(x, \{flag = 0\})$. Gives a basis for the kernel of the matrix x as columns of a matrix. The matrix can have entries of any type, provided they are compatible with the generic arithmetic operations ($+$, $\times$ and $/$).

If x is known to have integral entries, set $flag = 1$.

The library syntax is `GEN matker0(GEN x, long flag)`. Also available are `GEN ker(GEN x)` ($flag = 0$), `GEN keri(GEN x)` ($flag = 1$).

3.11.28 matkerint$(x, \{flag = 0\})$. Gives an LLL-reduced $\mathbf{Z}$-basis for the lattice equal to the kernel of the matrix x with rational entries.
flag

is deprecated, kept for backward compatibility.

The library syntax is `GEN matkerint0(GEN x, long flag)`. Use directly `GEN kerint(GEN x)` if x is known to have integer entries, and `Q_primpart` first otherwise.

3.11.29 matmuldiagonal(x, d). Product of the matrix x by the diagonal matrix whose diagonal entries are those of the vector d. Equivalent to, but much faster than $x * \texttt{matdiagonal}(d)$.

The library syntax is `GEN matmuldiagonal(GEN x, GEN d)`.

3.11.30 matmultodiagonal(x, y). Product of the matrices x and y assuming that the result is a diagonal matrix. Much faster than $x*y$ in that case. The result is undefined if $x*y$ is not diagonal.

The library syntax is `GEN matmultodiagonal(GEN x, GEN y)`.

3.11.31 matpascal$(n, \{q\})$. Creates as a matrix the lower triangular Pascal triangle of order $x+1$ (i.e. with binomial coefficients up to x). If q is given, compute the q-Pascal triangle (i.e. using q-binomial coefficients).

The library syntax is `GEN matqpascal(long n, GEN q = NULL)`. Also available is `GEN matpascal(GEN x)`.

3.11.32 matqr$(M, \{\textit{flag} = 0\})$. Returns $[Q, R]$, the QR-decomposition of the square invertible matrix M with real entries: Q is orthogonal and R upper triangular. If $\textit{flag} = 1$, the orthogonal matrix is returned as a sequence of Householder transforms: applying such a sequence is stabler and faster than multiplication by the corresponding Q matrix. More precisely, if

```
[Q,R] = matqr(M);
[q,r] = matqr(M, 1);
```

then $r = R$ and $\texttt{mathouseholder}(q, M)$ is (close to) R; furthermore

```
mathouseholder(q, matid(#M)) == Q~
```

the inverse of Q. This function raises an error if the precision is too low or x is singular.

The library syntax is `GEN matqr(GEN M, long flag, long prec)`.

3.11.33 matrank(x). Rank of the matrix x.

The library syntax is `long rank(GEN x)`.

3.11.34 matrix$(m, n, \{X\}, \{Y\}, \{\textit{expr} = 0\})$. Creation of the $m \times n$ matrix whose coefficients are given by the expression *expr*. There are two formal parameters in *expr*, the first one (X) corresponding to the rows, the second (Y) to the columns, and X goes from 1 to m, Y goes from 1 to n. If one of the last 3 parameters is omitted, fill the matrix with zeroes.

3.11.35 matrixqz$(A, \{p = 0\})$. A being an $m \times n$ matrix in $M_{m,n}(\mathbf{Q})$, let $\mathrm{Im}_{\mathbf{Q}} A$ (resp. $\mathrm{Im}_{\mathbf{Z}} A$) the $\mathbf{Q}$-vector space (resp. the $\mathbf{Z}$-module) spanned by the columns of A. This function has varying behavior depending on the sign of p:

If $p \geq 0$, A is assumed to have maximal rank $n \leq m$. The function returns a matrix $B \in M_{m,n}(\mathbf{Z})$, with $\mathrm{Im}_{\mathbf{Q}} B = \mathrm{Im}_{\mathbf{Q}} A$, such that the GCD of all its $n \times n$ minors is coprime to p; in particular, if $p = 0$ (default), this GCD is 1.

```
? minors(x) = vector(#x[,1], i, matdet(x[^i,]));
? A = [3,1/7; 5,3/7; 7,5/7]; minors(A)
%1 = [4/7, 8/7, 4/7]   \\ determinants of all 2x2 minors
? B = matrixqz(A)
%2 =
[3 1]

[5 2]

[7 3]
? minors(%)
%3 = [1, 2, 1]   \\ B integral with coprime minors
```

If $p = -1$, returns the HNF basis of the lattice $\mathbf{Z}^n \cap \mathrm{Im}_{\mathbf{Z}} A$.

If $p = -2$, returns the HNF basis of the lattice $\mathbf{Z}^n \cap \mathrm{Im}_{\mathbf{Q}} A$.

```
? matrixqz(A,-1)
%4 =
[8 5]

[4 3]

[0 1]

? matrixqz(A,-2)
%5 =
[2 -1]

[1 0]

[0 1]
```

The library syntax is `GEN matrixqz0(GEN A, GEN p = NULL)`.

3.11.36 matsize(x). x being a vector or matrix, returns a row vector with two components, the first being the number of rows (1 for a row vector), the second the number of columns (1 for a column vector).

The library syntax is `GEN matsize(GEN x)`.

3.11.37 matsnf$(X, \{flag = 0\})$. If X is a (singular or non-singular) matrix outputs the vector of elementary divisors of X, i.e. the diagonal of the Smith normal form of X, normalized so that $d_n \mid d_{n-1} \mid \ldots \mid d_1$.

The binary digits of *flag* mean:

1 (complete output): if set, outputs $[U, V, D]$, where U and V are two unimodular matrices such that UXV is the diagonal matrix D. Otherwise output only the diagonal of D. If X is not a square matrix, then D will be a square diagonal matrix padded with zeros on the left or the top.

2 (generic input): if set, allows polynomial entries, in which case the input matrix must be square. Otherwise, assume that X has integer coefficients with arbitrary shape.

4 (cleanup): if set, cleans up the output. This means that elementary divisors equal to 1 will be deleted, i.e. outputs a shortened vector D' instead of D. If complete output was required, returns $[U', V', D']$ so that $U'XV' = D'$ holds. If this flag is set, X is allowed to be of the form 'vector of elementary divisors' or $[U, V, D]$ as would normally be output with the cleanup flag unset.

The library syntax is `GEN matsnf0(GEN X, long flag)`.

3.11.38 matsolve(M, B). M being an invertible matrix and B a column vector, finds the solution X of $MX = B$, using Dixon p-adic lifting method if M and B are integral and Gaussian elimination otherwise. This has the same effect as, but is faster, than $M^{-1} * B$.

The library syntax is `GEN gauss(GEN M, GEN B)`. For integral input, the function `GEN ZM_gauss(GEN M, GEN B)` is also available.

3.11.39 matsolvemod$(M, D, B, \{flag = 0\})$. M being any integral matrix, D a column vector of non-negative integer moduli, and B an integral column vector, gives a small integer solution to the system of congruences $\sum_i m_{i,j} x_j \equiv b_i \pmod{d_i}$ if one exists, otherwise returns zero. Shorthand notation: B (resp. D) can be given as a single integer, in which case all the b_i (resp. d_i) above are taken to be equal to B (resp. D).

```
? M = [1,2;3,4];
? matsolvemod(M, [3,4]~, [1,2]~)
%2 = [-2, 0]~
? matsolvemod(M, 3, 1) \\ M X = [1,1]~ over F_3
%3 = [-1, 1]~
? matsolvemod(M, [3,0]~, [1,2]~) \\ x + 2y = 1 (mod 3), 3x + 4y = 2 (in Z)
%4 = [6, -4]~
```

If $flag = 1$, all solutions are returned in the form of a two-component row vector $[x, u]$, where x is a small integer solution to the system of congruences and u is a matrix whose columns give a basis of the homogeneous system (so that all solutions can be obtained by adding x to any linear combination of columns of u). If no solution exists, returns zero.

The library syntax is `GEN matsolvemod0(GEN M, GEN D, GEN B, long flag)`. Also available are `GEN gaussmodulo(GEN M, GEN D, GEN B)` ($flag = 0$) and `GEN gaussmodulo2(GEN M, GEN D, GEN B)` ($flag = 1$).

3.11.40 matsupplement(x). Assuming that the columns of the matrix x are linearly independent (if they are not, an error message is issued), finds a square invertible matrix whose first columns are the columns of x, i.e. supplement the columns of x to a basis of the whole space.

```
? matsupplement([1;2])
%1 =
[1 0]

[2 1]
```

Raises an error if x has 0 columns, since (due to a long standing design bug), the dimension of the ambient space (the number of rows) is unknown in this case:

```
? matsupplement(matrix(2,0))
  ***   at top-level: matsupplement(matrix
  ***                 ^--------------------
  *** matsupplement: sorry, suppl [empty matrix] is not yet implemented.
```

The library syntax is `GEN suppl(GEN x)`.

3.11.41 mattranspose(x). Transpose of x (also x~). This has an effect only on vectors and matrices.

The library syntax is `GEN gtrans(GEN x)`.

3.11.42 minpoly($A, \{v = ' x\}$). minimal polynomial of A with respect to the variable v., i.e. the monic polynomial P of minimal degree (in the variable v) such that $P(A) = 0$.

The library syntax is `GEN minpoly(GEN A, long v = -1)` where `v` is a variable number.

3.11.43 norml2(x). Square of the L^2-norm of x. More precisely, if x is a scalar, `norml2`(x) is defined to be the square of the complex modulus of x (real `t_QUAD`s are not supported). If x is a polynomial, a (row or column) vector or a matrix, `norml2`(x) is defined recursively as $\sum_i$ `norml2`(x_i), where (x_i) run through the components of x. In particular, this yields the usual $\sum |x_i|^2$ (resp. $\sum |x_{i,j}|^2$) if x is a polynomial or vector (resp. matrix) with complex components.

```
? norml2( [ 1, 2, 3 ] )      \\ vector
%1 = 14
? norml2( [ 1, 2; 3, 4] )   \\ matrix
%2 = 30
? norml2( 2*I + x )
%3 = 5
? norml2( [ [1,2], [3,4], 5, 6 ] )   \\ recursively defined
%4 = 91
```

The library syntax is `GEN gnorml2(GEN x)`.

3.11.44 normlp$(x, \{p = oo\})$. L^p-norm of x; sup norm if p is omitted or `+oo`. More precisely, if x is a scalar, `normlp`(x, p) is defined to be `abs`(x). If x is a polynomial, a (row or column) vector or a matrix:

• if p is omitted or `+oo`, then `normlp(`x`)` is defined recursively as $\max_i$ `normlp`(x_i)), where (x_i) run through the components of x. In particular, this yields the usual sup norm if x is a polynomial or vector with complex components.

• otherwise, `normlp(`x`, `p`)` is defined recursively as $(\sum_i$ `normlp`$^p(x_i, p))^{1/p}$. In particular, this yields the usual $(\sum |x_i|^p)^{1/p}$ if x is a polynomial or vector with complex components.

```
? v = [1,-2,3]; normlp(v)      \\ vector
%1 = 3
? normlp(v, +oo)               \\ same, more explicit
%2 = 3
? M = [1,-2;-3,4]; normlp(M)   \\ matrix
%3 = 4
? T = (1+I) + I*x^2; normlp(T)
%4 = 1.4142135623730950488016887242096980786
? normlp([[1,2], [3,4], 5, 6])   \\ recursively defined
%5 = 6

? normlp(v, 1)
%6 = 6
? normlp(M, 1)
%7 = 10
? normlp(T, 1)
%8 = 2.4142135623730950488016887242096980786
```

The library syntax is `GEN gnormlp(GEN x, GEN p = NULL, long prec)`.

3.11.45 qfauto$(G, \{fl\})$. G being a square and symmetric matrix with integer entries representing a positive definite quadratic form, outputs the automorphism group of the associate lattice. Since this requires computing the minimal vectors, the computations can become very lengthy as the dimension grows. G can also be given by an `qfisominit` structure. See `qfisominit` for the meaning of fl.

The output is a two-components vector $[o, g]$ where o is the group order and g is the list of generators (as a vector). For each generator H, the equality $G = {}^tHGH$ holds.

The interface of this function is experimental and will likely change in the future.

This function implements an algorithm of Plesken and Souvignier, following Souvignier's implementation.

The library syntax is `GEN qfauto0(GEN G, GEN fl = NULL)`. The function `GEN qfauto(GEN G, GEN fl)` is also available where G is a vector of `zm` matrices.

3.11.46 qfautoexport($qfa, \{flag\}$). *qfa* being an automorphism group as output by `qfauto`, export the underlying matrix group as a string suitable for (no flags or *flag* = 0) GAP or (*flag* = 1) Magma. The following example computes the size of the matrix group using GAP:

```
? G = qfauto([2,1;1,2])
%1 = [12, [[-1, 0; 0, -1], [0, -1; 1, 1], [1, 1; 0, -1]]]
? s = qfautoexport(G)
%2 = "Group([[-1, 0], [0, -1]], [[0, -1], [1, 1]], [[1, 1], [0, -1]])"
? extern("echo \"Order("s");\" | gap -q")
%3 = 12
```

The library syntax is `GEN qfautoexport(GEN qfa, long flag)`.

3.11.47 qfbil($x, y, \{q\}$). This function is obsolete, use `qfeval`.

The library syntax is `GEN qfbil(GEN x, GEN y, GEN q = NULL)`.

3.11.48 qfeval($\{q\}, x, \{y\}$). Evaluate the binary quadratic form q (given by a symmetric matrix) at the vector x; if y is present, evaluate the polar form at (x, y); if q omitted, use the standard Euclidean scalar product, corresponding to the identity matrix.

Roughly equivalent to `x~* q * y`, but a little faster and more convenient (does not distinguish between column and row vectors):

```
? x = [1,2,3]~; y = [-1,3,1]~; q = [1,2,3;2,2,-1;3,-1,9];
? qfeval(q,x,y)
%2 = 23
? for(i=1,10^6, qfeval(q,x,y))
time = 661ms
? for(i=1,10^6, x~*q*y)
time = 697ms
```

The speedup is noticeable for the quadratic form, compared to `x~* q * x`, since we save almost half the operations:

```
? for(i=1,10^6, qfeval(q,x))
time = 487ms
```

The special case $q = \mathrm{Id}$ is handled faster if we omit q altogether:

```
? qfeval(,x,y)
%1 = 2
? q = matid(#x);
? for(i=1,10^6, qfeval(q,x,y))
time = 529 ms.
? for(i=1,10^6, qfeval(,x,y))
time = 228 ms.
? for(i=1,10^6, x~*y)
time = 274 ms.
```

We also allow `t_MAT`s of compatible dimensions for x, and return `x~* q * x` in this case as well:

```
? M = [1,2,3;4,5,6;7,8,9]; qfeval(,M) \\ Gram matrix
```

```
%5 =
[66  78  90]

[78  93 108]

[90 108 126]

? q = [1,2,3;2,2,-1;3,-1,9];
? for(i=1,10^6, qfeval(q,M))
time = 2,008 ms.
? for(i=1,10^6, M~*q*M)
time = 2,368 ms.

? for(i=1,10^6, qfeval(,M))
time = 1,053 ms.
? for(i=1,10^6, M~*M)
time = 1,171 ms.
```

If q is a `t_QFI` or `t_QFR`, it is implicitly converted to the attached symmetric `t_MAT`. This is done more efficiently than by direct conversion, since we avoid introducing a denominator 2 and rational arithmetic:

```
? q = Qfb(2,3,4); x = [2,3];
? qfeval(q, x)
%2 = 62
? Q = Mat(q)
%3 =
 [  2 3/2]

 [3/2   4]
? qfeval(Q, x)
%4 = 62
? for (i=1, 10^6, qfeval(q,x))
time = 758 ms.
? for (i=1, 10^6, qfeval(Q,x))
time = 1,110 ms.
```

Finally, when x is a `t_MAT` with *integral* coefficients, we allow a `t_QFI` or `t_QFR` for q and return the binary quadratic form $q \circ M$. Again, the conversion to `t_MAT` is less efficient in this case:

```
? q = Qfb(2,3,4); Q = Mat(q); x = [1,2;3,4];
? qfeval(q, x)
%2 = Qfb(47, 134, 96)
? qfeval(Q,x)
%3 =
[47 67]

[67 96]
? for (i=1, 10^6, qfeval(q,x))
time = 701 ms.
? for (i=1, 10^6, qfeval(Q,x))
time = 1,639 ms.
```

The library syntax is `GEN qfeval0(GEN q = NULL, GEN x, GEN y = NULL)`.

3.11.49 qfgaussred(q). decomposition into squares of the quadratic form represented by the symmetric matrix q. The result is a matrix whose diagonal entries are the coefficients of the squares, and the off-diagonal entries on each line represent the bilinear forms. More precisely, if (a_{ij}) denotes the output, one has

$$q(x) = \sum_i a_{ii}(x_i + \sum_{j \neq i} a_{ij} x_j)^2$$

```
? qfgaussred([0,1;1,0])
%1 =
[1/2 1]

[-1 -1/2]
```

This means that $2xy = (1/2)(x+y)^2 - (1/2)(x-y)^2$. Singular matrices are supported, in which case some diagonal coefficients will vanish:

```
? qfgaussred([1,1;1,1])
%1 =
[1 1]

[1 0]
```

This means that $x^2 + 2xy + y^2 = (x+y)^2$.

The library syntax is `GEN qfgaussred(GEN q)`. `GEN qfgaussred_positive(GEN q)` assumes that q is positive definite and is a little faster; returns `NULL` if a vector with negative norm occurs (non positive matrix or too many rounding errors).

3.11.50 qfisom$(G, H, \{fl\})$. G, H being square and symmetric matrices with integer entries representing positive definite quadratic forms, return an invertible matrix S such that $G = {}^tSHS$. This defines a isomorphism between the corresponding lattices. Since this requires computing the minimal vectors, the computations can become very lengthy as the dimension grows. See `qfisominit` for the meaning of *fl*.

G can also be given by an `qfisominit` structure which is preferable if several forms H need to be compared to G.

This function implements an algorithm of Plesken and Souvignier, following Souvignier's implementation.

The library syntax is `GEN qfisom0(GEN G, GEN H, GEN fl = NULL)`. Also available is `GEN qfisom(GEN G, GEN H, GEN fl)` where G is a vector of `zm`, and H is a `zm`.

3.11.51 qfisominit$(G, \{fl\}, \{m\})$. G being a square and symmetric matrix with integer entries representing a positive definite quadratic form, return an `isom` structure allowing to compute isomorphisms between G and other quadratic forms faster.

The interface of this function is experimental and will likely change in future release.

If present, the optional parameter *fl* must be a `t_VEC` with two components. It allows to specify the invariants used, which can make the computation faster or slower. The components are

- `fl[1]` Depth of scalar product combination to use.
- `fl[2]` Maximum level of Bacher polynomials to use.

If present, m must be the set of vectors of norm up to the maximal of the diagonal entry of G, either as a matrix or as given by `qfminim`. Otherwise this function computes the minimal vectors so it become very lengthy as the dimension of G grows.

The library syntax is `GEN qfisominit0(GEN G, GEN fl = NULL, GEN m = NULL)`. Also available is `GEN qfisominit(GEN F, GEN fl)` where F is a vector of `zm`.

3.11.52 qfjacobi(A). Apply Jacobi's eigenvalue algorithm to the real symmetric matrix A. This returns $[L, V]$, where

- L is the vector of (real) eigenvalues of A, sorted in increasing order,
- V is the corresponding orthogonal matrix of eigenvectors of A.

```
? \p19
? A = [1,2;2,1]; mateigen(A)
%1 =
[-1 1]

[ 1 1]
? [L, H] = qfjacobi(A);
? L
%3 = [-1.000000000000000000, 3.000000000000000000]~
? H
%4 =
[ 0.7071067811865475245 0.7071067811865475244]

[-0.7071067811865475244 0.7071067811865475245]
? norml2( (A-L[1])*H[,1] )       \\ approximate eigenvector
%5 = 9.403954806578300064 E-38
? norml2(H*H~ - 1)
%6 = 2.350988701644575016 E-38   \\ close to orthogonal
```

The library syntax is `GEN jacobi(GEN A, long prec)`.

3.11.53 qflll$(x, \{flag = 0\})$. LLL algorithm applied to the *columns* of the matrix x. The columns of x may be linearly dependent. The result is a unimodular transformation matrix T such that $x \cdot T$ is an LLL-reduced basis of the lattice generated by the column vectors of x. Note that if x is not of maximal rank T will not be square. The LLL parameters are $(0.51, 0.99)$, meaning that the Gram-Schmidt coefficients for the final basis satisfy $\mu_{i,j} \leq |0.51|$, and the Lovász's constant is 0.99.

If *flag* $= 0$ (default), assume that x has either exact (integral or rational) or real floating point entries. The matrix is rescaled, converted to integers and the behavior is then as in *flag* $= 1$.

If *flag* $= 1$, assume that x is integral. Computations involving Gram-Schmidt vectors are approximate, with precision varying as needed (Lehmer's trick, as generalized by Schnorr). Adapted from Nguyen and Stehlé's algorithm and Stehlé's code (`fplll-1.3`).

If *flag* $= 2$, x should be an integer matrix whose columns are linearly independent. Returns a partially reduced basis for x, using an unpublished algorithm by Peter Montgomery: a basis is said to be *partially reduced* if $|v_i \pm v_j| \geq |v_i|$ for any two distinct basis vectors v_i, v_j.

This is faster than *flag* $= 1$, esp. when one row is huge compared to the other rows (knapsack-style), and should quickly produce relatively short vectors. The resulting basis is *not* LLL-reduced in general. If LLL reduction is eventually desired, avoid this partial reduction: applying LLL to the partially reduced matrix is significantly *slower* than starting from a knapsack-type lattice.

If *flag* $= 4$, as *flag* $= 1$, returning a vector $[K, T]$ of matrices: the columns of K represent a basis of the integer kernel of x (not LLL-reduced in general) and T is the transformation matrix such that $x \cdot T$ is an LLL-reduced $\mathbf{Z}$-basis of the image of the matrix x.

If *flag* $= 5$, case as case 4, but x may have polynomial coefficients.

If *flag* $= 8$, same as case 0, but x may have polynomial coefficients.

The library syntax is `GEN qflll0(GEN x, long flag)`. Also available are `GEN lll(GEN x)` (*flag* $= 0$), `GEN lllint(GEN x)` (*flag* $= 1$), and `GEN lllkerim(GEN x)` (*flag* $= 4$).

3.11.54 qflllgram$(G, \{flag = 0\})$. Same as `qflll`, except that the matrix $G =$ `x~ * x` is the Gram matrix of some lattice vectors x, and not the coordinates of the vectors themselves. In particular, G must now be a square symmetric real matrix, corresponding to a positive quadratic form (not necessarily definite: x needs not have maximal rank). The result is a unimodular transformation matrix T such that $x \cdot T$ is an LLL-reduced basis of the lattice generated by the column vectors of x. See `qflll` for further details about the LLL implementation.

If *flag* $= 0$ (default), assume that G has either exact (integral or rational) or real floating point entries. The matrix is rescaled, converted to integers and the behavior is then as in *flag* $= 1$.

If *flag* $= 1$, assume that G is integral. Computations involving Gram-Schmidt vectors are approximate, with precision varying as needed (Lehmer's trick, as generalized by Schnorr). Adapted from Nguyen and Stehlé's algorithm and Stehlé's code (`fplll-1.3`).

flag $= 4$: G has integer entries, gives the kernel and reduced image of x.

flag $= 5$: same as 4, but G may have polynomial coefficients.

The library syntax is `GEN qflllgram0(GEN G, long flag)`. Also available are `GEN lllgram(GEN G)` (*flag* $= 0$), `GEN lllgramint(GEN G)` (*flag* $= 1$), and `GEN lllgramkerim(GEN G)` (*flag* $= 4$).

3.11.55 qfminim$(x, \{b\}, \{m\}, \{flag = 0\})$. x being a square and symmetric matrix representing a positive definite quadratic form, this function deals with the vectors of x whose norm is less than or equal to b, enumerated using the Fincke-Pohst algorithm, storing at most m vectors (no limit if m is omitted). The function searches for the minimal non-zero vectors if b is omitted. The behavior is undefined if x is not positive definite (a "precision too low" error is most likely, although more precise error messages are possible). The precise behavior depends on *flag*.

If *flag* $= 0$ (default), returns at most $2m$ vectors. The result is a three-component vector, the first component being the number of vectors enumerated (which may be larger than $2m$), the second being the maximum norm found, and the last vector is a matrix whose columns are found vectors, only one being given for each pair $\pm v$ (at most m such pairs, unless m was omitted). The vectors are returned in no particular order.

If *flag* $= 1$, ignores m and returns $[N, v]$, where v is a non-zero vector of length $N \leq b$, or [] if no non-zero vector has length $\leq b$. If no explicit b is provided, return a vector of smallish norm (smallest vector in an LLL-reduced basis).

In these two cases, x must have *integral* entries. The implementation uses low precision floating point computations for maximal speed, which gives incorrect result when x has large entries. (The condition is checked in the code and the routine raises an error if large rounding errors occur.) A more robust, but much slower, implementation is chosen if the following flag is used:

If *flag* $= 2$, x can have non integral real entries. In this case, if b is omitted, the "minimal" vectors only have approximately the same norm. If b is omitted, m is an upper bound for the number of vectors that will be stored and returned, but all minimal vectors are nevertheless enumerated. If m is omitted, all vectors found are stored and returned; note that this may be a huge vector!

```
? x = matid(2);
? qfminim(x)  \\ 4 minimal vectors of norm 1: ±[0,1], ±[1,0]
%2 = [4, 1, [0, 1; 1, 0]]
? { x =
[4, 2, 0, 0, 0,-2, 0, 0, 0, 0, 0, 0, 1,-1, 0, 0, 0, 1, 0,-1, 0, 0, 0,-2;
 2, 4,-2,-2, 0,-2, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0, 0,-1, 0, 1,-1,-1;
 0,-2, 4, 0,-2, 0, 0, 0, 0, 0, 0, 0,-1, 1, 0, 0, 1, 0, 0, 1,-1,-1, 0, 0;
 0,-2, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 1,-1, 0, 0, 0, 1,-1, 0, 1,-1, 1, 0;
 0, 0,-2, 0, 4, 0, 0, 0, 1,-1, 0, 0, 1, 0, 0, 0,-2, 0, 0,-1, 1, 1, 0, 0;
-2, -2,0, 0, 0, 4,-2, 0,-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0,-1, 1, 1;
 0, 0, 0, 0, 0,-2, 4,-2, 0, 0, 0, 0, 0, 1, 0, 0, 0,-1, 0, 0, 0, 1,-1, 0;
 0, 0, 0, 0, 0, 0,-2, 4, 0, 0, 0, 0,-1, 0, 0, 0, 0, 0,-1,-1,-1, 0, 1, 0;
 0, 0, 0, 0, 1,-1, 0, 0, 4, 0,-2, 0, 1, 1, 0,-1, 0, 1, 0, 0, 0, 0, 0, 0;
 0, 0, 0, 0,-1, 0, 0, 0, 0, 4, 0, 0, 1, 1,-1, 1, 0, 0, 0, 1, 0, 0, 1, 0;
 0, 0, 0, 0, 0, 0, 0, 0,-2, 0, 4,-2, 0,-1, 0, 0, 0,-1, 0,-1, 0, 0, 0, 0;
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-2, 4,-1, 1, 0, 0,-1, 1, 0, 1, 1, 1,-1, 0;
 1, 0,-1, 1, 1, 0, 0,-1, 1, 1, 0,-1, 4, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1,-1;
-1,-1, 1,-1, 0, 0, 1, 0, 1, 1,-1, 1, 0, 4, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1;
 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 1, 4, 0, 0, 0, 1, 0, 0, 0, 0, 0;
 0, 0, 0, 0, 0, 0, 0, 0,-1, 1, 0, 0, 1, 1, 0, 4, 0, 0, 0, 0, 1, 1, 0, 0;
 0, 0, 1, 0,-2, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 0, 4, 1, 1, 1, 0, 0, 1, 1;
 1, 0, 0, 1, 0, 0,-1, 0, 1, 0,-1, 1, 1, 0, 0, 0, 1, 4, 0, 1, 1, 0, 1, 0;
 0, 0, 0,-1, 0, 1, 0,-1, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 4, 0, 1, 1, 0, 1;
-1, -1,1, 0,-1, 1, 0,-1, 0, 1,-1, 1, 0, 1, 0, 0, 1, 1, 0, 4, 0, 0, 1, 1;
```

```
 0, 0,-1, 1, 1, 0, 0,-1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 4, 1, 0, 1;
 0, 1,-1,-1, 1,-1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 4, 0, 1;
 0,-1, 0, 1, 0, 1,-1, 1, 0, 1, 0,-1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 4, 1;
-2,-1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0,-1, 1, 0, 0, 1, 0, 1, 1, 1, 1, 1, 4]; }
? qfminim(x,,0)  \\ the Leech lattice has 196560 minimal vectors of norm 4
time = 648 ms.
%4 = [196560, 4, [;]]
? qfminim(x,,0,2); \\ safe algorithm. Slower and unnecessary here.
time = 18,161 ms.
%5 = [196560, 4.000061035156250000, [;]]
```

In the last example, we store 0 vectors to limit memory use. All minimal vectors are nevertheless enumerated. Provided `parisize` is about 50MB, `qfminim(x)` succeeds in 2.5 seconds.

The library syntax is `GEN qfminim0(GEN x, GEN b = NULL, GEN m = NULL, long flag, long prec)`. Also available are `GEN minim(GEN x, GEN b = NULL, GEN m = NULL)` (*flag* $= 0$), `GEN minim2(GEN x, GEN b = NULL, GEN m = NULL)` (*flag* $= 1$). `GEN minim_raw(GEN x, GEN b = NULL, GEN m = NULL)` (do not perform LLL reduction on x and return `NULL` on accuracy error).

3.11.56 qfnorm$(x, \{q\})$. This function is obsolete, use `qfeval`.

The library syntax is `GEN qfnorm(GEN x, GEN q = NULL)`.

3.11.57 qforbits(G, V). Return the orbits of V under the action of the group of linear transformation generated by the set G. It is assumed that G contains minus identity, and only one vector in $\{v, -v\}$ should be given. If G does not stabilize V, the function return 0.

In the example below, we compute representatives and lengths of the orbits of the vectors of norm ≤ 3 under the automorphisms of the lattice A_1^6.

```
?  Q=matid(6); G=qfauto(Q); V=qfminim(Q,3);
?  apply(x->[x[1],#x],qforbits(G,V))
%2 = [[[0,0,0,0,0,1]~,6],[[0,0,0,0,1,-1]~,30],[[0,0,0,1,-1,-1]~,80]]
```

The library syntax is `GEN qforbits(GEN G, GEN V)`.

3.11.58 qfparam$(G, sol, \{flag = 0\})$. Coefficients of binary quadratic forms that parametrize the solutions of the ternary quadratic form G, using the particular solution *sol*. *flag*is optional and can be 1, 2, or 3, in which case the *flag*-th form is reduced. The default is *flag*=0 (no reduction).

```
? G = [1,0,0;0,1,0;0,0,-34];
? M = qfparam(G, qfsolve(G))
%2 =
[ 3 -10 -3]

[-5  -6  5]

[ 1   0  1]
```

Indeed, the solutions can be parametrized as

$$(3x^2 - 10xy - 3y^2)^2 + (-5x^2 - 6xy + 5y^2)^2 - 34(x^2 + y^2)^2 = 0.$$

```
? v = y^2 * M*[1,x/y,(x/y)^2]~
```

```
%3 = [3*x^2 - 10*y*x - 3*y^2, -5*x^2 - 6*y*x + 5*y^2, -x^2 - y^2]~
? v~*G*v
%4 = 0
```

The library syntax is `GEN qfparam(GEN G, GEN sol, long flag)`.

3.11.59 qfperfection(G). G being a square and symmetric matrix with integer entries representing a positive definite quadratic form, outputs the perfection rank of the form. That is, gives the rank of the family of the s symmetric matrices $v_i v_i^t$, where s is half the number of minimal vectors and the v_i $(1 \leq i \leq s)$ are the minimal vectors.

Since this requires computing the minimal vectors, the computations can become very lengthy as the dimension of x grows.

The library syntax is `GEN perf(GEN G)`.

3.11.60 qfrep$(q, B, \{flag = 0\})$. q being a square and symmetric matrix with integer entries representing a positive definite quadratic form, count the vectors representing successive integers.

• If $flag = 0$, count all vectors. Outputs the vector whose i-th entry, $1 \leq i \leq B$ is half the number of vectors v such that $q(v) = i$.

• If $flag = 1$, count vectors of even norm. Outputs the vector whose i-th entry, $1 \leq i \leq B$ is half the number of vectors such that $q(v) = 2i$.

```
? q = [2, 1; 1, 3];
? qfrep(q, 5)
%2 = Vecsmall([0, 1, 2, 0, 0]) \\ 1 vector of norm 2, 2 of norm 3, etc.
? qfrep(q, 5, 1)
%3 = Vecsmall([1, 0, 0, 1, 0]) \\ 1 vector of norm 2, 0 of norm 4, etc.
```

This routine uses a naive algorithm based on `qfminim`, and will fail if any entry becomes larger than 2^{31} (or 2^{63}).

The library syntax is `GEN qfrep0(GEN q, GEN B, long flag)`.

3.11.61 qfsign(x). Returns $[p, m]$ the signature of the quadratic form represented by the symmetric matrix x. Namely, p (resp. m) is the number of positive (resp. negative) eigenvalues of x. The result is computed using Gaussian reduction.

The library syntax is `GEN qfsign(GEN x)`.

3.11.62 qfsolve(G). Given a square symmetric matrix G of dimension $n \geq 1$, solve over $\mathbf{Q}$ the quadratic equation $X^t G X = 0$. The matrix G must have rational coefficients. The solution might be a single non-zero vector (vectorv) or a matrix (whose columns generate a totally isotropic subspace).

If no solution exists, returns an integer, that can be a prime p such that there is no local solution at p, or -1 if there is no real solution, or -2 if $n = 2$ and $-\det G$ is positive but not a square (which implies there is a real solution, but no local solution at some p dividing $\det G$).

```
? G = [1,0,0;0,1,0;0,0,-34];
? qfsolve(G)
%1 = [-3, -5, 1]~
? qfsolve([1,0; 0,2])
```

```
%2 = -1   \\ no real solution
? qfsolve([1,0,0;0,3,0; 0,0,-2])
%3 = 3    \\ no solution in Q_3
? qfsolve([1,0; 0,-2])
%4 = -2   \\ no solution, n = 2
```

The library syntax is `GEN qfsolve(GEN G)`.

3.11.63 seralgdep(s, p, r). finds a linear relation between powers $(1, s, \ldots, s^p)$ of the series s, with polynomial coefficients of degree $\leq r$. In case no relation is found, return 0.

```
? s = 1 + 10*y - 46*y^2 + 460*y^3 - 5658*y^4 + 77740*y^5 + O(y^6);
? seralgdep(s, 2, 2)
%2 = -x^2 + (8*y^2 + 20*y + 1)
? subst(%, x, s)
%3 = O(y^6)
? seralgdep(s, 1, 3)
%4 = (-77*y^2 - 20*y - 1)*x + (310*y^3 + 231*y^2 + 30*y + 1)
? seralgdep(s, 1, 2)
%5 = 0
```

The series main variable must not be x, so as to be able to express the result as a polynomial in x.

The library syntax is `GEN seralgdep(GEN s, long p, long r)`.

3.11.64 setbinop$(f, X, \{Y\})$. The set whose elements are the f(x,y), where x,y run through X,Y. respectively. If Y is omitted, assume that $X = Y$ and that f is symmetric: $f(x, y) = f(y, x)$ for all x, y in X.

```
? X = [1,2,3]; Y = [2,3,4];
? setbinop((x,y)->x+y, X,Y) \\ set X + Y
%2 = [3, 4, 5, 6, 7]
? setbinop((x,y)->x-y, X,Y) \\ set X - Y
%3 = [-3, -2, -1, 0, 1]
? setbinop((x,y)->x+y, X)   \\ set 2X = X + X
%2 = [2, 3, 4, 5, 6]
```

The library syntax is `GEN setbinop(GEN f, GEN X, GEN Y = NULL)`.

3.11.65 setintersect(x, y). Intersection of the two sets x and y (see `setisset`). If x or y is not a set, the result is undefined.

The library syntax is `GEN setintersect(GEN x, GEN y)`.

3.11.66 setisset(x). Returns true (1) if x is a set, false (0) if not. In PARI, a set is a row vector whose entries are strictly increasing with respect to a (somewhat arbitrary) universal comparison function. To convert any object into a set (this is most useful for vectors, of course), use the function `Set`.

```
? a = [3, 1, 1, 2];
? setisset(a)
%2 = 0
? Set(a)
%3 = [1, 2, 3]
```

The library syntax is `long setisset(GEN x)`.

3.11.67 setminus(x, y). Difference of the two sets x and y (see `setisset`), i.e. set of elements of x which do not belong to y. If x or y is not a set, the result is undefined.

The library syntax is `GEN setminus(GEN x, GEN y)`.

3.11.68 setsearch$(S, x, \{flag = 0\})$. Determines whether x belongs to the set S (see `setisset`).

We first describe the default behaviour, when *flag* is zero or omitted. If x belongs to the set S, returns the index j such that $S[j] = x$, otherwise returns 0.

```
? T = [7,2,3,5]; S = Set(T);
? setsearch(S, 2)
%2 = 1
? setsearch(S, 4)      \\ not found
%3 = 0
? setsearch(T, 7)      \\ search in a randomly sorted vector
%4 = 0 \\ WRONG !
```

If S is not a set, we also allow sorted lists with respect to the `cmp` sorting function, without repeated entries, as per `listsort`$(L, 1)$; otherwise the result is undefined.

```
? L = List([1,4,2,3,2]); setsearch(L, 4)
%1 = 0 \\ WRONG !
? listsort(L, 1); L    \\ sort L first
%2 = List([1, 2, 3, 4])
? setsearch(L, 4)
%3 = 4                 \\ now correct
```

If *flag* is non-zero, this function returns the index j where x should be inserted, and 0 if it already belongs to S. This is meant to be used for dynamically growing (sorted) lists, in conjunction with `listinsert`.

```
? L = List([1,5,2,3,2]); listsort(L,1); L
%1 = List([1,2,3,5])
? j = setsearch(L, 4, 1)  \\ 4 should have been inserted at index j
%2 = 4
? listinsert(L, 4, j); L
%3 = List([1, 2, 3, 4, 5])
```

The library syntax is `long setsearch(GEN S, GEN x, long flag)`.

3.11.69 setunion(x, y). Union of the two sets x and y (see `setisset`). If x or y is not a set, the result is undefined.

The library syntax is `GEN setunion(GEN x, GEN y)`.

3.11.70 trace(x). This applies to quite general x. If x is not a matrix, it is equal to the sum of x and its conjugate, except for polmods where it is the trace as an algebraic number.

For x a square matrix, it is the ordinary trace. If x is a non-square matrix (but not a vector), an error occurs.

The library syntax is `GEN gtrace(GEN x)`.

3.11.71 vecextract$(x, y, \{z\})$. Extraction of components of the vector or matrix x according to y. In case x is a matrix, its components are the *columns* of x. The parameter y is a component specifier, which is either an integer, a string describing a range, or a vector.

If y is an integer, it is considered as a mask: the binary bits of y are read from right to left, but correspond to taking the components from left to right. For example, if $y = 13 = (1101)_2$ then the components 1,3 and 4 are extracted.

If y is a vector (`t_VEC`, `t_COL` or `t_VECSMALL`), which must have integer entries, these entries correspond to the component numbers to be extracted, in the order specified.

If y is a string, it can be

• a single (non-zero) index giving a component number (a negative index means we start counting from the end).

• a range of the form "$a..b$", where a and b are indexes as above. Any of a and b can be omitted; in this case, we take as default values $a = 1$ and $b = -1$, i.e. the first and last components respectively. We then extract all components in the interval $[a, b]$, in reverse order if $b < a$.

In addition, if the first character in the string is `^`, the complement of the given set of indices is taken.

If z is not omitted, x must be a matrix. y is then the *row* specifier, and z the *column* specifier, where the component specifier is as explained above.

```
? v = [a, b, c, d, e];
? vecextract(v, 5)          \\ mask
%1 = [a, c]
? vecextract(v, [4, 2, 1]) \\ component list
%2 = [d, b, a]
? vecextract(v, "2..4")     \\ interval
%3 = [b, c, d]
? vecextract(v, "-1..-3")   \\ interval + reverse order
%4 = [e, d, c]
? vecextract(v, "^2")       \\ complement
%5 = [a, c, d, e]
? vecextract(matid(3), "2..", "..")
%6 =
[0 1 0]

[0 0 1]
```

The range notations `v[i..j]` and `v[^i]` (for `t_VEC` or `t_COL`) and `M[i..j, k..l]` and friends (for `t_MAT`) implement a subset of the above, in a simpler and *faster* way, hence should be preferred in most common situations. The following features are not implemented in the range notation:

• reverse order,

• omitting either a or b in $a..b$.

The library syntax is `GEN extract0(GEN x, GEN y, GEN z = NULL)`.

3.11.72 vecsearch$(v, x, \{cmpf\})$. Determines whether x belongs to the sorted vector or list v: return the (positive) index where x was found, or 0 if it does not belong to v.

If the comparison function cmpf is omitted, we assume that v is sorted in increasing order, according to the standard comparison function `lex`, thereby restricting the possible types for x and the elements of v (integers, fractions, reals, and vectors of such).

If `cmpf` is present, it is understood as a comparison function and we assume that v is sorted according to it, see `vecsort` for how to encode comparison functions.

```
? v = [1,3,4,5,7];
? vecsearch(v, 3)
%2 = 2
? vecsearch(v, 6)
%3 = 0 \\ not in the list
? vecsearch([7,6,5], 5) \\ unsorted vector: result undefined
%4 = 0
```

By abuse of notation, x is also allowed to be a matrix, seen as a vector of its columns; again by abuse of notation, a `t_VEC` is considered as part of the matrix, if its transpose is one of the matrix columns.

```
? v = vecsort([3,0,2; 1,0,2]) \\ sort matrix columns according to lex order
%1 =
[0 2 3]

[0 2 1]
? vecsearch(v, [3,1]~)
%2 = 3
? vecsearch(v, [3,1])  \\ can search for x or x~
%3 = 3
? vecsearch(v, [1,2])
%4 = 0 \\ not in the list
```

The library syntax is `long vecsearch(GEN v, GEN x, GEN cmpf = NULL)`.

3.11.73 vecsort$(x, \{cmpf\}, \{flag = 0\})$. Sorts the vector x in ascending order, using a mergesort method. x must be a list, vector or matrix (seen as a vector of its columns). Note that mergesort is stable, hence the initial ordering of "equal" entries (with respect to the sorting criterion) is not changed.

If `cmpf` is omitted, we use the standard comparison function `lex`, thereby restricting the possible types for the elements of x (integers, fractions or reals and vectors of those). If `cmpf` is present, it is understood as a comparison function and we sort according to it. The following possibilities exist:

• an integer k: sort according to the value of the k-th subcomponents of the components of x.

• a vector: sort lexicographically according to the components listed in the vector. For example, if `cmpf` $= [2, 1, 3]$, sort with respect to the second component, and when these are equal, with respect to the first, and when these are equal, with respect to the third.

• a comparison function (`t_CLOSURE`), with two arguments x and y, and returning an integer which is < 0, > 0 or $= 0$ if $x < y$, $x > y$ or $x = y$ respectively. The `sign` function is very useful in this context:

```
? vecsort([3,0,2; 1,0,2]) \\ sort columns according to lex order
%1 =
[0 2 3]

[0 2 1]
? vecsort(v, (x,y)->sign(y-x))            \\ reverse sort
? vecsort(v, (x,y)->sign(abs(x)-abs(y)))  \\ sort by increasing absolute value
? cmpf(x,y) = my(dx = poldisc(x), dy = poldisc(y)); sign(abs(dx) - abs(dy))
? vecsort([x^2+1, x^3-2, x^4+5*x+1], cmpf)
```

The last example used the named `cmpf` instead of an anonymous function, and sorts polynomials with respect to the absolute value of their discriminant. A more efficient approach would use precomputations to ensure a given discriminant is computed only once:

```
? DISC = vector(#v, i, abs(poldisc(v[i])));
? perm = vecsort(vector(#v,i,i), (x,y)->sign(DISC[x]-DISC[y]))
? vecextract(v, perm)
```

Similar ideas apply whenever we sort according to the values of a function which is expensive to compute.

The binary digits of *flag* mean:

• 1: indirect sorting of the vector x, i.e. if x is an n-component vector, returns a permutation of $[1, 2, \ldots, n]$ which applied to the components of x sorts x in increasing order. For example, `vecextract(x, vecsort(x,,1))` is equivalent to `vecsort(x)`.

• 4: use descending instead of ascending order.

• 8: remove "duplicate" entries with respect to the sorting function (keep the first occurring entry). For example:

```
? vecsort([Pi,Mod(1,2),z], (x,y)->0, 8)    \\ make everything compare equal
%1 = [3.1415926535897932384626433832]
? vecsort([[2,3],[0,1],[0,3]], 2, 8)
%2 = [[0, 1], [2, 3]]
```

The library syntax is `GEN vecsort0(GEN x, GEN cmpf = NULL, long flag)`.

3.11.74 vecsum(v). Return the sum of the components of the vector v. Return 0 on an empty vector.

```
? vecsum([1,2,3])
%1 = 6
? vecsum([])
%2 = 0
```

The library syntax is `GEN vecsum(GEN v)`.

3.11.75 vector$(n, \{X\}, \{expr = 0\})$. Creates a row vector (type `t_VEC`) with n components whose components are the expression *expr* evaluated at the integer points between 1 and n. If one of the last two arguments is omitted, fill the vector with zeroes.

```
? vector(3,i, 5*i)
%1 = [5, 10, 15]
? vector(3)
%2 = [0, 0, 0]
```

The variable X is lexically scoped to each evaluation of *expr*. Any change to X within *expr* does not affect subsequent evaluations, it still runs 1 to n. A local change allows for example different indexing:

```
vector(10, i, i=i-1; f(i)) \\ i = 0, ..., 9
vector(10, i, i=2*i; f(i)) \\ i = 2, 4, ..., 20
```

This per-element scope for X differs from `for` loop evaluations, as the following example shows:

```
n = 3
v = vector(n); vector(n, i, i++)            ----> [2, 3, 4]
v = vector(n); for (i = 1, n, v[i] = i++)   ----> [2, 0, 4]
```

3.11.76 vectorsmall$(n, \{X\}, \{expr = 0\})$. Creates a row vector of small integers (type `t_VECSMALL`) with n components whose components are the expression *expr* evaluated at the integer points between 1 and n. If one of the last two arguments is omitted, fill the vector with zeroes.

3.11.77 vectorv$(n, \{X\}, \{expr = 0\})$. As `vector`, but returns a column vector (type `t_COL`).

3.12 Sums, products, integrals and similar functions.

Although the `gp` calculator is programmable, it is useful to have a number of preprogrammed loops, including sums, products, and a certain number of recursions. Also, a number of functions from numerical analysis like numerical integration and summation of series will be described here.

One of the parameters in these loops must be the control variable, hence a simple variable name. In the descriptions, the letter X will always denote any simple variable name, and represents the formal parameter used in the function. The expression to be summed, integrated, etc. is any legal PARI expression, including of course expressions using loops.

Library mode. Since it is easier to program directly the loops in library mode, these functions are mainly useful for GP programming. On the other hand, numerical routines code a function (to be integrated, summed, etc.) with two parameters named

```
GEN (*eval)(void*,GEN)
void *E;  \\ context: eval(E, x) must evaluate your function at x.
```

see the Libpari manual for details.

Numerical integration. Starting with version 2.2.9 the "double exponential" univariate integration method is implemented in `intnum` and its variants. Romberg integration is still available under the name `intnumromb`, but superseded. It is possible to compute numerically integrals to thousands of decimal places in reasonable time, as long as the integrand is regular. It is also reasonable to compute numerically integrals in several variables, although more than two becomes lengthy. The integration domain may be non-compact, and the integrand may have reasonable singularities at endpoints. To use `intnum`, you must split the integral into a sum of subintegrals where the function has no singularities except at the endpoints. Polynomials in logarithms are not considered singular, and neglecting these logs, singularities are assumed to be algebraic (asymptotic to $C(x-a)^{-\alpha}$ for some $\alpha > -1$ when x is close to a), or to correspond to simple discontinuities of some (higher) derivative of the function. For instance, the point 0 is a singularity of $\mathrm{abs}(x)$.

See also the discrete summation methods below, sharing the prefix `sum`.

3.12.1 asympnum$(expr, \{k = 20\}, \{alpha = 1\})$. Asymptotic expansion of *expr*, corresponding to a sequence $u(n)$, assuming it has the shape

$$u(n) \approx \sum_{i \geq 0} a_i n^{-i\alpha}$$

with rational coefficients a_i with reasonable height; the algorithm is heuristic and performs repeated calls to limitnum, with `k` and `alpha` are as in `limitnum`

```
? f(n) = n! / (n^n*exp(-n)*sqrt(n));
? asympnum(f)
%2 = []   \\ failure !
? l = limitnum(f)
%3 = 2.5066282746310005024157652848110452530
? asympnum(n->f(n)/l) \\ normalize
%4 = [1, 1/12, 1/288, -139/51840]
```

and we indeed get a few terms of Stirling's expansion. Note that it helps to normalize with a limit computed to higher accuracy:

```
? \p100
? L = limitnum(f)
? \p38
? asympnum(n->f(n)/L) \\ we get more terms!
%6 = [1, 1/12, 1/288, -139/51840, -571/2488320, 163879/209018880,\
      5246819/75246796800, -534703531/902961561600]
```

If `alpha` is not an integer, loss of accuracy is expected, so it should be precomputed to double accuracy, say:

```
? \p38
? asympnum(n->-log(1-1/n^Pi),,Pi)
%1 = [0, 1, 1/2, 1/3]
? asympnum(n->-log(1-1/sqrt(n)),,1/2)
%2 = [0, 1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/11, 1/12, \
  1/13, 1/14, 1/15, 1/16, 1/17, 1/18, 1/19, 1/20, 1/21, 1/22]
? localprec(100); a = Pi;
? asympnum(n->-log(1-1/n^a),,a) \\ better !
```

```
%4 = [0, 1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/11, 1/12]
```

The library syntax is **asympnum**(`void *E, GEN (*u)(void *,GEN,long), long muli, GEN alpha, long prec`), where `u(E, n, prec)` must return $u(n)$ in precision `prec`. Also available is `GEN asympnum0(GEN u, long muli, GEN alpha, long prec)`, where u must be a vector of sufficient length as above.

3.12.2 contfraceval$(CF, t, \{lim = -1\})$. Given a continued fraction `CF` output by `contfracinit`, evaluate the first `lim` terms of the continued fraction at `t` (all terms if `lim` is negative or omitted; if positive, `lim` must be less than or equal to the length of `CF`.

The library syntax is `GEN contfraceval(GEN CF, GEN t, long lim)`.

3.12.3 contfracinit$(M, \{lim = -1\})$. Given M representing the power series $S = \sum_{n\geq 0} M[n+1]z^n$, transform it into a continued fraction; restrict to $n \leq$ `lim` if latter is non-negative. M can be a vector, a power series, a polynomial, or a rational function. The result is a 2-component vector $[A, B]$ such that $S = M[1]/(1 + A[1]z + B[1]z^2/(1 + A[2]z + B[2]z^2/(1 + ...1/(1 + A[lim/2]z))))$. Does not work if any coefficient of M vanishes, nor for series for which certain partial denominators vanish.

The library syntax is `GEN contfracinit(GEN M, long lim)`.

3.12.4 derivnum$(X = a, expr)$. Numerical derivation of *expr* with respect to X at $X = a$.

```
? derivnum(x=0,sin(exp(x))) - cos(1)
%1 = -1.262177448 E-29
```

A clumsier approach, which would not work in library mode, is

```
? f(x) = sin(exp(x))
? f'(0) - cos(1)
%1 = -1.262177448 E-29
```

When a is a power series, compute `derivnum(t=a,f)` as $f'(a) = (f(a))'/a'$.

The library syntax is **derivnum**(`void *E, GEN (*eval)(void*,GEN), GEN a, long prec`). Also available is `GEN derivfun(void *E, GEN (*eval)(void *, GEN), GEN a, long prec)`, which also allows power series for a.

3.12.5 intcirc$(X = a, R, expr, \{tab\})$. Numerical integration of $(2i\pi)^{-1} expr$ with respect to X on the circle $|X - a| = R$. In other words, when *expr* is a meromorphic function, sum of the residues in the corresponding disk; *tab* is as in `intnum`, except that if computed with `intnuminit` it should be with the endpoints `[-1, 1]`.

```
? \p105
? intcirc(s=1, 0.5, zeta(s)) - 1
time = 496 ms.
%1 = 1.2883911040127271720 E-101 + 0.E-118*I
```

The library syntax is **intcirc**(`void *E, GEN (*eval)(void*,GEN), GEN a,GEN R,GEN tab, long prec`).

3.12.6 intfuncinit$(t = a, b, f, \{m = 0\})$. Initialize tables for use with integral transforms such Fourier, Laplace or Mellin transforms, in order to compute

$$\int_a^b f(t)k(t,z)\,dt$$

for some kernel $k(t,z)$. The endpoints a and b are coded as in `intnum`, f is the function to which the integral transform is to be applied and the non-negative integer m is as in `intnum`: multiply the number of sampling points roughly by 2^m, hopefully increasing the accuracy. This function is particularly useful when the function f is hard to compute, such as a gamma product.

Limitation. the endpoints a and b must be at infinity, with the same asymptotic behaviour. Oscillating types are not supported. This is easily overcome by integrating vectors of functions, see example below.

Examples.

- numerical Fourier transform

$$F(z) = \int_{-\infty}^{+\infty} f(t)e^{-2i\pi zt}\,dt.$$

First the easy case, assume that f decrease exponentially:

```
f(t) = exp(-t^2);
A = [-oo,1];
B = [+oo,1];
\p200
T = intfuncinit(t = A,B , f(t));
F(z) =
{ my(a = -2*I*Pi*z);
  intnum(t = A,B, exp(a*t), T);
}
? F(1) - sqrt(Pi)*exp(-Pi^2)
%1 = -1.3... E-212
```

Now the harder case, f decrease slowly: we must specify the oscillating behaviour. Thus, we cannot precompute usefully since everything depends on the point we evaluate at:

```
f(t) = 1 / (1+ abs(t));
\p200
\\ Fourier cosine transform
FC(z) =
{ my(a = 2*Pi*z);
  intnum(t = [-oo, a*I], [+oo, a*I], cos(a*t)*f(t));
}
FC(1)
```

- Fourier coefficients: we must integrate over a period, but `intfuncinit` does not support finite endpoints. The solution is to integrate a vector of functions !

```
FourierSin(f, T, k) =  \\ first k sine Fourier coeffs
{
```

```
  my (w = 2*Pi/T);
  my (v = vector(k+1));
  intnum(t = -T/2, T/2,
     my (z = exp(I*w*t));
     v[1] = z;
     for (j = 2, k, v[j] = v[j-1]*z);
     f(t) * imag(v)) * 2/T;
}
FourierSin(t->sin(2*t), 2*Pi, 10)
```

The same technique can be used instead of `intfuncinit` to integrate $f(t)k(t,z)$ whenever the list of z-values is known beforehand.

Note that the above code includes an unrelated optimization: the $\sin(jwt)$ are computed as imaginary parts of $\exp(ijwt)$ and the latter by successive multiplications.

- numerical Mellin inversion

$$F(z) = (2i\pi)^{-1} \int_{c-i\infty}^{c+i\infty} f(s) z^{-s}\, ds = (2\pi)^{-1} \int_{-\infty}^{+\infty} f(c+it) e^{-\log z(c+it)}\, dt.$$

We take $c = 2$ in the program below:

```
f(s) = gamma(s)^3;  \\ f(c+it) decrease as exp(-3Pi|t|/2)
c = 2; \\ arbitrary
A = [-oo,3*Pi/2];
B = [+oo,3*Pi/2];
T = intfuncinit(t=A,B, f(c + I*t));
F(z) =
{ my (a = -log(z));
  intnum(t=A,B, exp(a*I*t), T)*exp(a*c) / (2*Pi);
}
```

The library syntax is **intfuncinit**(`void *E, GEN (*eval)(void*,GEN), GEN a,GEN b,long m, long prec`).

3.12.7 intnum$(X = a, b, expr, \{tab\})$. Numerical integration of *expr* on $]a,b[$ with respect to X, using the double-exponential method, and thus $O(D \log D)$ evaluation of the integrand in precision D. The integrand may have values belonging to a vector space over the real numbers; in particular, it can be complex-valued or vector-valued. But it is assumed that the function is regular on $]a,b[$. If the endpoints a and b are finite and the function is regular there, the situation is simple:

```
? intnum(x = 0,1, x^2)
%1 = 0.33333333333333333333333333333
? intnum(x = 0,Pi/2, [cos(x), sin(x)])
%2 = [1.0000000000000000000000000000, 1.0000000000000000000000000000]
```

An endpoint equal to $\pm\infty$ is coded as `+oo` or `-oo`, as expected:

```
? intnum(x = 1,+oo, 1/x^2)
%3 = 1.0000000000000000000000000000
```

In basic usage, it is assumed that the function does not decrease exponentially fast at infinity:

```
? intnum(x=0,+oo, exp(-x))
  ***   at top-level: intnum(x=0,+oo,exp(-
  ***                 ^--------------------
  *** exp: overflow in expo().
```

We shall see in a moment how to avoid that last problem, after describing the last *optional* argument *tab*.

The *tab* argument. The routine uses weights w_i, which are mostly independent of the function being integrated, evaluated at many sampling points x_i and approximates the integral by $\sum w_i f(x_i)$. If *tab* is

• a non-negative integer m, we multiply the number of sampling points by 2^m, hopefully increasing accuracy. Note that the running time increases roughly by a factor 2^m. One may try consecutive values of m until they give the same value up to an accepted error.

• a set of integration tables containing precomputed x_i and w_i as output by `intnuminit`. This is useful if several integrations of the same type are performed (on the same kind of interval and functions, for a given accuracy): we skip a precomputation of $O(D \log D)$ elementary functions in accuracy D, whose running time has the same order of magnitude as the evaluation of the integrand. This is in particular usefule for multivariate integrals.

Specifying the behavior at endpoints. This is done as follows. An endpoint a is either given as such (a scalar, real or complex, `oo` or `-oo` for $\pm\infty$), or as a two component vector $[a, \alpha]$, to indicate the behavior of the integrand in a neighborhood of a.

If a is finite, the code $[a, \alpha]$ means the function has a singularity of the form $(x-a)^\alpha$, up to logarithms. (If $\alpha \geq 0$, we only assume the function is regular, which is the default assumption.) If a wrong singularity exponent is used, the result will lose a catastrophic number of decimals:

```
? intnum(x=0, 1, x^(-1/2))          \\ assume x^(-1/2) is regular at 0
%1 = 1.99999999999999999999999999827931660
? intnum(x=[0,-1/2], 1, x^(-1/2))   \\ no, it's not
%2 = 2.00000000000000000000000000000000000
? intnum(x=[0,-1/10], 1, x^(-1/2)) \\ using a wrong exponent is bad
%3 = 1.99999999999999999999999999999901912
```

If a is $\pm\infty$, which is coded as `+oo` or `-oo`, the situation is more complicated, and $[\pm\infty, \alpha]$ means:

• $\alpha = 0$ (or no α at all, i.e. simply $\pm\infty$) assumes that the integrand tends to zero moderately quickly, at least as $O(x^{-2})$ but not exponentially fast.

• $\alpha > 0$ assumes that the function tends to zero exponentially fast approximately as $\exp(-\alpha x)$. This includes oscillating but quickly decreasing functions such as $\exp(-x)\sin(x)$.

```
? intnum(x=0, +oo, exp(-2*x))
  ***   at top-level: intnum(x=0,+oo,exp(-
  ***                 ^--------------------
  *** exp: exponent (expo) overflow
? intnum(x=0, [+oo, 2], exp(-2*x))  \\ OK!
%1 = 0.50000000000000000000000000000000000
? intnum(x=0, [+oo, 3], exp(-2*x))  \\ imprecise exponent, still OK !
%2 = 0.50000000000000000000000000000000000
```

```
? intnum(x=0, [+oo, 10], exp(-2*x)) \\ wrong exponent ⇒ disaster
%3 = 0.49999999999952372962457451698256707393
```

As the last exemple shows, the exponential decrease rate *must* be indicated to avoid overflow, but the method is robust enough for a rough guess to be acceptable.

• $\alpha < -1$ assumes that the function tends to 0 slowly, like x^α. Here the algorithm is less robust and it is essential to give a sharp α, unless $\alpha \leq -2$ in which case we use the default algorithm as if α were missing (or equal to 0).

```
? intnum(x=1, +oo, x^(-3/2))          \\ default
%1 = 1.99999999999999999999999999999646391207
? intnum(x=1, [+oo,-3/2], x^(-3/2))  \\ precise decrease rate
%2 = 2.0000000000000000000000000000000000000
? intnum(x=1, [+oo,-11/10], x^(-3/2)) \\ worse than default
%3 = 2.0000000000000000000000000089298011973
```

The last two codes are reserved for oscillating functions. Let $k > 0$ real, and $g(x)$ a non-oscillating function tending slowly to 0 (e.g. like a negative power of x), then

• $\alpha = k * I$ assumes that the function behaves like $\cos(kx)g(x)$.

• $\alpha = -k * I$ assumes that the function behaves like $\sin(kx)g(x)$.

Here it is critical to give the exact value of k. If the oscillating part is not a pure sine or cosine, one must expand it into a Fourier series, use the above codings, and sum the resulting contributions. Otherwise you will get nonsense. Note that $\cos(kx)$, and similarly $\sin(kx)$, means that very function, and not a translated version such as $\cos(kx + a)$.

Note. If $f(x) = \cos(kx)g(x)$ where $g(x)$ tends to zero exponentially fast as $\exp(-\alpha x)$, it is up to the user to choose between $[\pm\mathtt{oo}, \alpha]$ and $[\pm\mathtt{oo}, k * I]$, but a good rule of thumb is that if the oscillations are weaker than the exponential decrease, choose $[\pm\mathtt{oo}, \alpha]$, otherwise choose $[\pm\mathtt{oo}, k * I]$, although the latter can reasonably be used in all cases, while the former cannot. To take a specific example, in the inverse Mellin transform, the integrand is almost always a product of an exponentially decreasing and an oscillating factor. If we choose the oscillating type of integral we perhaps obtain the best results, at the expense of having to recompute our functions for a different value of the variable z giving the transform, preventing us to use a function such as `intfuncinit`. On the other hand using the exponential type of integral, we obtain less accurate results, but we skip expensive recomputations. See `intfuncinit` for more explanations.

We shall now see many examples to get a feeling for what the various parameters achieve. All examples below assume precision is set to 115 decimal digits. We first type

```
? \p 115
```

Apparent singularities. In many cases, apparent singularities can be ignored. For instance, if $f(x) = 1/(\exp(x) - 1) - \exp(-x)/x$, then $\int_0^\infty f(x)\,dx = \gamma$, Euler's constant `Euler`. But

```
? f(x) = 1/(exp(x)-1) - exp(-x)/x
? intnum(x = 0, [oo,1],  f(x)) - Euler
%1 = 0.E-115
```

But close to 0 the function f is computed with an enormous loss of accuracy, and we are in fact lucky that it get multiplied by weights which are sufficiently close to 0 to hide this:

```
? f(1e-200)
%2 = -3.885337784451458142 E84
```

A more robust solution is to define the function differently near special points, e.g. by a Taylor expansion

```
? F = truncate( f(t + O(t^10)) ); \\ expansion around t = 0
? poldegree(F)
%4 = 7
? g(x) = if (x > 1e-18, f(x), subst(F,t,x)); \\ note that 7 · 18 > 105
? intnum(x = 0, [oo,1],  g(x)) - Euler
%2 = 0.E-115
```

It is up to the user to determine constants such as the 10^{-18} and 10 used above.

True singularities. With true singularities the result is worse. For instance

```
? intnum(x = 0, 1,  x^(-1/2)) - 2
%1 = -3.5... E-68 \\ only 68 correct decimals
? intnum(x = [0,-1/2], 1,  x^(-1/2)) - 2
%2 = 0.E-114 \\ better
```

Oscillating functions.

```
? intnum(x = 0, oo, sin(x) / x) - Pi/2
%1 = 16.19.. \\ nonsense
? intnum(x = 0, [oo,1], sin(x)/x) - Pi/2
%2 = -0.006.. \\ bad
? intnum(x = 0, [oo,-I], sin(x)/x) - Pi/2
%3 = 0.E-115 \\ perfect
? intnum(x = 0, [oo,-I], sin(2*x)/x) - Pi/2  \\ oops, wrong k
%4 = 0.06...
? intnum(x = 0, [oo,-2*I], sin(2*x)/x) - Pi/2
%5 = 0.E-115 \\ perfect
? intnum(x = 0, [oo,-I], sin(x)^3/x) - Pi/4
%6 = -0.0008... \\ bad
? sin(x)^3 - (3*sin(x)-sin(3*x))/4
%7 = O(x^17)
```

We may use the above linearization and compute two oscillating integrals with endpoints `[oo, -I]` and `[oo, -3*I]` respectively, or notice the obvious change of variable, and reduce to the single integral $\frac{1}{2}\int_0^\infty \sin(x)/x\,dx$. We finish with some more complicated examples:

```
? intnum(x = 0, [oo,-I], (1-cos(x))/x^2) - Pi/2
```

```
%1 = -0.0003... \\ bad
? intnum(x = 0, 1, (1-cos(x))/x^2) \
+ intnum(x = 1, oo, 1/x^2) - intnum(x = 1, [oo,I], cos(x)/x^2) - Pi/2
%2 = 0.E-115 \\ perfect
? intnum(x = 0, [oo, 1], sin(x)^3*exp(-x)) - 0.3
%3 = -7.34... E-55 \\ bad
? intnum(x = 0, [oo,-I], sin(x)^3*exp(-x)) - 0.3
%4 = 8.9... E-103 \\ better. Try higher m
? tab = intnuminit(0,[oo,-I], 1); \\ double number of sampling points
? intnum(x = 0, oo, sin(x)^3*exp(-x), tab) - 0.3
%6 = 0.E-115 \\ perfect
```

Warning. Like `sumalt`, `intnum` often assigns a reasonable value to diverging integrals. Use these values at your own risk! For example:

```
? intnum(x = 0, [oo, -I], x^2*sin(x))
%1 = -2.0000000000...
```

Note the formula

$$\int_0^\infty \sin(x)/x^s\,dx = \cos(\pi s/2)\Gamma(1-s)\,,$$

a priori valid only for $0 < \Re(s) < 2$, but the right hand side provides an analytic continuation which may be evaluated at $s = -2\ldots$

Multivariate integration. Using successive univariate integration with respect to different formal parameters, it is immediate to do naive multivariate integration. But it is important to use a suitable `intnuminit` to precompute data for the *internal* integrations at least!

For example, to compute the double integral on the unit disc $x^2 + y^2 \leq 1$ of the function $x^2 + y^2$, we can write

```
? tab = intnuminit(-1,1);
? intnum(x=-1,1, intnum(y=-sqrt(1-x^2),sqrt(1-x^2), x^2+y^2, tab),tab) - Pi/2
%2 = -7.1... E-115 \\ OK
```

The first *tab* is essential, the second optional. Compare:

```
? tab = intnuminit(-1,1);
time = 4 ms.
? intnum(x=-1,1, intnum(y=-sqrt(1-x^2),sqrt(1-x^2), x^2+y^2));
time = 3,092 ms. \\ slow
? intnum(x=-1,1, intnum(y=-sqrt(1-x^2),sqrt(1-x^2), x^2+y^2, tab), tab);
time = 252 ms.  \\ faster
? intnum(x=-1,1, intnum(y=-sqrt(1-x^2),sqrt(1-x^2), x^2+y^2, tab));
time = 261 ms.  \\ the internal integral matters most
```

The library syntax is **intnum**(`void *E, GEN (*eval)(void*,GEN), GEN a,GEN b,GEN tab, long prec`), where an omitted *tab* is coded as `NULL`.

3.12.8 intnumgauss$(X = a, b, expr, \{tab\})$. Numerical integration of *expr* on the compact interval $[a, b]$ with respect to X using Gauss-Legendre quadrature; `tab` is either omitted or precomputed with `intnumgaussinit`. As a convenience, it can be an integer n in which case we call `intnumgaussinit`(n) and use n-point quadrature.

```
? test(n, b = 1) = T=intnumgaussinit(n);\
    intnumgauss(x=-b,b, 1/(1+x^2),T) - 2*atan(b);
? test(0) \\ default
%1 = -9.490148553624725335 E-22
? test(40)
%2 = -6.186629001816965717 E-31
? test(50)
%3 = -1.1754943508222875080 E-38
? test(50, 2) \\ double interval length
%4 = -4.891779568527713636 E-21
? test(90, 2) \\ n must almost be doubled as well!
%5 = -9.403954806578300064 E-38
```

On the other hand, we recommend to split the integral and change variables rather than increasing n too much:

```
? f(x) = 1/(1+x^2);
? b = 100;
? intnumgauss(x=0,1, f(x)) + intnumgauss(x=1,1/b, f(1/x)*(-1/x^2)) - atan(b)
%3 = -1.0579449157400587572 E-37
```

The library syntax is `GEN intnumgauss0(GEN X, GEN b, GEN expr, GEN tab = NULL, long prec)`.

3.12.9 intnumgaussinit$(\{n\})$. Initialize tables for n-point Gauss-Legendre integration of a smooth function f lon a compact interval $[a, b]$ at current `realprecision`. If n is omitted, make a default choice $n \approx$ `realprecision`, suitable for analytic functions on $[-1, 1]$. The error is bounded by

$$\frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi), \qquad a < \xi < b$$

so, if the interval length increases, n should be increased as well.

```
? T = intnumgaussinit();
? intnumgauss(t=-1,1,exp(t), T) - exp(1)+exp(-1)
%1 = -5.877471754111437540 E-39
? intnumgauss(t=-10,10,exp(t), T) - exp(10)+exp(-10)
%2 = -8.358367809712546836 E-35
? intnumgauss(t=-1,1,1/(1+t^2), T) - Pi/2
%3 = -9.490148553624725335 E-22
? T = intnumgaussinit(50);
? intnumgauss(t=-1,1,1/(1+t^2), T) - Pi/2
%5 = -1.1754943508222875080 E-38
? intnumgauss(t=-5,5,1/(1+t^2), T) - 2*atan(5)
%6 = -1.2[...]E-8
```

On the other hand, we recommend to split the integral and change variables rather than increasing n too much, see `intnumgauss`.

The library syntax is `GEN intnumgaussinit(long n, long prec)`.

3.12.10 intnuminit$(a, b, \{m = 0\})$. Initialize tables for integration from a to b, where a and b are coded as in `intnum`. Only the compactness, the possible existence of singularities, the speed of decrease or the oscillations at infinity are taken into account, and not the values. For instance `intnuminit(-1,1)` is equivalent to `intnuminit(0,Pi)`, and `intnuminit([0,-1/2],oo)` is equivalent to `intnuminit([-1,-1/2], -oo)`; on the other hand, the order matters and `intnuminit([0,-1/2], [1,-1/3])` is *not* equivalent to `intnuminit([0,-1/3], [1,-1/2])` !

If m is present, it must be non-negative and we multiply the default number of sampling points by 2^m (increasing the running time by a similar factor).

The result is technical and liable to change in the future, but we document it here for completeness. Let $x = \phi(t)$, $t \in]-\infty, \infty[$ be an internally chosen change of variable, achieving double exponential decrease of the integrand at infinity. The integrator `intnum` will compute

$$h \sum_{|n|<N} \phi'(nh) F(\phi(nh))$$

for some integration step h and truncation parameter N. In basic use, let

```
[h, x0, w0, xp, wp, xm, wm] = intnuminit(a,b);
```

- h is the integration step
- $x_0 = \phi(0)$ and $w_0 = \phi'(0)$,
- *xp* contains the $\phi(nh)$, $0 < n < N$,
- *xm* contains the $\phi(nh)$, $0 < -n < N$, or is empty.
- *wp* contains the $\phi'(nh)$, $0 < n < N$,
- *wm* contains the $\phi'(nh)$, $0 < -n < N$, or is empty.

The arrays *xm* and *wm* are left empty when ϕ is an odd function. In complicated situations when non-default behaviour is specified at end points, `intnuminit` may return up to 3 such arrays, corresponding to a splitting of up to 3 integrals of basic type.

If the functions to be integrated later are of the form $F = f(t)k(t, z)$ for some kernel k (e.g. Fourier, Laplace, Mellin, ...), it is useful to also precompute the values of $f(\phi(nh))$, which is accomplished by `intfuncinit`. The hard part is to determine the behaviour of F at endpoints, depending on z.

The library syntax is `GEN intnuminit(GEN a, GEN b, long m, long prec)`.

3.12.11 intnumromb$(X = a, b, expr, \{flag = 0\})$. Numerical integration of *expr* (smooth in $]a, b[$), with respect to X. Suitable for low accuracy; if *expr* is very regular (e.g. analytic in a large region) and high accuracy is desired, try `intnum` first.

Set *flag* $= 0$ (or omit it altogether) when a and b are not too large, the function is smooth, and can be evaluated exactly everywhere on the interval $[a, b]$.

If *flag* $= 1$, uses a general driver routine for doing numerical integration, making no particular assumption (slow).

flag $= 2$ is tailored for being used when a or b are infinite using the change of variable $t = 1/X$. One *must* have $ab > 0$, and in fact if for example $b = +\infty$, then it is preferable to have a as large as possible, at least $a \geq 1$.

If *flag* $= 3$, the function is allowed to be undefined at a (but right continuous) or b (left continuous), for example the function $\sin(x)/x$ between $x = 0$ and 1.

The user should not require too much accuracy: `realprecision` about 30 decimal digits (`realbitprecision` about 100 bits) is OK, but not much more. In addition, analytical cleanup of the integral must have been done: there must be no singularities in the interval or at the boundaries. In practice this can be accomplished with a change of variable. Furthermore, for improper integrals, where one or both of the limits of integration are plus or minus infinity, the function must decrease sufficiently rapidly at infinity, which can often be accomplished through integration by parts. Finally, the function to be integrated should not be very small (compared to the current precision) on the entire interval. This can of course be accomplished by just multiplying by an appropriate constant.

Note that infinity can be represented with essentially no loss of accuracy by an appropriate huge number. However beware of real underflow when dealing with rapidly decreasing functions. For example, in order to compute the $\int_0^\infty e^{-x^2}\,dx$ to 28 decimal digits, then one can set infinity equal to 10 for example, and certainly not to `1e1000`.

The library syntax is **intnumromb'bitprec**(`void *E, GEN (*eval)(void*,GEN), GEN a, GEN b, long flag, long bitprec`), where `eval`(x, E) returns the value of the function at x. You may store any additional information required by `eval` in E, or set it to `NULL`. The historical variant The library syntax is **intnumromb**(`..., long prec`), where `prec` is expressed in words, not bits, is obsolete and should no longer be used.

3.12.12 limitnum$(expr, \{k = 20\}, \{alpha = 1\})$. Lagrange-Zagier numerical extrapolation of *expr*, corresponding to a sequence u_n, either given by a closure `n->u(n)` or by a vector of values I.e., assuming that u_n tends to a finite limit ℓ, try to determine ℓ. This routine is purely numerical and heuristic, thus may or may not work on your examples; k is ignored if u is given by a vector, and otherwise is a multiplier such that we extrapolate from $u(kn)$.

Assume that u_n has an asymptotic expansion in $n^{-\alpha}$:

$$u_n = \ell + \sum_{i \geq 1} a_i n^{-i\alpha}$$

for some a_i.

```
? limitnum(n -> n*sin(1/n))
%1 = 1.0000000000000000000000000000000000000
? limitnum(n -> (1+1/n)^n) - exp(1)
```

```
%2 = 0.E-37

? limitnum(n -> 2^(4*n+1)*(n!)^4 / (2*n)! /(2*n+1)! )
%3 = 3.14159265358979323846264338327950288842
? Pi
%4 = 3.14159265358979323846264338327950288842
```

If u_n is given by a vector, it must be long enough for the extrapolation to make sense: at least k times the current `realprecision`. The preferred format is thus a closure, although it becomes inconvenient when u_n cannot be directly computed in time polynomial in $\log n$, for instance if it is defined as a sum or by induction. In that case, passing a vector of values is the best option. It usually pays off to interpolate $u(kn)$ for some $k > 1$:

```
? limitnum(vector(10,n,(1+1/n)^n))
 ***                  ^--------------------
 *** limitnum: non-existent component in limitnum: index < 20
\\ at this accuracy, we must have at least 20 values
? limitnum(vector(20,n,(1+1/n)^n)) - exp(1)
%5 = -2.05... E-20
? limitnum(vector(20,n, m=10*n;(1+1/m)^m)) - exp(1) \\ better accuracy
%6 = 0.E-37

? v = vector(20); s = 0;
? for(i=1,#v, s += 1/i; v[i]= s - log(i));
? limitnum(v) - Euler
%9 = -1.6... E-19

? V = vector(200); s = 0;
? for(i=1,#V, s += 1/i; V[i]= s);
? v = vector(#V \ 10, i, V[10*i] - log(10*i));
? limitnum(v) - Euler
%13 = 6.43... E-29
```

The library syntax is **limitnum**(`void *E, GEN (*u)(void *,GEN,long), long muli, GEN alpha, long prec`), where `u(E, n, prec)` must return $u(n)$ in precision `prec`. Also available is `GEN limitnum0(GEN u, long muli, GEN alpha, long prec)`, where u must be a vector of sufficient length as above.

3.12.13 prod$(X = a, b, expr, \{x = 1\})$. Product of expression *expr*, initialized at x, the formal parameter X going from a to b. As for `sum`, the main purpose of the initialization parameter x is to force the type of the operations being performed. For example if it is set equal to the integer 1, operations will start being done exactly. If it is set equal to the real 1., they will be done using real numbers having the default precision. If it is set equal to the power series $1 + O(X^k)$ for a certain k, they will be done using power series of precision at most k. These are the three most common initializations.

As an extreme example, compare

```
? prod(i=1, 100, 1 - X^i);  \\ this has degree 5050 !!
time = 128 ms.
? prod(i=1, 100, 1 - X^i, 1 + O(X^101))
time = 8 ms.
%2 = 1 - X - X^2 + X^5 + X^7 - X^12 - X^15 + X^22 + X^26 - X^35 - X^40 + \
```

```
X^51 + X^57 - X^70 - X^77 + X^92 + X^100 + O(X^101)
```

Of course, in this specific case, it is faster to use `eta`, which is computed using Euler's formula.

```
? prod(i=1, 1000, 1 - X^i, 1 + O(X^1001));
time = 589 ms.
? \ps1000
seriesprecision = 1000 significant terms
? eta(X) - %
time = 8ms.
%4 = O(X^1001)
```

The library syntax is **produit**(`GEN a, GEN b, char *expr, GEN x`).

3.12.14 prodeuler$(X = a, b, expr)$. Product of expression *expr*, initialized at 1. (i.e. to a *real* number equal to 1 to the current `realprecision`), the formal parameter X ranging over the prime numbers between a and b.

The library syntax is **prodeuler**(`void *E, GEN (*eval)(void*,GEN), GEN a,GEN b, long prec`).

3.12.15 prodinf$(X = a, expr, \{flag = 0\})$. infinite product of expression *expr*, the formal parameter X starting at a. The evaluation stops when the relative error of the expression minus 1 is less than the default precision. In particular, non-convergent products result in infinite loops. The expressions must always evaluate to an element of **C**.

If $flag = 1$, do the product of the $(1 + expr)$ instead.

The library syntax is **prodinf**(`void *E, GEN (*eval)(void*,GEN), GEN a, long prec`) ($flag = 0$), or `prodinf1` with the same arguments ($flag = 1$).

3.12.16 solve$(X = a, b, expr)$. Find a real root of expression *expr* between a and b, under the condition $expr(X = a) * expr(X = b) \leq 0$. (You will get an error message `roots must be bracketed in solve` if this does not hold.) This routine uses Brent's method and can fail miserably if *expr* is not defined in the whole of $[a, b]$ (try `solve(x=1, 2, tan(x))`).

The library syntax is **zbrent**(`void *E,GEN (*eval)(void*,GEN),GEN a,GEN b,long prec`).

3.12.17 solvestep$(X = a, b, step, expr, \{flag = 0\})$. Find zeros of a continuous function in the real interval $[a, b]$ by naive interval splitting. This function is heuristic and may or may not find the intended zeros. Binary digits of *flag* mean

- 1: return as soon as one zero is found, otherwise return all zeros found;
- 2: refine the splitting until at least one zero is found (may loop indefinitely if there are no zeros);
- 4: do a multiplicative search (we must have $a > 0$ and $step > 1$), otherwise an additive search; *step* is the multiplicative or additive step.
- 8: refine the splitting until at least one zero is very close to an integer.

```
? solvestep(X=0,10,1,sin(X^2),1)
%1 = 1.772453850905516027298167483341145182
? solvestep(X=1,12,2,besselj(4,X),4)
```

```
%2 = [7.588342434..., 11.064709488...]
```

The library syntax is **solvestep**(`void *E, GEN (*eval)(void*,GEN), GEN a,GEN b, GEN step,long flag,long prec`).

3.12.18 sum$(X = a, b, expr, \{x = 0\})$. Sum of expression *expr*, initialized at x, the formal parameter going from a to b. As for `prod`, the initialization parameter x may be given to force the type of the operations being performed.

As an extreme example, compare

```
? sum(i=1, 10^4, 1/i); \\ rational number: denominator has 4345 digits.
time = 236 ms.
? sum(i=1, 5000, 1/i, 0.)
time = 8 ms.
%2 = 9.787606036044382264178477904
```

The library syntax is **somme**(`GEN a, GEN b, char *expr, GEN x`).

3.12.19 sumalt$(X = a, expr, \{flag = 0\})$. Numerical summation of the series *expr*, which should be an alternating series $(-1)^k a_k$, the formal variable X starting at a. Use an algorithm of Cohen, Villegas and Zagier (*Experiment. Math.* **9** (2000), no. 1, 3–12).

If $flag = 0$, assuming that the a_k are the moments of a positive measure on $[0, 1]$, the relative error is $O(3 + \sqrt{8})^{-n}$ after using a_k for $k \leq n$. If `realprecision` is p, we thus set $n = \log(10)p/\log(3 + \sqrt{8}) \approx 1.3p$; besides the time needed to compute the a_k, $k \leq n$, the algorithm overhead is negligible: time $O(p^2)$ and space $O(p)$.

If $flag = 1$, use a variant with more complicated polynomials, see `polzagier`. If the a_k are the moments of $w(x)dx$ where w (or only $xw(x^2)$) is a smooth function extending analytically to the whole complex plane, convergence is in $O(14.4^{-n})$. If $xw(x^2)$ extends analytically to a smaller region, we still have exponential convergence, with worse constants. Usually faster when the computation of a_k is expensive. If `realprecision` is p, we thus set $n = \log(10)p/\log(14.4) \approx 0.86p$; besides the time needed to compute the a_k, $k \leq n$, the algorithm overhead is *not* negligible: time $O(p^3)$ and space $O(p^2)$. Thus, even if the analytic conditions for rigorous use are met, this variant is only worthwile if the a_k are hard to compute, at least $O(p^2)$ individually on average: otherwise we gain a small constant factor (1.5, say) in the number of needed a_k at the expense of a large overhead.

The conditions for rigorous use are hard to check but the routine is best used heuristically: even divergent alternating series can sometimes be summed by this method, as well as series which are not exactly alternating (see for example Section 2.7). It should be used to try and guess the value of an infinite sum. (However, see the example at the end of Section 2.7.1.)

If the series already converges geometrically, `suminf` is often a better choice:

```
? \p28
? sumalt(i = 1, -(-1)^i / i)  - log(2)
time = 0 ms.
%1 = -2.524354897 E-29
? suminf(i = 1, -(-1)^i / i)    \\ Had to hit C-C
  ***   at top-level: suminf(i=1,-(-1)^i/i)
  ***                                 ^------
```

```
  *** suminf: user interrupt after 10min, 20,100 ms.
? \p1000
? sumalt(i = 1, -(-1)^i / i)  - log(2)
time = 90 ms.
%2 = 4.459597722 E-1002
? sumalt(i = 0, (-1)^i / i!) - exp(-1)
time = 670 ms.
%3 = -4.036987814906334831564973613521906157943533385918978305 87 E-944
? suminf(i = 0, (-1)^i / i!) - exp(-1)
time = 110 ms.
%4 = -8.39147638 E-1000   \\  faster and more accurate
```

The library syntax is **sumalt**(`void *E, GEN (*eval)(void*,GEN),GEN a,long prec`). Also available is `sumalt2` with the same arguments (*flag* = 1).

3.12.20 sumdiv$(n, X, expr)$. Sum of expression *expr* over the positive divisors of n. This function is a trivial wrapper essentially equivalent to

```
D = divisors(n);
for (i = 1, #D, X = D[i]; eval(expr))
```

(except that `X` is lexically scoped to the `sumdiv` loop). If *expr* is a multiplicative function, use `sumdivmult`.

3.12.21 sumdivmult$(n, d, expr)$. Sum of *multiplicative* expression *expr* over the positive divisors d of n. Assume that *expr* evaluates to $f(d)$ where f is multiplicative: $f(1) = 1$ and $f(ab) = f(a)f(b)$ for coprime a and b.

3.12.22 suminf$(X = a, expr)$. infinite sum of expression *expr*, the formal parameter X starting at a. The evaluation stops when the relative error of the expression is less than the default precision for 3 consecutive evaluations. The expressions must always evaluate to a complex number.

If the series converges slowly, make sure `realprecision` is low (even 28 digits may be too much). In this case, if the series is alternating or the terms have a constant sign, `sumalt` and `sumpos` should be used instead.

```
? \p28
? suminf(i = 1, -(-1)^i / i)   \\ Had to hit C-C
  ***   at top-level: suminf(i=1,-(-1)^i/i)
  ***                                 ^------
  *** suminf: user interrupt after 10min, 20,100 ms.
? sumalt(i = 1, -(-1)^i / i) - log(2)
time = 0 ms.
%1 = -2.524354897 E-29
```

The library syntax is **suminf**(`void *E, GEN (*eval)(void*,GEN), GEN a, long prec`).

3.12.23 sumnum$(n = a, f, \{tab\})$. Numerical summation of $f(n)$ at high accuracy using Euler-MacLaurin, the variable n taking values from a to $+\infty$, where f is assumed to have positive values and is a C^∞ function; `a` must be an integer and `tab`, if given, is the output of `sumnuminit`. The latter precomputes abcissas and weights, speeding up the computation; it also allows to specify the behaviour at infinity via `sumnuminit([+oo, asymp])`.

```
? \p500
? z3 = zeta(3);
? sumpos(n = 1, n^-3) - z3
time = 2,332 ms.
%2 = 2.438468843 E-501
? sumnum(n = 1, n^-3) - z3 \\ here slower than sumpos
time = 2,752 ms.
%3 = 0.E-500
```

Complexity. The function f will be evaluated at $O(D \log D)$ real arguments, where $D \approx$ `realprecision` $\cdot \log(10)$. The routine is geared towards slowly decreasing functions: if f decreases exponentially fast, then one of `suminf` or `sumpos` should be preferred. If f satisfies the stronger hypotheses required for Monien summation, i.e. if $f(1/z)$ is holomorphic in a complex neighbourhood of $[0, 1]$, then `sumnummonien` will be faster since it only requires $O(D/\log D)$ evaluations:

```
? sumnummonien(n = 1, 1/n^3) - z3
time = 1,985 ms.
%3 = 0.E-500
```

The `tab` argument precomputes technical data not depending on the expression being summed and valid for a given accuracy, speeding up immensely later calls:

```
? tab = sumnuminit();
time = 2,709 ms.
? sumnum(n = 1, 1/n^3, tab) - z3 \\ now much faster than sumpos
time = 40 ms.
%5 = 0.E-500

? tabmon = sumnummonieninit(); \\ Monien summation allows precomputations too
time = 1,781 ms.
? sumnummonien(n = 1, 1/n^3, tabmon) - z3
time = 2 ms.
%7 = 0.E-500
```

The speedup due to precomputations becomes less impressive when the function f is expensive to evaluate, though:

```
? sumnum(n = 1, lngamma(1+1/n)/n, tab);
time = 14,180 ms.

? sumnummonien(n = 1, lngamma(1+1/n)/n, tabmon); \\ fewer evaluations
time = 717 ms.
```

Behaviour at infinity. By default, `sumnum` assumes that *expr* decreases slowly at infinity, but at least like $O(n^{-2})$. If the function decreases like n^α for some $-2 < \alpha < -1$, then it must be indicated via

```
tab = sumnuminit([+oo, alpha]); /* alpha < 0 slow decrease */
```

otherwise loss of accuracy is expected. If the functions decreases quickly, like $\exp(-\alpha n)$ for some $\alpha > 0$, then it must be indicated via

```
tab = sumnuminit([+oo, alpha]); /* alpha  > 0 exponential decrease */
```

otherwise exponent overflow will occur.

```
? sumnum(n=1,2^-n)
 ***   at top-level: sumnum(n=1,2^-n)
 ***                               ^----
 *** _^_: overflow in expo().
? tab = sumnuminit([+oo,log(2)]); sumnum(n=1,2^-n, tab)
%1 = 1.000[...]
```

As a shortcut, one can also input

```
sumnum(n = [a, asymp], f)
```

instead of

```
tab = sumnuminit(asymp);
sumnum(n = a, f, tab)
```

Further examples.

```
? \p200
? sumnum(n = 1, n^(-2)) - zeta(2) \\ accurate, fast
time = 200 ms.
%1 = -2.376364457868949779 E-212
? sumpos(n = 1, n^(-2)) - zeta(2)  \\ even faster
time = 96 ms.
%2 = 0.E-211
? sumpos(n=1,n^(-4/3)) - zeta(4/3)   \\ now much slower
time = 13,045 ms.
%3 = -9.980730723049589073 E-210
? sumnum(n=1,n^(-4/3)) - zeta(4/3)  \\ fast but inaccurate
time = 365 ms.
%4 = -9.85[...]E-85
? sumnum(n=[1,-4/3],n^(-4/3)) - zeta(4/3) \\ with decrease rate, now accurate
time = 416 ms.
%5 = -4.134874156691972616 E-210
? tab = sumnuminit([+oo,-4/3]);
time = 196 ms.
? sumnum(n=1, n^(-4/3), tab) - zeta(4/3) \\ faster with precomputations
time = 216 ms.
%5 = -4.134874156691972616 E-210
? sumnum(n=1,-log(n)*n^(-4/3), tab) - zeta'(4/3)
time = 321 ms.
```

```
%7 = 7.224147951921607329 E-210
```

Note that in the case of slow decrease ($\alpha < 0$), the exact decrease rate must be indicated, while in the case of exponential decrease, a rough value will do. In fact, for exponentially decreasing functions, `sumnum` is given for completeness and comparison purposes only: one of `suminf` or `sumpos` should always be preferred.

```
? sumnum(n=[1, 1], 2^-n) \\ pretend we decrease as exp(-n)
time = 240 ms.
%8 = 1.000[...] \\ perfect
? sumpos(n=1, 2^-n)
%9 = 1.000[...] \\ perfect and instantaneous
```

The library syntax is **sumnum**(`(void *E, GEN (*eval)(void*, GEN), GEN a, GEN tab, long prec)`) where an omitted *tab* is coded as `NULL`.

3.12.24 sumnuminit({*asymp*}). Initialize tables for Euler–MacLaurin delta summation of a series with positive terms. If given, `asymp` is of the form $[+\texttt{oo}, \alpha]$, as in `intnum` and indicates the decrease rate at infinity of functions to be summed. A positive $\alpha > 0$ encodes an exponential decrease of type $\exp(-\alpha n)$ and a negative $-2 < \alpha < -1$ encodes a slow polynomial decrease of type n^α.

```
? \p200
? sumnum(n=1, n^-2);
time = 200 ms.
? tab = sumnuminit();
time = 188 ms.
? sumnum(n=1, n^-2, tab); \\ faster
time = 8 ms.
? tab = sumnuminit([+oo, log(2)]); \\ decrease like 2^-n
time = 200 ms.
? sumnum(n=1, 2^-n, tab)
time = 44 ms.
? tab = sumnuminit([+oo, -4/3]); \\ decrease like n^(-4/3)
time = 200 ms.
? sumnum(n=1, n^(-4/3), tab);
time = 221 ms.
```

The library syntax is `GEN sumnuminit(GEN asymp = NULL, long prec)`.

3.12.25 sumnummonien($n = a, f, \{tab\}$). Numerical summation $\sum_{n \geq a} f(n)$ at high accuracy, the variable n taking values from the integer a to $+\infty$ using Monien summation, which assumes that $f(1/z)$ has a complex analytic continuation in a (complex) neighbourhood of the segment $[0, 1]$.

The function f is evaluated at $O(D/\log D)$ real arguments, where $D \approx$ `realprecision`$\cdot\log(10)$. By default, assume that $f(n) = O(n^{-2})$ and has a non-zero asymptotic expansion

$$f(n) = \sum_{i \geq 2} a_i n^{-i}$$

at infinity. To handle more complicated behaviours and allow time-saving precomputations (for a given `realprecision`), see `sumnummonieninit`.

The library syntax is `GEN sumnummonien0(GEN n, GEN f, GEN tab = NULL, long prec)`.

3.12.26 sumnummonieninit($\{asymp\}, \{w\}, \{n0 = 1\}$). Initialize tables for Monien summation of a series $\sum_{n \geq n_0} f(n)$ where $f(1/z)$ has a complex analytic continuation in a (complex) neighbourhood of the segment $[0, 1]$.

By default, assume that $f(n) = O(n^{-2})$ and has a non-zero asymptotic expansion

$$f(n) = \sum_{i \geq 2} a_i / n^i$$

at infinity. Note that the sum starts at $i = 2$! The argument `asymp` allows to specify different expansions:

- a real number $\alpha > 1$ means
$$f(n) = \sum_{i \geq 1} a_i / n^{\alpha i}$$

(Now the summation starts at 1.)

- a vector $[\alpha, \beta]$ of reals, where we must have $\alpha > 0$ and $\alpha + \beta > 1$ to ensure convergence, means that
$$f(n) = \sum_{i \geq 1} a_i / n^{\alpha i + \beta}$$

Note that `asymp` $= [\alpha, \alpha]$ is equivalent to `asymp` $= \alpha$.

```
? \p57
? s = sumnum(n = 1, sin(1/sqrt(n)) / n); \\ reference point

? \p38
? sumnummonien(n = 1, sin(1/sqrt(n)) / n) - s
%2 = -0.001[...] \\ completely wrong

? t = sumnummonieninit([1,1/2]);  \\ f(n) = sum_i 1 / n^(i/2+1)
? sumnummonien(n = 1, sin(1/sqrt(n)) / n, t) - s
%3 = 0.E-37 \\ now correct
```

(As a matter of fact, in the above summation, the result given by `sumnum` at `\p38` is slighly incorrect, so we had to increase the accuracy to `\p57`.)

The argument w is used to sum expressions of the form

$$\sum_{n \geq n_0} f(n) w(n),$$

for varying f *as above*, and fixed weight function w, where we further assume that the auxiliary sums

$$g_w(m) = \sum_{n \geq n_0} w(n) / n^{\alpha m + \beta}$$

converge for all $m \geq 1$. Note that for non-negative integers k, and weight $w(n) = (\log n)^k$, the function $g_w(m) = \zeta^{(k)}(\alpha m + \beta)$ has a simple expression; for general weights, g_w is computed using `sumnum`. The following variants are available

- an integer $k \geq 0$, to code $w(n) = (\log n)^k$; only the cases $k = 0, 1$ are presently implemented; due to a poor implementation of ζ derivatives, it is not currently worth it to exploit the special shape of g_w when $k > 0$;

• a `t_CLOSURE` computing the values $w(n)$, where we assume that $w(n) = O(n^\epsilon)$ for all $\epsilon > 0$;

• a vector $[w, \texttt{fast}]$, where w is a closure as above and `fast` is a scalar; we assume that $w(n) = O(n^{\texttt{fast}+\epsilon})$; note that $\texttt{w} = [w, 0]$ is equivalent to $\texttt{w} = w$.

• a vector $[w, \texttt{oo}]$, where w is a closure as above; we assume that $w(n)$ decreases exponentially. Note that in this case, `sumnummonien` is provided for completeness and comparison purposes only: one of `suminf` or `sumpos` should be preferred in practice.

The cases where w is a closure or $w(n) = \log n$ are the only ones where n_0 is taken into account and stored in the result. The subsequent call to `sumnummonien` *must* use the same value.

```
? \p300
? sumnummonien(n = 1, n^-2*log(n)) + zeta'(2)
time = 536 ms.
%1 = -1.323[...]E-6 \\ completely wrong, f does not satisfy hypotheses !
? tab = sumnummonieninit(, 1); \\ codes w(n) = log(n)
time = 18,316 ms.
? sumnummonien(n = 1, n^-2, tab) + zeta'(2)
time = 44 ms.
%3 = -5.562684646268003458 E-309  \\ now perfect
? tab = sumnummonieninit(, n->log(n)); \\ generic, about as fast
time = 18,693 ms.
? sumnummonien(n = 1, n^-2, tab) + zeta'(2)
time = 40 ms.
%5 = -5.562684646268003458 E-309  \\ identical result
```

The library syntax is `GEN sumnummonieninit(GEN asymp = NULL, GEN w = NULL, GEN n0 = NULL, long prec)`.

3.12.27 sumpos$(X = a, expr, \{flag = 0\})$. Numerical summation of the series *expr*, which must be a series of terms having the same sign, the formal variable X starting at a. The algorithm used is Van Wijngaarden's trick for converting such a series into an alternating one, then we use `sumalt`. For regular functions, the function `sumnum` is in general much faster once the initializations have been made using `sumnuminit`.

The routine is heuristic and assumes that *expr* is more or less a decreasing function of X. In particular, the result will be completely wrong if *expr* is 0 too often. We do not check either that all terms have the same sign. As `sumalt`, this function should be used to try and guess the value of an infinite sum.

If $flag = 1$, use `sumalt`$(,1)$ instead of `sumalt`$(,0)$, see Section 3.12.19. Requiring more stringent analytic properties for rigorous use, but allowing to compute fewer series terms.

To reach accuracy 10^{-p}, both algorithms require $O(p^2)$ space; furthermore, assuming the terms decrease polynomially (in $O(n^{-C})$), both need to compute $O(p^2)$ terms. The `sumpos`$(,1)$ variant has a smaller implied constant (roughly 1.5 times smaller). Since the `sumalt`$(,1)$ overhead is now small compared to the time needed to compute series terms, this last variant should be about 1.5 faster. On the other hand, the achieved accuracy may be much worse: as for `sumalt`, since conditions for rigorous use are hard to check, the routine is best used heuristically.

The library syntax is **sumpos**`(void *E, GEN (*eval)(void*,GEN),GEN a,long prec)`. Also available is `sumpos2` with the same arguments ($flag = 1$).

3.13 Plotting functions.

Although plotting is not even a side purpose of PARI, a number of plotting functions are provided. Moreover, a lot of people suggested ideas or submitted patches for this section of the code. There are three types of graphic functions.

3.13.1 High-level plotting functions. (all the functions starting with `ploth`) in which the user has little to do but explain what type of plot he wants, and whose syntax is similar to the one used in the preceding section.

3.13.2 Low-level plotting functions. (called *rectplot* functions, sharing the prefix `plot`), where every drawing primitive (point, line, box, etc.) is specified by the user. These low-level functions work as follows. You have at your disposal 16 virtual windows which are filled independently, and can then be physically ORed on a single window at user-defined positions. These windows are numbered from 0 to 15, and must be initialized before being used by the function `plotinit`, which specifies the height and width of the virtual window (called a *rectwindow* in the sequel). At all times, a virtual cursor (initialized at $[0,0]$) is attached to the window, and its current value can be obtained using the function `plotcursor`.

A number of primitive graphic objects (called *rect* objects) can then be drawn in these windows, using a default color attached to that window (which can be changed using the `plotcolor` function) and only the part of the object which is inside the window will be drawn, with the exception of polygons and strings which are drawn entirely. The ones sharing the prefix `plotr` draw relatively to the current position of the virtual cursor, the others use absolute coordinates. Those having the prefix `plotrecth` put in the rectwindow a large batch of rect objects corresponding to the output of the related `ploth` function.

Finally, the actual physical drawing is done using `plotdraw`. The rectwindows are preserved so that further drawings using the same windows at different positions or different windows can be done without extra work. To erase a window, use `plotkill`. It is not possible to partially erase a window: erase it completely, initialize it again, then fill it with the graphic objects that you want to keep.

In addition to initializing the window, you may use a scaled window to avoid unnecessary conversions. For this, use `plotscale`. As long as this function is not called, the scaling is simply the number of pixels, the origin being at the upper left and the y-coordinates going downwards.

Plotting functions are platform independent, but a number of graphical drivers are available for screen output: X11-windows (hence also for GUI's based on X11 such as Openwindows and Motif), and the Qt and FLTK graphical libraries. The physical window opened by `plotdraw` or any of the `ploth*` functions is completely separated from `gp` (technically, a `fork` is done, and the non-graphical memory is immediately freed in the child process), which means you can go on working in the current `gp` session, without having to kill the window first. This window can be closed, enlarged or reduced using the standard window manager functions. No zooming procedure is implemented though (yet).

3.13.3 Functions for PostScript output. in the same way that `printtex` allows you to have a TEX output corresponding to printed results, the functions starting with `ps` allow you to have `PostScript` output of the plots. This will not be identical with the screen output, but sufficiently close. Note that you can use PostScript output even if you do not have the plotting routines enabled. The PostScript output is written in a file whose name is derived from the `psfile` default (`./pari.ps` if you did not tamper with it). Each time a new PostScript output is asked for, the PostScript output is appended to that file. Hence you probably want to remove this file, or change the value of `psfile`, in between plots. On the other hand, in this manner, as many plots as desired can be kept in a single file.

3.13.4 Library mode. *None of the graphic functions are available within the PARI library, you must be under* `gp` *to use them.* The reason for that is that you really should not use PARI for heavy-duty graphical work, there are better specialized alternatives around. This whole set of routines was only meant as a convenient, but simple-minded, visual aid. If you really insist on using these in your program (we warned you), the source (`plot*.c`) should be readable enough for you to achieve something.

3.13.5 plot$(X = a, b, expr, \{Ymin\}, \{Ymax\})$. Crude ASCII plot of the function represented by expression *expr* from a to b, with Y ranging from *Ymin* to *Ymax*. If *Ymin* (resp. *Ymax*) is not given, the minimum (resp. the maximum) of the computed values of the expression is used instead.

The library syntax is `void pariplot(GEN X, GEN b, GEN expr, GEN Ymin = NULL, GEN Ymax = NULL, long prec)`.

3.13.6 plotbox$(w, x2, y2)$. Let $(x1, y1)$ be the current position of the virtual cursor. Draw in the rectwindow w the outline of the rectangle which is such that the points $(x1, y1)$ and $(x2, y2)$ are opposite corners. Only the part of the rectangle which is in w is drawn. The virtual cursor does *not* move.

3.13.7 plotclip(w). 'clips' the content of rectwindow w, i.e remove all parts of the drawing that would not be visible on the screen. Together with `plotcopy` this function enables you to draw on a scratchpad before committing the part you're interested in to the final picture.

3.13.8 plotcolor(w, c). Set default color to c in rectwindow w. This is only implemented for the X-windows, fltk and Qt graphing engines. Possible values for c are given by the `graphcolormap` default, factory setting are

1=black, 2=blue, 3=violetred, 4=red, 5=green, 6=grey, 7=gainsborough.

but this can be considerably extended.

3.13.9 plotcopy$(sourcew, destw, dx, dy, \{flag = 0\})$. Copy the contents of rectwindow *sourcew* to rectwindow *destw* with offset (dx,dy). If flag's bit 1 is set, dx and dy express fractions of the size of the current output device, otherwise dx and dy are in pixels. dx and dy are relative positions of northwest corners if other bits of flag vanish, otherwise of: 2: southwest, 4: southeast, 6: northeast corners

3.13.10 plotcursor(w). Give as a 2-component vector the current (scaled) position of the virtual cursor corresponding to the rectwindow w.

3.13.11 plotdraw$(list, \{flag = 0\})$. Physically draw the rectwindows given in $list$ which must be a vector whose number of components is divisible by 3. If $list = [w1, x1, y1, w2, x2, y2, \ldots]$, the windows $w1$, $w2$, etc. are physically placed with their upper left corner at physical position $(x1, y1)$, $(x2, y2)$,... respectively, and are then drawn together. Overlapping regions will thus be drawn twice, and the windows are considered transparent. Then display the whole drawing in a special window on your screen. If $flag \neq 0$, x1, y1 etc. express fractions of the size of the current output device

3.13.12 ploth$(X = a, b, expr, \{flags = 0\}, \{n = 0\})$. High precision plot of the function $y = f(x)$ represented by the expression $expr$, x going from a to b. This opens a specific window (which is killed whenever you click on it), and returns a four-component vector giving the coordinates of the bounding box in the form $[xmin, xmax, ymin, ymax]$.

Important note. `ploth` may evaluate `expr` thousands of times; given the relatively low resolution of plotting devices, few significant digits of the result will be meaningful. Hence you should keep the current precision to a minimum (e.g. 9) before calling this function.

n specifies the number of reference point on the graph, where a value of 0 means we use the hardwired default values (1000 for general plot, 1500 for parametric plot, and 8 for recursive plot).

If no *flag* is given, *expr* is either a scalar expression $f(X)$, in which case the plane curve $y = f(X)$ will be drawn, or a vector $[f_1(X), \ldots, f_k(X)]$, and then all the curves $y = f_i(X)$ will be drawn in the same window.

The binary digits of *flag* mean:

• 1 = `Parametric`: *parametric plot.* Here *expr* must be a vector with an even number of components. Successive pairs are then understood as the parametric coordinates of a plane curve. Each of these are then drawn.

For instance:

```
ploth(X=0,2*Pi,[sin(X),cos(X)], "Parametric")
ploth(X=0,2*Pi,[sin(X),cos(X)])
ploth(X=0,2*Pi,[X,X,sin(X),cos(X)], "Parametric")
```

draw successively a circle, two entwined sinusoidal curves and a circle cut by the line $y = x$.

• 2 = `Recursive`: *recursive plot.* If this flag is set, only *one* curve can be drawn at a time, i.e. *expr* must be either a two-component vector (for a single parametric curve, and the parametric flag *has* to be set), or a scalar function. The idea is to choose pairs of successive reference points, and if their middle point is not too far away from the segment joining them, draw this as a local approximation to the curve. Otherwise, add the middle point to the reference points. This is fast, and usually more precise than usual plot. Compare the results of

```
ploth(X=-1,1, sin(1/X), "Recursive")
ploth(X=-1,1, sin(1/X))
```

for instance. But beware that if you are extremely unlucky, or choose too few reference points, you may draw some nice polygon bearing little resemblance to the original curve. For instance you should *never* plot recursively an odd function in a symmetric interval around 0. Try

```
ploth(x = -20, 20, sin(x), "Recursive")
```

to see why. Hence, it's usually a good idea to try and plot the same curve with slightly different parameters.

The other values toggle various display options:

• 4 = `no_Rescale`: do not rescale plot according to the computed extrema. This is used in conjunction with `plotscale` when graphing multiple functions on a rectwindow (as a `plotrecth` call):

```
s = plothsizes();
plotinit(0, s[2]-1, s[2]-1);
plotscale(0, -1,1, -1,1);
plotrecth(0, t=0,2*Pi, [cos(t),sin(t)], "Parametric|no_Rescale")
plotdraw([0, -1,1]);
```

This way we get a proper circle instead of the distorted ellipse produced by

```
ploth(t=0,2*Pi, [cos(t),sin(t)], "Parametric")
```

• 8 = `no_X_axis`: do not print the x-axis.

• 16 = `no_Y_axis`: do not print the y-axis.

• 32 = `no_Frame`: do not print frame.

• 64 = `no_Lines`: only plot reference points, do not join them.

• 128 = `Points_too`: plot both lines and points.

• 256 = `Splines`: use splines to interpolate the points.

• 512 = `no_X_ticks`: plot no x-ticks.

• 1024 = `no_Y_ticks`: plot no y-ticks.

• 2048 = `Same_ticks`: plot all ticks with the same length.

• 4096 = `Complex`: is a parametric plot but where each member of `expr` is considered a complex number encoding the two coordinates of a point. For instance:

```
ploth(X=0,2*Pi,exp(I*X), "Complex")
ploth(X=0,2*Pi,[(1+I)*X,exp(I*X)], "Complex")
```

will draw respectively a circle and a circle cut by the line $y = x$.

3.13.13 plothraw(*listx*, *listy*, {*flag* = 0}). Given *listx* and *listy* two vectors of equal length, plots (in high precision) the points whose (x, y)-coordinates are given in *listx* and *listy*. Automatic positioning and scaling is done, but with the same scaling factor on x and y. If *flag* is 1, join points, other non-0 flags toggle display options and should be combinations of bits 2^k, $k \geq 3$ as in `ploth`.

3.13.14 plothsizes({*flag* = 0}). Return data corresponding to the output window in the form of a 6-component vector: window width and height, sizes for ticks in horizontal and vertical directions (this is intended for the `gnuplot` interface and is currently not significant), width and height of characters.

If *flag* = 0, sizes of ticks and characters are in pixels, otherwise are fractions of the screen size

3.13.15 plotinit$(w, \{x\}, \{y\}, \{flag = 0\})$. Initialize the rectwindow w, destroying any rect objects you may have already drawn in w. The virtual cursor is set to $(0, 0)$. The rectwindow size is set to width x and height y; omitting either x or y means we use the full size of the device in that direction. If $flag = 0$, x and y represent pixel units. Otherwise, x and y are understood as fractions of the size of the current output device (hence must be between 0 and 1) and internally converted to pixels.

The plotting device imposes an upper bound for x and y, for instance the number of pixels for screen output. These bounds are available through the `plothsizes` function. The following sequence initializes in a portable way (i.e independent of the output device) a window of maximal size, accessed through coordinates in the $[0, 1000] \times [0, 1000]$ range:

```
s = plothsizes();
plotinit(0, s[1]-1, s[2]-1);
plotscale(0, 0,1000, 0,1000);
```

3.13.16 plotkill(w). Erase rectwindow w and free the corresponding memory. Note that if you want to use the rectwindow w again, you have to use `plotinit` first to specify the new size. So it's better in this case to use `plotinit` directly as this throws away any previous work in the given rectwindow.

3.13.17 plotlines$(w, X, Y, \{flag = 0\})$. Draw on the rectwindow w the polygon such that the (x,y)-coordinates of the vertices are in the vectors of equal length X and Y. For simplicity, the whole polygon is drawn, not only the part of the polygon which is inside the rectwindow. If *flag* is non-zero, close the polygon. In any case, the virtual cursor does not move.

X and Y are allowed to be scalars (in this case, both have to). There, a single segment will be drawn, between the virtual cursor current position and the point (X, Y). And only the part thereof which actually lies within the boundary of w. Then *move* the virtual cursor to (X, Y), even if it is outside the window. If you want to draw a line from $(x1, y1)$ to $(x2, y2)$ where $(x1, y1)$ is not necessarily the position of the virtual cursor, use `plotmove(w,x1,y1)` before using this function.

3.13.18 plotlinetype$(w, type)$. This function is obsolete and currently a no-op.

Change the type of lines subsequently plotted in rectwindow w. *type* -2 corresponds to frames, -1 to axes, larger values may correspond to something else. $w = -1$ changes highlevel plotting.

3.13.19 plotmove(w, x, y). Move the virtual cursor of the rectwindow w to position (x, y).

3.13.20 plotpoints(w, X, Y). Draw on the rectwindow w the points whose (x, y)-coordinates are in the vectors of equal length X and Y and which are inside w. The virtual cursor does *not* move. This is basically the same function as `plothraw`, but either with no scaling factor or with a scale chosen using the function `plotscale`.

As was the case with the `plotlines` function, X and Y are allowed to be (simultaneously) scalar. In this case, draw the single point (X, Y) on the rectwindow w (if it is actually inside w), and in any case *move* the virtual cursor to position (x, y).

3.13.21 plotpointsize$(w, size)$. This function is obsolete. It is currently a no-op.

Changes the "size" of following points in rectwindow w. If $w = -1$, change it in all rectwindows.

3.13.22 plotpointtype$(w, type)$. This function is obsolete and currently a no-op.

change the type of points subsequently plotted in rectwindow w. $type = -1$ corresponds to a dot, larger values may correspond to something else. $w = -1$ changes highlevel plotting.

3.13.23 plotrbox(w, dx, dy). Draw in the rectwindow w the outline of the rectangle which is such that the points $(x1, y1)$ and $(x1 + dx, y1 + dy)$ are opposite corners, where $(x1, y1)$ is the current position of the cursor. Only the part of the rectangle which is in w is drawn. The virtual cursor does *not* move.

3.13.24 plotrecth$(w, X = a, b, expr, \{flag = 0\}, \{n = 0\})$. Writes to rectwindow w the curve output of `ploth`$(w, X = a, b, expr, flag, n)$. Returns a vector for the bounding box.

3.13.25 plotrecthraw$(w, data, \{flags = 0\})$. Plot graph(s) for $data$ in rectwindow w. *flag* has the same significance here as in `ploth`, though recursive plot is no more significant.
data

is a vector of vectors, each corresponding to a list a coordinates. If parametric plot is set, there must be an even number of vectors, each successive pair corresponding to a curve. Otherwise, the first one contains the x coordinates, and the other ones contain the y-coordinates of curves to plot.

3.13.26 plotrline(w, dx, dy). Draw in the rectwindow w the part of the segment $(x1, y1) - (x1 + dx, y1 + dy)$ which is inside w, where $(x1, y1)$ is the current position of the virtual cursor, and move the virtual cursor to $(x1 + dx, y1 + dy)$ (even if it is outside the window).

3.13.27 plotrmove(w, dx, dy). Move the virtual cursor of the rectwindow w to position $(x1 + dx, y1 + dy)$, where $(x1, y1)$ is the initial position of the cursor (i.e. to position (dx, dy) relative to the initial cursor).

3.13.28 plotrpoint(w, dx, dy). Draw the point $(x1 + dx, y1 + dy)$ on the rectwindow w (if it is inside w), where $(x1, y1)$ is the current position of the cursor, and in any case move the virtual cursor to position $(x1 + dx, y1 + dy)$.

3.13.29 plotscale$(w, x1, x2, y1, y2)$. Scale the local coordinates of the rectwindow w so that x goes from $x1$ to $x2$ and y goes from $y1$ to $y2$ ($x2 < x1$ and $y2 < y1$ being allowed). Initially, after the initialization of the rectwindow w using the function `plotinit`, the default scaling is the graphic pixel count, and in particular the y axis is oriented downwards since the origin is at the upper left. The function `plotscale` allows to change all these defaults and should be used whenever functions are graphed.

3.13.30 plotstring$(w, x, \{flags = 0\})$. Draw on the rectwindow w the String x (see Section 2.9), at the current position of the cursor.
flag

is used for justification: bits 1 and 2 regulate horizontal alignment: left if 0, right if 2, center if 1. Bits 4 and 8 regulate vertical alignment: bottom if 0, top if 8, v-center if 4. Can insert additional small gap between point and string: horizontal if bit 16 is set, vertical if bit 32 is set (see the tutorial for an example).

3.13.31 psdraw($list, \{flag = 0\}$). Same as `plotdraw`, except that the output is a PostScript program appended to the `psfile`, and flag!=0 scales the plot from size of the current output device to the standard PostScript plotting size

3.13.32 psploth($X = a, b, expr, \{flags = 0\}, \{n = 0\}$). Same as `ploth`, except that the output is a PostScript program appended to the `psfile`.

3.13.33 psplothraw($listx, listy, \{flag = 0\}$). Same as `plothraw`, except that the output is a PostScript program appended to the `psfile`.

3.14 Programming in GP: control statements.

A number of control statements are available in GP. They are simpler and have a syntax slightly different from their C counterparts, but are quite powerful enough to write any kind of program. Some of them are specific to GP, since they are made for number theorists. As usual, X will denote any simple variable name, and *seq* will always denote a sequence of expressions, including the empty sequence.

Caveat. In constructs like

```
for (X = a,b, seq)
```

the variable `X` is lexically scoped to the loop, leading to possibly unexpected behavior:

```
n = 5;
for (n = 1, 10,
  if (something_nice(), break);
);
\\  at this point n is 5 !
```

If the sequence `seq` modifies the loop index, then the loop is modified accordingly:

```
? for (n = 1, 10, n += 2; print(n))
3
6
9
12
```

3.14.1 break($\{n = 1\}$). Interrupts execution of current *seq*, and immediately exits from the n innermost enclosing loops, within the current function call (or the top level loop); the integer n must be positive. If n is greater than the number of enclosing loops, all enclosing loops are exited.

3.14.2 breakpoint(). Interrupt the program and enter the breakloop. The program continues when the breakloop is exited.

```
? f(N,x)=my(z=x^2+1);breakpoint();gcd(N,z^2+1-z);
? f(221,3)
  ***   at top-level: f(221,3)
  ***                 ^--------
  ***   in function f: my(z=x^2+1);breakpoint();gcd(N,z
  ***                               ^--------------------
  ***   Break loop: type <Return> to continue; 'break' to go back to GP
break> z
10
break>
%2 = 13
```

3.14.3 dbg_down($\{n = 1\}$). (In the break loop) go down n frames. This allows to cancel a previous call to `dbg_up`.

3.14.4 dbg_err(). In the break loop, return the error data of the current error, if any. See `iferr` for details about error data. Compare:

```
? iferr(1/(Mod(2,12019)^(6!)-1),E,Vec(E))
%1 = ["e_INV", "Fp_inv", Mod(119, 12019)]
? 1/(Mod(2,12019)^(6!)-1)
  ***   at top-level: 1/(Mod(2,12019)^(6!)-
  ***                  ^--------------------
  *** _/_: impossible inverse in Fp_inv: Mod(119, 12019).
  ***   Break loop: type 'break' to go back to GP prompt
break> Vec(dbg_err())
["e_INV", "Fp_inv", Mod(119, 12019)]
```

3.14.5 dbg_up($\{n = 1\}$). (In the break loop) go up n frames. This allows to inspect data of the parent function. To cancel a `dbg_up` call, use `dbg_down`

3.14.6 dbg_x($A\{, n\}$). Print the inner structure of `A`, complete if `n` is omitted, up to level `n` otherwise. This is useful for debugging. This is similar to `\x` but does not require `A` to be an history entry. In particular, it can be used in the break loop.

3.14.7 for($X = a, b, seq$). Evaluates *seq*, where the formal variable X goes from a to b. Nothing is done if $a > b$. a and b must be in $\mathbf{R}$. If b is set to `+oo`, the loop will not stop; it is expected that the caller will break out of the loop itself at some point, using `break` or `return`.

3.14.8 forcomposite$(n = a, \{b\}, seq)$. Evaluates *seq*, where the formal variable n ranges over the composite numbers between the non-negative real numbers a to b, including a and b if they are composite. Nothing is done if $a > b$.

```
? forcomposite(n = 0, 10, print(n))
4
6
8
9
10
```

Omitting b means we will run through all composites $\geq a$, starting an infinite loop; it is expected that the user will break out of the loop himself at some point, using `break` or `return`.

Note that the value of n cannot be modified within *seq*:

```
? forcomposite(n = 2, 10, n = [])
 ***   at top-level: forcomposite(n=2,10,n=[])
 ***                                       ^---
 ***   index read-only: was changed to [].
```

3.14.9 fordiv(n, X, seq). Evaluates *seq*, where the formal variable X ranges through the divisors of n (see `divisors`, which is used as a subroutine). It is assumed that `factor` can handle n, without negative exponents. Instead of n, it is possible to input a factorization matrix, i.e. the output of `factor(n)`.

This routine uses `divisors` as a subroutine, then loops over the divisors. In particular, if n is an integer, divisors are sorted by increasing size.

To avoid storing all divisors, possibly using a lot of memory, the following (much slower) routine loops over the divisors using essentially constant space:

```
FORDIV(N)=
{ my(P, E);
  P = factor(N); E = P[,2]; P = P[,1];
  forvec( v = vector(#E, i, [0,E[i]]),
  X = factorback(P, v)
  \\ ...
);
}
? for(i=1,10^5, FORDIV(i))
time = 3,445 ms.
? for(i=1,10^5, fordiv(i, d, ))
time = 490 ms.
```

3.14.10 forell$(E, a, b, seq, \{flag = 0\})$. Evaluates *seq*, where the formal variable $E = [name, M, G]$ ranges through all elliptic curves of conductors from a to b. In this notation *name* is the curve name in Cremona's elliptic curve database, M is the minimal model, G is a $\mathbf{Z}$-basis of the free part of the Mordell-Weil group $E(\mathbf{Q})$. If flag is non-zero, select only the first curve in each isogeny class.

```
? forell(E, 1, 500, my([name,M,G] = E); \
    if (#G > 1, print(name)))
389a1
433a1
446d1
? c = 0; forell(E, 1, 500, c++); c   \\ number of curves
%2 = 2214
? c = 0; forell(E, 1, 500, c++, 1); c \\ number of isogeny classes
%3 = 971
```

The `elldata` database must be installed and contain data for the specified conductors.

The library syntax is **forell**(`void *data, long (*call)(void*,GEN), long a, long b, long flag`).

3.14.11 forpart$(X = k, seq, \{a = k\}, \{n = k\})$. Evaluate *seq* over the partitions $X = [x_1, \dots x_n]$ of the integer k, i.e. increasing sequences $x_1 \le x_2 \dots \le x_n$ of sum $x_1 + \dots + x_n = k$. By convention, 0 admits only the empty partition and negative numbers have no partitions. A partition is given by a `t_VECSMALL`, where parts are sorted in nondecreasing order:

```
? forpart(X=3, print(X))
Vecsmall([3])
Vecsmall([1, 2])
Vecsmall([1, 1, 1])
```

Optional parameters n and a are as follows:

• $n = nmax$ (resp. $n = [nmin, nmax]$) restricts partitions to length less than $nmax$ (resp. length between $nmin$ and $nmax$), where the *length* is the number of nonzero entries.

• $a = amax$ (resp. $a = [amin, amax]$) restricts the parts to integers less than $amax$ (resp. between $amin$ and $amax$).

By default, parts are positive and we remove zero entries unless $amin \le 0$, in which case X is of constant length $nmax$.

```
\\ at most 3 non-zero parts, all <= 4
? forpart(v=5,print(Vec(v)),4,3)
[1, 4]
[2, 3]
[1, 1, 3]
[1, 2, 2]

\\ between 2 and 4 parts less than 5, fill with zeros
? forpart(v=5,print(Vec(v)),[0,5],[2,4])
[0, 0, 1, 4]
[0, 0, 2, 3]
[0, 1, 1, 3]
[0, 1, 2, 2]
```

```
[1, 1, 1, 2]
```

The behavior is unspecified if X is modified inside the loop.

The library syntax is **forpart**(`void *data, long (*call)(void*,GEN), long k, GEN a, GEN n`).

3.14.12 forprime$(p = a, \{b\}, seq)$. Evaluates *seq*, where the formal variable p ranges over the prime numbers between the real numbers a to b, including a and b if they are prime. More precisely, the value of p is incremented to `nextprime(`p `+ 1)`, the smallest prime strictly larger than p, at the end of each iteration. Nothing is done if $a > b$.

```
? forprime(p = 4, 10, print(p))
5
7
```

Setting b to `+oo` means we will run through all primes $\geq a$, starting an infinite loop; it is expected that the caller will break out of the loop itself at some point, using `break` or `return`.

Note that the value of p cannot be modified within *seq*:

```
? forprime(p = 2, 10, p = [])
 ***   at top-level: forprime(p=2,10,p=[])
 ***                                    ^---
 ***   prime index read-only: was changed to [].
```

3.14.13 forstep$(X = a, b, s, seq)$. Evaluates *seq*, where the formal variable X goes from a to b, in increments of s. Nothing is done if $s > 0$ and $a > b$ or if $s < 0$ and $a < b$. s must be in $\mathbf{R}^*$ or a vector of steps $[s_1, \ldots, s_n]$. In the latter case, the successive steps are used in the order they appear in s.

```
? forstep(x=5, 20, [2,4], print(x))
5
7
11
13
17
19
```

Setting b to `+oo` will start an infinite loop; it is expected that the caller will break out of the loop itself at some point, using `break` or `return`.

3.14.14 forsubgroup$(H = G, \{bound\}, seq)$. Evaluates *seq* for each subgroup H of the *abelian* group G (given in SNF form or as a vector of elementary divisors).

If *bound* is present, and is a positive integer, restrict the output to subgroups of index less than *bound*. If *bound* is a vector containing a single positive integer B, then only subgroups of index exactly equal to B are computed

The subgroups are not ordered in any obvious way, unless G is a p-group in which case Birkhoff's algorithm produces them by decreasing index. A subgroup is given as a matrix whose columns give its generators on the implicit generators of G. For example, the following prints all subgroups of index less than 2 in $G = \mathbf{Z}/2\mathbf{Z}g_1 \times \mathbf{Z}/2\mathbf{Z}g_2$:

```
? G = [2,2]; forsubgroup(H=G, 2, print(H))
```

```
[1; 1]
[1; 2]
[2; 1]
[1, 0; 1, 1]
```

The last one, for instance is generated by $(g_1, g_1 + g_2)$. This routine is intended to treat huge groups, when `subgrouplist` is not an option due to the sheer size of the output.

For maximal speed the subgroups have been left as produced by the algorithm. To print them in canonical form (as left divisors of G in HNF form), one can for instance use

```
? G = matdiagonal([2,2]); forsubgroup(H=G, 2, print(mathnf(concat(G,H))))
[2, 1; 0, 1]
[1, 0; 0, 2]
[2, 0; 0, 1]
[1, 0; 0, 1]
```

Note that in this last representation, the index $[G : H]$ is given by the determinant. See `galoissubcyclo` and `galoisfixedfield` for applications to Galois theory.

The library syntax is **forsubgroup**(`void *data, long (*call)(void*,GEN), GEN G, GEN bound`).

3.14.15 forvec$(X = v, seq, \{flag = 0\})$. Let v be an n-component vector (where n is arbitrary) of two-component vectors $[a_i, b_i]$ for $1 \leq i \leq n$, where all entries a_i, b_i are real numbers. This routine lets X vary over the n-dimensional hyperrectangle given by v, that is, X is an n-dimensional vector taking successively its entries $X[i]$ in the range $[a_i, b_i]$ with lexicographic ordering. (The component with the highest index moves the fastest.) The type of X is the same as the type of v: `t_VEC` or `t_COL`.

The expression *seq* is evaluated with the successive values of X.

If *flag* $= 1$, generate only nondecreasing vectors X, and if *flag* $= 2$, generate only strictly increasing vectors X.

```
? forvec (X=[[0,1],[-1,1]], print(X));
[0, -1]
[0, 0]
[0, 1]
[1, -1]
[1, 0]
[1, 1]
? forvec (X=[[0,1],[-1,1]], print(X), 1);
[0, 0]
[0, 1]
[1, 1]
? forvec (X=[[0,1],[-1,1]], print(X), 2)
[0, 1]
```

3.14.16 if$(a, \{seq1\}, \{seq2\})$. Evaluates the expression sequence *seq1* if a is non-zero, otherwise the expression *seq2*. Of course, *seq1* or *seq2* may be empty:

`if (`a`,`seq`)` evaluates *seq* if a is not equal to zero (you don't have to write the second comma), and does nothing otherwise,

`if (`a`,,`seq`)` evaluates *seq* if a is equal to zero, and does nothing otherwise. You could get the same result using the `!` (`not`) operator: `if (!`a`,`seq`)`.

The value of an `if` statement is the value of the branch that gets evaluated: for instance

```
x = if(n % 4 == 1, y, z);
```

sets x to y if n is 1 modulo 4, and to z otherwise.

Successive 'else' blocks can be abbreviated in a single compound `if` as follows:

```
if (test1, seq1,
    test2, seq2,
    ...
    testn, seqn,
    seqdefault);
```

is equivalent to

```
if (test1, seq1
         , if (test2, seq2
                    , ...
                      if (testn, seqn, seqdefault)...));
```

For instance, this allows to write traditional switch / case constructions:

```
if (x == 0, do0(),
    x == 1, do1(),
    x == 2, do2(),
    dodefault());
```

Remark. The boolean operators `&&` and `||` are evaluated according to operator precedence as explained in Section 2.4, but, contrary to other operators, the evaluation of the arguments is stopped as soon as the final truth value has been determined. For instance

```
if (x != 0 && f(1/x), ...)
```

is a perfectly safe statement.

Remark. Functions such as `break` and `next` operate on *loops*, such as `for`*xxx*, `while`, `until`. The `if` statement is *not* a loop. (Obviously!)

3.14.17 iferr(*seq1*, *E*, *seq2*{, *pred*}). Evaluates the expression sequence *seq1*. If an error occurs, set the formal parameter *E* set to the error data. If *pred* is not present or evaluates to true, catch the error and evaluate *seq2*. Both *pred* and *seq2* can reference *E*. The error type is given by `errname(E)`, and other data can be accessed using the `component` function. The code *seq2* should check whether the error is the one expected. In the negative the error can be rethrown using `error(E)` (and possibly caught by an higher `iferr` instance). The following uses `iferr` to implement Lenstra's ECM factoring method

```
? ecm(N, B = 1000!, nb = 100)=
  {
    for(a = 1, nb,
      iferr(ellmul(ellinit([a,1]*Mod(1,N)), [0,1]*Mod(1,N), B),
        E, return(gcd(lift(component(E,2)),N)),
        errname(E)=="e_INV" && type(component(E,2)) == "t_INTMOD"))
  }
? ecm(2^101-1)
%2 = 7432339208719
```

The return value of `iferr` itself is the value of *seq2* if an error occurs, and the value of *seq1* otherwise. We now describe the list of valid error types, and the attached error data *E*; in each case, we list in order the components of *E*, accessed via `component(E,1)`, `component(E,2)`, etc.

Internal errors, "system" errors.

• `"e_ARCH"`. A requested feature *s* is not available on this architecture or operating system. *E* has one component (`t_STR`): the missing feature name *s*.

• `"e_BUG"`. A bug in the PARI library, in function *s*. *E* has one component (`t_STR`): the function name *s*.

• `"e_FILE"`. Error while trying to open a file. *E* has two components, 1 (`t_STR`): the file type (input, output, etc.), 2 (`t_STR`): the file name.

• `"e_IMPL"`. A requested feature *s* is not implemented. *E* has one component, 1 (`t_STR`): the feature name *s*.

• `"e_PACKAGE"`. Missing optional package *s*. *E* has one component, 1 (`t_STR`): the package name *s*.

Syntax errors, type errors.

• "e_DIM". The dimensions of arguments x and y submitted to function s does not match up. E.g., multiplying matrices of inconsistent dimension, adding vectors of different lengths,... E has three component, 1 (t_STR): the function name s, 2: the argument x, 3: the argument y.

• "e_FLAG". A flag argument is out of bounds in function s. E has one component, 1 (t_STR): the function name s.

• "e_NOTFUNC". Generated by the PARI evaluator; tried to use a GEN x which is not a t_CLOSURE in a function call syntax (as in f = 1; f(2);). E has one component, 1: the offending GEN x.

• "e_OP". Impossible operation between two objects than cannot be typecast to a sensible common domain for deeper reasons than a type mismatch, usually for arithmetic reasons. As in O(2) + O(3): it is valid to add two t_PADICs, provided the underlying prime is the same; so the addition is not forbidden a priori for type reasons, it only becomes so when inspecting the objects and trying to perform the operation. E has three components, 1 (t_STR): the operator name op, 2: first argument, 3: second argument.

• "e_TYPE". An argument x of function s had an unexpected type. (As in factor("blah").) E has two components, 1 (t_STR): the function name s, 2: the offending argument x.

• "e_TYPE2". Forbidden operation between two objects than cannot be typecast to a sensible common domain, because their types do not match up. (As in Mod(1,2) + Pi.) E has three components, 1 (t_STR): the operator name op, 2: first argument, 3: second argument.

• "e_PRIORITY". Object o in function s contains variables whose priority is incompatible with the expected operation. E.g. Pol([x,1], 'y): this raises an error because it's not possible to create a polynomial whose coefficients involve variables with higher priority than the main variable. E has four components: 1 (t_STR): the function name s, 2: the offending argument o, 3 (t_STR): an operator op describing the priority error, 4 (t_POL): the variable v describing the priority error. The argument satisfies variable(x) opvariable(v).

• "e_VAR". The variables of arguments x and y submitted to function s does not match up. E.g., considering the algebraic number Mod(t,t^2+1) in nfinit(x^2+1). E has three component, 1 (t_STR): the function name s, 2 (t_POL): the argument x, 3 (t_POL): the argument y.

Overflows.

• "e_COMPONENT". Trying to access an inexistent component in a vector/matrix/list in a function: the index is less than 1 or greater than the allowed length. E has four components, 1 (t_STR): the function name 2 (t_STR): an operator op ($<$ or $>$), 2 (t_GEN): a numerical limit l bounding the allowed range, 3 (GEN): the index x. It satisfies x op l.

• "e_DOMAIN". An argument is not in the function's domain. E has five components, 1 (t_STR): the function name, 2 (t_STR): the mathematical name of the out-of-domain argument 3 (t_STR): an operator op describing the domain error, 4 (t_GEN): the numerical limit l describing the domain error, 5 (GEN): the out-of-domain argument x. The argument satisfies x op l, which prevents it from belonging to the function's domain.

• "e_MAXPRIME". A function using the precomputed list of prime numbers ran out of primes. E has one component, 1 (t_INT): the requested prime bound, which overflowed primelimit or 0 (bound is unknown).

• "e_MEM". A call to pari_malloc or pari_realloc failed. E has no component.

• "e_OVERFLOW". An object in function s becomes too large to be represented within PARI's hardcoded limits. (As in 2^2^2^10 or exp(1e100), which overflow in lg and expo.) E has one component, 1 (t_STR): the function name s.

• "e_PREC". Function s fails because input accuracy is too low. (As in floor(1e100) at default accuracy.) E has one component, 1 (t_STR): the function name s.

• "e_STACK". The PARI stack overflows. E has no component.

Errors triggered intentionally.

• "e_ALARM". A timeout, generated by the alarm function. E has one component (t_STR): the error message to print.

• "e_USER". A user error, as triggered by error$(g_1, \dots, g_n)$. E has one component, 1 (t_VEC): the vector of n arguments given to error.

Mathematical errors.

• "e_CONSTPOL". An argument of function s is a constant polynomial, which does not make sense. (As in galoisinit(Pol(1)).) E has one component, 1 (t_STR): the function name s.

• "e_COPRIME". Function s expected coprime arguments, and did receive x, y, which were not. E has three component, 1 (t_STR): the function name s, 2: the argument x, 3: the argument y.

• "e_INV". Tried to invert a non-invertible object x in function s. E has two components, 1 (t_STR): the function name s, 2: the non-invertible x. If $x =$ Mod(a, b) is a t_INTMOD and a is not 0 mod b, this allows to factor the modulus, as gcd(a, b) is a non-trivial divisor of b.

• "e_IRREDPOL". Function s expected an irreducible polynomial, and did receive T, which was not. (As in nfinit(x^2-1).) E has two component, 1 (t_STR): the function name s, 2 (t_POL): the polynomial x.

• "e_MISC". Generic uncategorized error. E has one component (t_STR): the error message to print.

• "e_MODULUS". moduli x and y submitted to function s are inconsistent. As in

```
nfalgtobasis(nfinit(t^3-2), Mod(t,t^2+1)
```

E has three component, 1 (t_STR): the function s, 2: the argument x, 3: the argument x.

• "e_PRIME". Function s expected a prime number, and did receive p, which was not. (As in idealprimedec(nf, 4).) E has two component, 1 (t_STR): the function name s, 2: the argument p.

• "e_ROOTS0". An argument of function s is a zero polynomial, and we need to consider its roots. (As in polroots(0).) E has one component, 1 (t_STR): the function name s.

• "e_SQRTN". Trying to compute an n-th root of x, which does not exist, in function s. (As in sqrt(Mod(-1,3)).) E has two components, 1 (t_STR): the function name s, 2: the argument x.

3.14.18 next($\{n = 1\}$). Interrupts execution of current *seq*, resume the next iteration of the innermost enclosing loop, within the current function call (or top level loop). If n is specified, resume at the n-th enclosing loop. If n is bigger than the number of enclosing loops, all enclosing loops are exited.

3.14.19 return($\{x = 0\}$). Returns from current subroutine, with result x. If x is omitted, return the `(void)` value (return no result, like `print`).

3.14.20 until(a, *seq*). Evaluates *seq* until a is not equal to 0 (i.e. until a is true). If a is initially not equal to 0, *seq* is evaluated once (more generally, the condition on a is tested *after* execution of the *seq*, not before as in `while`).

3.14.21 while(a, *seq*). While a is non-zero, evaluates the expression sequence *seq*. The test is made *before* evaluating the *seq*, hence in particular if a is initially equal to zero the *seq* will not be evaluated at all.

3.15 Programming in GP: other specific functions.

In addition to the general PARI functions, it is necessary to have some functions which will be of use specifically for `gp`, though a few of these can be accessed under library mode. Before we start describing these, we recall the difference between *strings* and *keywords* (see Section 2.9): the latter don't get expanded at all, and you can type them without any enclosing quotes. The former are dynamic objects, where everything outside quotes gets immediately expanded.

3.15.1 Strprintf(*fmt*, $\{x\}*$). Returns a string built from the remaining arguments according to the format fmt. The format consists of ordinary characters (not %), printed unchanged, and conversions specifications. See `printf`.

3.15.2 addhelp(*sym*, *str*). Changes the help message for the symbol `sym`. The string *str* is expanded on the spot and stored as the online help for `sym`. It is recommended to document global variables and user functions in this way, although `gp` will not protest if you don't.

You can attach a help text to an alias, but it will never be shown: aliases are expanded by the `?` help operator and we get the help of the symbol the alias points to. Nothing prevents you from modifying the help of built-in PARI functions. But if you do, we would like to hear why you needed it!

Without `addhelp`, the standard help for user functions consists of its name and definition.

```
gp> f(x) = x^2;
gp> ?f
f =
  (x)->x^2
```

Once addhelp is applied to f, the function code is no longer included. It can still be consulted by typing the function name:

```
gp> addhelp(f, "Square")
gp> ?f
Square

gp> f
%2 = (x)->x^2
```

The library syntax is `void addhelp(const char *sym, const char *str)`.

3.15.3 alarm($\{s = 0\}, \{code\}$). If *code* is omitted, trigger an *e_ALARM* exception after s seconds, cancelling any previously set alarm; stop a pending alarm if $s = 0$ or is omitted.

Otherwise, if s is positive, the function evaluates *code*, aborting after s seconds. The return value is the value of *code* if it ran to completion before the alarm timeout, and a `t_ERROR` object otherwise.

```
? p = nextprime(10^25); q = nextprime(10^26); N = p*q;
? E = alarm(1, factor(N));
? type(E)
%3 = "t_ERROR"
? print(E)
%4 = error("alarm interrupt after 964 ms.")
? alarm(10, factor(N));   \\ enough time
%5 =
[ 10000000000000000000000013 1]

[10000000000000000000000067 1]
```

Here is a more involved example: the function `timefact(N,sec)` below tries to factor N and gives up after *sec* seconds, returning a partial factorisation.

```
\\ Time-bounded partial factorization
default(factor_add_primes,1);
timefact(N,sec)=
{
  F = alarm(sec, factor(N));
  if (type(F) == "t_ERROR", factor(N, 2^24), F);
}
```

We either return the factorization directly, or replace the `t_ERROR` result by a simple bounded factorization `factor(N, 2^24)`. Note the `factor_add_primes` trick: any prime larger than 2^{24} discovered while attempting the initial factorization is stored and remembered. When the alarm rings, the subsequent bounded factorization finds it right away.

Caveat. It is not possible to set a new alarm *within* another `alarm` code: the new timer erases the parent one.

The library syntax is `GEN gp_alarm(long s, GEN code = NULL)`.

3.15.4 alias(*newsym*, *sym*). Defines the symbol *newsym* as an alias for the symbol *sym*:

```
? alias("det", "matdet");
? det([1,2;3,4])
%1 = -2
```

You are not restricted to ordinary functions, as in the above example: to alias (from/to) member functions, prefix them with '`_.`'; to alias operators, use their internal name, obtained by writing `_` in lieu of the operators argument: for instance, `_!` and `!_` are the internal names of the factorial and the logical negation, respectively.

```
? alias("mod", "_.mod");
? alias("add", "_+_");
? alias("_.sin", "sin");
```

```
? mod(Mod(x,x^4+1))
%2 = x^4 + 1
? add(4,6)
%3 = 10
? Pi.sin
%4 = 0.E-37
```

Alias expansion is performed directly by the internal GP compiler. Note that since alias is performed at compilation-time, it does not require any run-time processing, however it only affects GP code compiled *after* the alias command is evaluated. A slower but more flexible alternative is to use variables. Compare

```
? fun = sin;
? g(a,b) = intnum(t=a,b,fun(t));
? g(0, Pi)
%3 = 2.0000000000000000000000000000000000000
? fun = cos;
? g(0, Pi)
%5 = 1.8830410776607851098 E-39
```

with

```
? alias(fun, sin);
? g(a,b) = intnum(t=a,b,fun(t));
? g(0,Pi)
%2 = 2.0000000000000000000000000000000000000
? alias(fun, cos);  \\ Oops. Does not affect *previous* definition!
? g(0,Pi)
%3 = 2.0000000000000000000000000000000000000
? g(a,b) = intnum(t=a,b,fun(t)); \\ Redefine, taking new alias into account
? g(0,Pi)
%5 = 1.8830410776607851098 E-39
```

A sample alias file `misc/gpalias` is provided with the standard distribution.

The library syntax is `void alias0(const char *newsym, const char *sym)`.

3.15.5 allocatemem($\{s = 0\}$). This special operation changes the stack size *after* initialization. x must be a non-negative integer. If $x > 0$, a new stack of at least x bytes is allocated. We may allocate more than x bytes if x is way too small, or for alignment reasons: the current formula is $\max(16 * \lceil x/16 \rceil, 500032)$ bytes.

If $x = 0$, the size of the new stack is twice the size of the old one.

This command is much more useful if `parisizemax` is non-zero, and we describe this case first. With `parisizemax` enabled, there are three sizes of interest:

• a virtual stack size, `parisizemax`, which is an absolute upper limit for the stack size; this is set by `default(parisizemax, ...)`.

• the desired typical stack size, `parisize`, that will grow as needed, up to `parisizemax`; this is set by `default(parisize, ...)`.

• the current stack size, which is less that `parisizemax`, typically equal to `parisize` but possibly larger and increasing dynamically as needed; `allocatemem` allows to change that one explicitly.

The `allocatemem` command forces stack usage to increase temporarily (up to `parisizemax` of course); for instance if you notice using `\gm2` that we seem to collect garbage a lot, e.g.

```
? \gm2
  debugmem = 2
? default(parisize,"32M")
 ***   Warning: new stack size = 32000000 (30.518 Mbytes).
? bnfinit('x^2+10^30-1)
 *** bnfinit: collecting garbage in hnffinal, i = 1.
 *** bnfinit: collecting garbage in hnffinal, i = 2.
 *** bnfinit: collecting garbage in hnffinal, i = 3.
```

and so on for hundred of lines. Then, provided the `breakloop` default is set, you can interrupt the computation, type `allocatemem(100*10^6)` at the break loop prompt, then let the computation go on by typing `<Enter>`. Back at the `gp` prompt, the desired stack size of `parisize` is restored. Note that changing either `parisize` or `parisizemax` at the break loop prompt would interrupt the computation, contrary to the above.

In most cases, `parisize` will increase automatically (up to `parisizemax`) and there is no need to perform the above maneuvers. But that the garbage collector is sufficiently efficient that a given computation can still run without increasing the stack size, albeit very slowly due to the frequent garbage collections.

Deprecated: when `parisizemax` is unset. This is currently still the default behavior in order not to break backward compatibility. The rest of this section documents the behavior of `allocatemem` in that (deprecated) situation: it becomes a synonym for `default(parisize,...)`. In that case, there is no notion of a virtual stack, and the stack size is always equal to `parisize`. If more memory is needed, the PARI stack overflows, aborting the computation.

Thus, increasing `parisize` via `allocatemem` or `default(parisize,...)` before a big computation is important. Unfortunately, either must be typed at the `gp` prompt in interactive usage, or left by itself at the start of batch files. They cannot be used meaningfully in loop-like constructs, or as part of a larger expression sequence, e.g

```
allocatemem(); x = 1;    \\ This will not set x!
```

In fact, all loops are immediately exited, user functions terminated, and the rest of the sequence following `allocatemem()` is silently discarded, as well as all pending sequences of instructions. We just go on reading the next instruction sequence from the file we are in (or from the user). In particular, we have the following possibly unexpected behavior: in

```
read("file.gp"); x = 1
```

were `file.gp` contains an `allocatemem` statement, the `x = 1` is never executed, since all pending instructions in the current sequence are discarded.

The reason for these unfortunate side-effects is that, with `parisizemax` disabled, increasing the stack size physically moves the stack, so temporary objects created during the current expression evaluation are not correct anymore. (In particular byte-compiled expressions, which are allocated on the stack.) To avoid accessing obsolete pointers to the old stack, this routine ends by a `longjmp`.

The library syntax is `void gp_allocatemem(GEN s = NULL)`.

3.15.6 apply(f, A). Apply the `t_CLOSURE` `f` to the entries of `A`. If `A` is a scalar, return `f(A)`. If `A` is a polynomial or power series, apply `f` on all coefficients. If `A` is a vector or list, return the elements $f(x)$ where x runs through `A`. If `A` is a matrix, return the matrix whose entries are the $f($`A[i,j]`$)$.

```
? apply(x->x^2, [1,2,3,4])
%1 = [1, 4, 9, 16]
? apply(x->x^2, [1,2;3,4])
%2 =
[1 4]

[9 16]
? apply(x->x^2, 4*x^2 + 3*x+ 2)
%3 = 16*x^2 + 9*x + 4
```

Note that many functions already act componentwise on vectors or matrices, but they almost never act on lists; in this case, `apply` is a good solution:

```
? L = List([Mod(1,3), Mod(2,4)]);
? lift(L)
  ***   at top-level: lift(L)
  ***                 ^-------
  *** lift: incorrect type in lift.
? apply(lift, L);
%2 = List([1, 2])
```

Remark. For v a `t_VEC`, `t_COL`, `t_LIST` or `t_MAT`, the alternative set-notations

```
[g(x) | x <- v, f(x)]
[x | x <- v, f(x)]
[g(x) | x <- v]
```

are available as shortcuts for

```
apply(g, select(f, Vec(v)))
select(f, Vec(v))
apply(g, Vec(v))
```

respectively:

```
? L = List([Mod(1,3), Mod(2,4)]);
? [ lift(x) | x<-L ]
%2 = [1, 2]
```

The library syntax is **genapply**(`void *E, GEN (*fun)(void*,GEN), GEN a`).

3.15.7 call(f, A). $A = [a_1, \ldots, a_n]$ being a vector and f being a function, returns the evaluation of $f(a_1, \ldots, a_n)$. f can also be the name of a built-in GP function. If $\#A = 1$, `call`(f, A) = `apply`(f, A)[1]. If f is variadic, the variadic arguments must grouped in a vector in the last component of A.

This function is useful

• when writing a variadic function, to call another one:

```
fprintf(file,format,args[..]) = write(file,call(Strprintf,[format,args]))
```

• when dealing with function arguments with unspecified arity

The function below implements a global memoization interface:

```
memo=Map();
memoize(f,A[..])=
{
  my(res);
  if(!mapisdefined(memo, [f,A], &res),
    res = call(f,A);
    mapput(memo,[f,A],res));
 res;
}
```

for example:

```
? memoize(factor,2^128+1)
%3 = [59649589127497217,1;5704689200685129054721,1]
? ##
  ***   last result computed in 76 ms.
? memoize(factor,2^128+1)
%4 = [59649589127497217,1;5704689200685129054721,1]
? ##
  ***   last result computed in 0 ms.
? memoize(ffinit,3,3)
%5 = Mod(1,3)*x^3+Mod(1,3)*x^2+Mod(1,3)*x+Mod(2,3)
? fibo(n)=if(n==0,0,n==1,1,memoize(fibo,n-2)+memoize(fibo,n-1));
? fibo(100)
%7 = 354224848179261915075
```

• to call operators through their internal names without using `alias`

```
matnbelts(M) = call("_*_",matsize(M))
```

The library syntax is `GEN call0(GEN f, GEN A)`.

3.15.8 default$(\{key\}, \{val\})$. Returns the default corresponding to keyword *key*. If *val* is present, sets the default to *val* first (which is subject to string expansion first). Typing `default()` (or `\d`) yields the complete default list as well as their current values. See Section 2.12 for an introduction to GP defaults, Section 3.17 for a list of available defaults, and Section 2.13 for some shortcut alternatives. Note that the shortcuts are meant for interactive use and usually display more information than `default`.

The library syntax is `GEN default0(const char *key = NULL, const char *val = NULL)`.

3.15.9 errname(E). Returns the type of the error message `E` as a string.

The library syntax is `GEN errname(GEN E)`.

3.15.10 error($\{str\}*$). Outputs its argument list (each of them interpreted as a string), then interrupts the running gp program, returning to the input prompt. For instance

```
error("n = ", n, " is not squarefree!")
```

3.15.11 extern(str). The string str is the name of an external command (i.e. one you would type from your UNIX shell prompt). This command is immediately run and its output fed into gp, just as if read from a file.

The library syntax is `GEN gpextern(const char *str)`.

3.15.12 externstr(str). The string str is the name of an external command (i.e. one you would type from your UNIX shell prompt). This command is immediately run and its output is returned as a vector of GP strings, one component per output line.

The library syntax is `GEN externstr(const char *str)`.

3.15.13 fold(f, A). Apply the `t_CLOSURE f` of arity 2 to the entries of `A`, in order to return `f(...f(f(A[1],A[2]),A[3])...,A[#A])`.

```
? fold((x,y)->x*y, [1,2,3,4])
%1 = 24
? fold((x,y)->[x,y], [1,2,3,4])
%2 = [[[1, 2], 3], 4]
? fold((x,f)->f(x), [2,sqr,sqr,sqr])
%3 = 256
? fold((x,y)->(x+y)/(1-x*y),[1..5])
%4 = -9/19
? bestappr(tan(sum(i=1,5,atan(i))))
%5 = -9/19
```

The library syntax is `GEN fold0(GEN f, GEN A)`. Also available is `GEN genfold(void *E, GEN (*fun)(void*, GEN, GEN), GEN A)`.

3.15.14 getabstime(). Returns the CPU time (in milliseconds) elapsed since gp startup. This provides a reentrant version of `gettime`:

```
my (t = getabstime());
...
print("Time: ", getabstime() - t);
```

For a version giving wall-clock time, see `getwalltime`.

The library syntax is `long getabstime()`.

3.15.15 getenv(s). Return the value of the environment variable `s` if it is defined, otherwise return 0.

The library syntax is `GEN gp_getenv(const char *s)`.

3.15.16 getheap(). Returns a two-component row vector giving the number of objects on the heap and the amount of memory they occupy in long words. Useful mainly for debugging purposes.

The library syntax is `GEN getheap()`.

3.15.17 getrand(). Returns the current value of the seed used by the pseudo-random number generator `random`. Useful mainly for debugging purposes, to reproduce a specific chain of computations. The returned value is technical (reproduces an internal state array), and can only be used as an argument to `setrand`.

The library syntax is `GEN getrand()`.

3.15.18 getstack(). Returns the current value of `top` − `avma`, i.e. the number of bytes used up to now on the stack. Useful mainly for debugging purposes.

The library syntax is `long getstack()`.

3.15.19 gettime(). Returns the CPU time (in milliseconds) used since either the last call to `gettime`, or to the beginning of the containing GP instruction (if inside `gp`), whichever came last.

For a reentrant version, see `getabstime`.

For a version giving wall-clock time, see `getwalltime`.

The library syntax is `long gettime()`.

3.15.20 getwalltime(). Returns the time (in milliseconds) elapsed since the UNIX Epoch (1970-01-01 00:00:00 (UTC)).

```
my (t = getwalltime());
...
print("Time: ", getwalltime() - t);
```

The library syntax is `GEN getwalltime()`.

3.15.21 global(*list of variables*). Obsolete. Scheduled for deletion.

3.15.22 inline(x, ..., z). (Experimental) declare $x, \ldots, z$ as inline variables. Such variables behave like lexically scoped variable (see my()) but with unlimited scope. It is however possible to exit the scope by using `uninline()`. When used in a GP script, it is recommended to call `uninline()` before the script's end to avoid inline variables leaking outside the script.

3.15.23 input(). Reads a string, interpreted as a GP expression, from the input file, usually standard input (i.e. the keyboard). If a sequence of expressions is given, the result is the result of the last expression of the sequence. When using this instruction, it is useful to prompt for the string by using the `print1` function. Note that in the present version 2.19 of `pari.el`, when using `gp` under GNU Emacs (see Section 2.16) one *must* prompt for the string, with a string which ends with the same prompt as any of the previous ones (a `"? "` will do for instance).

The library syntax is `GEN gp_input()`.

3.15.24 install(*name*, *code*, {*gpname*}, {*lib*}). Loads from dynamic library *lib* the function *name*. Assigns to it the name *gpname* in this gp session, with *prototype code* (see below). If *gpname* is omitted, uses *name*. If *lib* is omitted, all symbols known to gp are available: this includes the whole of `libpari.so` and possibly others (such as `libc.so`).

Most importantly, `install` gives you access to all non-static functions defined in the PARI library. For instance, the function

```
GEN addii(GEN x, GEN y)
```

adds two PARI integers, and is not directly accessible under gp (it is eventually called by the + operator of course):

```
? install("addii", "GG")
? addii(1, 2)
%1 = 3
```

It also allows to add external functions to the gp interpreter. For instance, it makes the function `system` obsolete:

```
? install(system, vs, sys,/*omitted*/)
? sys("ls gp*")
gp.c            gp.h            gp_rl.c
```

This works because `system` is part of `libc.so`, which is linked to gp. It is also possible to compile a shared library yourself and provide it to gp in this way: use `gp2c`, or do it manually (see the `modules_build` variable in `pari.cfg` for hints).

Re-installing a function will print a warning and update the prototype code if needed. However, it will not reload a symbol from the library, even if the latter has been recompiled.

Prototype. We only give a simplified description here, covering most functions, but there are many more possibilities. The full documentation is available in `libpari.dvi`, see

```
??prototype
```

• First character `i`, `l`, `v` : return type int / long / void. (Default: `GEN`)

• One letter for each mandatory argument, in the same order as they appear in the argument list: `G` (`GEN`), `&` (`GEN*`), `L` (`long`), `s` (`char *`), `n` (variable).

• `p` to supply `realprecision` (usually `long prec` in the argument list), `P` to supply `seriesprecision` (usually `long precdl`).

We also have special constructs for optional arguments and default values:

• `DG` (optional `GEN`, `NULL` if omitted),

• `D&` (optional `GEN*`, `NULL` if omitted),

• `Dn` (optional variable, -1 if omitted),

For instance the prototype corresponding to

```
long issquareall(GEN x, GEN *n = NULL)
```

is `lGD&`.

Caution. This function may not work on all systems, especially when gp has been compiled statically. In that case, the first use of an installed function will provoke a Segmentation Fault (this should never happen with a dynamically linked executable). If you intend to use this function, please check first on some harmless example such as the one above that it works properly on your machine.

The library syntax is `void gpinstall(const char *name, const char *code, const char *gpname, const char *lib)`.

3.15.25 kill(*sym*). Restores the symbol `sym` to its "undefined" status, and deletes any help messages attached to `sym` using `addhelp`. Variable names remain known to the interpreter and keep their former priority: you cannot make a variable "less important" by killing it!

```
? z = y = 1; y
%1 = 1
? kill(y)
? y            \\ restored to ``undefined'' status
%2 = y
? variable()
%3 = [x, y, z] \\ but the variable name y is still known, with y > z !
```

For the same reason, killing a user function (which is an ordinary variable holding a `t_CLOSURE`) does not remove its name from the list of variable names.

If the symbol is attached to a variable — user functions being an important special case —, one may use the quote operator `a = 'a` to reset variables to their starting values. However, this will not delete a help message attached to `a`, and is also slightly slower than `kill(a)`.

```
? x = 1; addhelp(x, "foo"); x
%1 = 1
? x = 'x; x   \\ same as 'kill', except we don't delete help.
%2 = x
? ?x
foo
```

On the other hand, `kill` is the only way to remove aliases and installed functions.

```
? alias(fun, sin);
? kill(fun);
? install(addii, GG);
? kill(addii);
```

The library syntax is `void kill0(const char *sym)`.

3.15.26 listcreate({n}). This function is obsolete, use `List`.

Creates an empty list. This routine used to have a mandatory argument, which is now ignored (for backward compatibility).

3.15.27 listinsert(L, x, n). Inserts the object x at position n in L (which must be of type `t_LIST`). This has complexity $O(\#L - n + 1)$: all the remaining elements of *list* (from position $n+1$ onwards) are shifted to the right.

The library syntax is `GEN listinsert(GEN L, GEN x, long n)`.

3.15.28 listkill(L). Obsolete, retained for backward compatibility. Just use `L = List()` instead of `listkill(L)`. In most cases, you won't even need that, e.g. local variables are automatically cleared when a user function returns.

The library syntax is `void listkill(GEN L)`.

3.15.29 listpop($list, \{n\}$). Removes the n-th element of the list *list* (which must be of type `t_LIST`). If n is omitted, or greater than the list current length, removes the last element. If the list is already empty, do nothing. This runs in time $O(\#L - n + 1)$.

The library syntax is `void listpop0(GEN list, long n)`.

3.15.30 listput($list, x, \{n\}$). Sets the n-th element of the list *list* (which must be of type `t_LIST`) equal to x. If n is omitted, or greater than the list length, appends x. The function returns the inserted element.

```
? L = List();
? listput(L, 1)
%2 = 1
? listput(L, 2)
%3 = 2
? L
%4 = List([1, 2])
```

You may put an element into an occupied cell (not changing the list length), but it is easier to use the standard `list[n] = x` construct.

```
? listput(L, 3, 1) \\ insert at position 1
%5 = 3
? L
%6 = List([3, 2])
? L[2] = 4 \\ simpler
%7 = List([3, 4])
? L[10] = 1  \\ can't insert beyond the end of the list
 ***   at top-level: L[10]=1
 ***                 ^------
 ***   non-existent component: index > 2
? listput(L, 1, 10) \\ but listput can
%8 = 1
? L
%9 = List([3, 2, 1])
```

This function runs in time $O(\#L)$ in the worst case (when the list must be reallocated), but in time $O(1)$ on average: any number of successive `listput`s run in time $O(\#L)$, where $\#L$ denotes the list *final* length.

The library syntax is `GEN listput0(GEN list, GEN x, long n)`.

3.15.31 listsort$(L, \{flag = 0\})$. Sorts the `t_LIST` *list* in place, with respect to the (somewhat arbitrary) universal comparison function `cmp`. In particular, the ordering is the same as for sets and `setsearch` can be used on a sorted list.

```
? L = List([1,2,4,1,3,-1]); listsort(L); L
%1 = List([-1, 1, 1, 2, 3, 4])
? setsearch(L, 4)
%2 = 6
? setsearch(L, -2)
%3 = 0
```

This is faster than the `vecsort` command since the list is sorted in place: no copy is made. No value returned.

If *flag* is non-zero, suppresses all repeated coefficients.

The library syntax is `void listsort(GEN L, long flag)`.

3.15.32 localbitprec(p). Set the real precision to p bits in the dynamic scope. All computations are performed as if `realbitprecision` was p: transcendental constants (e.g. `Pi`) and conversions from exact to floating point inexact data use p bits, as well as iterative routines implicitly using a floating point accuracy as a termination criterion (e.g. `solve` or `intnum`). But `realbitprecision` itself is unaffected and is "unmasked" when we exit the dynamic (*not* lexical) scope. In effect, this is similar to

```
my(bit = default(realbitprecision));
default(realbitprecision,p);
...
default(realbitprecision, bit);
```

but is both less cumbersome, cleaner (no need to manipulate a global variable, which in fact never changes and is only temporarily masked) and more robust: if the above computation is interrupted or an exception occurs, `realbitprecision` will not be restored as intended.

Such `localbitprec` statements can be nested, the innermost one taking precedence as expected. Beware that `localbitprec` follows the semantic of `local`, not `my`: a subroutine called from `localbitprec` scope uses the local accuracy:

```
? f()=bitprecision(1.0);
? f()
%2 = 128
? localbitprec(1000); f()
%3 = 1024
```

Note that the bit precision of *data* (`1.0` in the above example) increases by steps of 64 (32 on a 32-bit machine) so we get 1024 instead of the expected 1000; `localbitprec` bounds the relative error exactly as specified in functions that support that granularity (e.g. `lfun`), and rounded to the next multiple of 64 (resp. 32) everywhere else.

Warning. Changing `realbitprecision` or `realprecision` in programs is deprecated in favor of `localbitprec` and `localprec`. Think about the `realprecision` and `realbitprecision` defaults as interactive commands for the `gp` interpreter, best left out of GP programs. Indeed, the above rules imply that mixing both constructs yields surprising results:

```
? \p38
? localprec(19); default(realprecision,1000);  Pi
%1 = 3.141592653589793239
? \p
  realprecision = 1001 significant digits (1000 digits displayed)
```

Indeed, `realprecision` itself is ignored within `localprec` scope, so `Pi` is computed to a low accuracy. And when we leave the `localprec` scope, `realprecision` only regains precedence, it is not "restored" to the original value.

3.15.33 localprec(p). Set the real precision to p in the dynamic scope. All computations are performed as if `realprecision` was p: transcendental constants (e.g. `Pi`) and conversions from exact to floating point inexact data use p decimal digits, as well as iterative routines implicitly using a floating point accuracy as a termination criterion (e.g. `solve` or `intnum`). But `realprecision` itself is unaffected and is "unmasked" when we exit the dynamic (*not* lexical) scope. In effect, this is similar to

```
my(prec = default(realprecision));
default(realprecision,p);
...
default(realprecision, prec);
```

but is both less cumbersome, cleaner (no need to manipulate a global variable, which in fact never changes and is only temporarily masked) and more robust: if the above computation is interrupted or an exception occurs, `realprecision` will not be restored as intended.

Such `localprec` statements can be nested, the innermost one taking precedence as expected. Beware that `localprec` follows the semantic of `local`, not `my`: a subroutine called from `localprec` scope uses the local accuracy:

```
? f()=precision(1.);
? f()
%2 = 38
? localprec(19); f()
%3 = 19
```

Warning. Changing `realprecision` itself in programs is now deprecated in favor of `localprec`. Think about the `realprecision` default as an interactive command for the `gp` interpreter, best left out of GP programs. Indeed, the above rules imply that mixing both constructs yields surprising results:

```
? \p38
? localprec(19); default(realprecision,100);  Pi
%1 = 3.141592653589793239
? \p
    realprecision = 115 significant digits (100 digits displayed)
```

Indeed, `realprecision` itself is ignored within `localprec` scope, so `Pi` is computed to a low accuracy. And when we leave `localprec` scope, `realprecision` only regains precedence, it is not "restored" to the original value.

3.15.34 mapdelete(M, x). Removes x from the domain of the map M.

```
? M = Map(["a",1; "b",3; "c",7]);
? mapdelete(M,"b");
? Mat(M)
["a" 1]

["c" 7]
```

The library syntax is `void mapdelete(GEN M, GEN x)`.

3.15.35 mapget(M, x). Returns the image of x by the map M.

```
? M=Map(["a",23;"b",43]);
? mapget(M,"a")
%2 = 23
? mapget(M,"b")
%3 = 43
```

Raises an exception when the key x is not present in M.

```
? mapget(M,"c")
  ***   at top-level: mapget(M,"c")
  ***                 ^-------------
  *** mapget: non-existent component in mapget: index not in map
```

The library syntax is `GEN mapget(GEN M, GEN x)`.

3.15.36 mapisdefined($M, x, \{\&z\}$). Returns true (1) if x has an image by the map M, false (0) otherwise. If z is present, set z to the image of x, if it exists.

```
? M1 = Map([1, 10; 2, 20]);
? mapisdefined(M1,3)
%1 = 0
? mapisdefined(M1, 1, &z)
%2 = 1
? z
%3 = 10
```

```
? M2 = Map(); N = 19;
? for (a=0, N-1, mapput(M2, a^3%N, a));
? {for (a=0, N-1,
     if (mapisdefined(M2, a, &b),
       printf("%d is the cube of %d mod %d\n",a,b,N)));}
0 is the cube of 0 mod 19
1 is the cube of 11 mod 19
7 is the cube of 9 mod 19
8 is the cube of 14 mod 19
11 is the cube of 17 mod 19
12 is the cube of 15 mod 19
18 is the cube of 18 mod 19
```

The library syntax is `GEN mapisdefined(GEN M, GEN x, GEN *z = NULL)`.

3.15.37 mapput(M, x, y). Associates x to y in the map M. The value y can be retrieved with mapget.

```
? M = Map();
? mapput(M, "foo", 23);
? mapput(M, 7718, "bill");
? mapget(M, "foo")
%4 = 23
? mapget(M, 7718)
%5 = "bill"
? Vec(M)  \\ keys
%6 = [7718, "foo"]
? Mat(M)
%7 =
[ 7718 "bill"]

["foo"     23]
```

The library syntax is `void mapput(GEN M, GEN x, GEN y)`.

3.15.38 print($\{str\}*$). Outputs its (string) arguments in raw format, ending with a newline.

3.15.39 print1($\{str\}*$). Outputs its (string) arguments in raw format, without ending with a newline. Note that you can still embed newlines within your strings, using the \n notation !

3.15.40 printf(*fmt*, {*x*}∗). This function is based on the C library command of the same name. It prints its arguments according to the format *fmt*, which specifies how subsequent arguments are converted for output. The format is a character string composed of zero or more directives:

• ordinary characters (not `%`), printed unchanged,

• conversions specifications (`%` followed by some characters) which fetch one argument from the list and prints it according to the specification.

More precisely, a conversion specification consists in a `%`, one or more optional flags (among `#`, `0`, `-`, `+`, ‘ ’), an optional decimal digit string specifying a minimal field width, an optional precision in the form of a period (‘`.`’) followed by a decimal digit string, and the conversion specifier (among `d`,`i`, `o`, `u`, `x`,`X`, `p`, `e`,`E`, `f`, `g`,`G`, `s`).

The flag characters. The character `%` is followed by zero or more of the following flags:

• `#`: the value is converted to an “alternate form”. For `o` conversion (octal), a `0` is prefixed to the string. For `x` and `X` conversions (hexa), respectively `0x` and `0X` are prepended. For other conversions, the flag is ignored.

• `0`: the value should be zero padded. For `d`, `i`, `o`, `u`, `x`, `X` `e`, `E`, `f`, `F`, `g`, and `G` conversions, the value is padded on the left with zeros rather than blanks. (If the `0` and `-` flags both appear, the `0` flag is ignored.)

• `-`: the value is left adjusted on the field boundary. (The default is right justification.) The value is padded on the right with blanks, rather than on the left with blanks or zeros. A `-` overrides a `0` if both are given.

• ‘ ’ (a space): a blank is left before a positive number produced by a signed conversion.

• `+`: a sign (+ or -) is placed before a number produced by a signed conversion. A `+` overrides a space if both are used.

The field width. An optional decimal digit string (whose first digit is non-zero) specifying a *minimum* field width. If the value has fewer characters than the field width, it is padded with spaces on the left (or right, if the left-adjustment flag has been given). In no case does a small field width cause truncation of a field; if the value is wider than the field width, the field is expanded to contain the conversion result. Instead of a decimal digit string, one may write `*` to specify that the field width is given in the next argument.

The precision. An optional precision in the form of a period (‘`.`’) followed by a decimal digit string. This gives the number of digits to appear after the radix character for `e`, `E`, `f`, and `F` conversions, the maximum number of significant digits for `g` and `G` conversions, and the maximum number of characters to be printed from an `s` conversion. Instead of a decimal digit string, one may write `*` to specify that the field width is given in the next argument.

The length modifier. This is ignored under `gp`, but necessary for `libpari` programming. Description given here for completeness:

• `l`: argument is a `long` integer.

• `P`: argument is a `GEN`.

The conversion specifier. A character that specifies the type of conversion to be applied.

• `d`, `i`: a signed integer.

• `o`, `u`, `x`, `X`: an unsigned integer, converted to unsigned octal (`o`), decimal (`u`) or hexadecimal (`x` or `X`) notation. The letters `abcdef` are used for `x` conversions; the letters `ABCDEF` are used for `X` conversions.

• `e`, `E`: the (real) argument is converted in the style `[ -]d.ddd e[ -]dd`, where there is one digit before the decimal point, and the number of digits after it is equal to the precision; if the precision is missing, use the current `realprecision` for the total number of printed digits. If the precision is explicitly 0, no decimal-point character appears. An `E` conversion uses the letter `E` rather than `e` to introduce the exponent.

• `f`, `F`: the (real) argument is converted in the style `[ -]ddd.ddd`, where the number of digits after the decimal point is equal to the precision; if the precision is missing, use the current `realprecision` for the total number of printed digits. If the precision is explicitly 0, no decimal-point character appears. If a decimal point appears, at least one digit appears before it.

• `g`, `G`: the (real) argument is converted in style `e` or `f` (or `E` or `F` for `G` conversions) `[ -]ddd.ddd`, where the total number of digits printed is equal to the precision; if the precision is missing, use the current `realprecision`. If the precision is explicitly 0, it is treated as 1. Style `e` is used when the decimal exponent is < -4, to print `0.`, or when the integer part cannot be decided given the known significant digits, and the `f` format otherwise.

• `c`: the integer argument is converted to an unsigned char, and the resulting character is written.

• `s`: convert to a character string. If a precision is given, no more than the specified number of characters are written.

• `p`: print the address of the argument in hexadecimal (as if by `%#x`).

• `%`: a `%` is written. No argument is converted. The complete conversion specification is `%%`.

Examples:

```
? printf("floor: %d, field width 3: %3d, with sign: %+3d\n", Pi, 1, 2);
floor: 3, field width 3:   1, with sign:  +2

? printf("%.5g %.5g %.5g\n",123,123/456,123456789);
123.00 0.26974 1.2346 e8

? printf("%-2.5s:%2.5s:%2.5s\n", "P", "PARI", "PARIGP");
P :PARI:PARIG

\\ min field width and precision given by arguments
? x = 23; y=-1/x; printf("x=%+06.2f y=%+0*.*f\n", x, 6, 2, y);
x=+23.00 y=-00.04

\\ minimum fields width 5, pad left with zeroes
? for (i = 2, 5, printf("%05d\n", 10^i))
00100
01000
10000
100000  \\ don't truncate fields whose length is larger than the minimum width
? printf("%.2f  |%06.2f|", Pi,Pi)
```

```
3.14  |  3.14|
```

All numerical conversions apply recursively to the entries of vectors and matrices:

```
? printf("%4d", [1,2,3]);
[   1,   2,   3]
? printf("%5.2f", mathilbert(3));
[ 1.00  0.50  0.33]

[ 0.50  0.33  0.25]

[ 0.33  0.25  0.20]
```

Technical note. Our implementation of `printf` deviates from the C89 and C99 standards in a few places:

• whenever a precision is missing, the current `realprecision` is used to determine the number of printed digits (C89: use 6 decimals after the radix character).

• in conversion style `e`, we do not impose that the exponent has at least two digits; we never write a `+` sign in the exponent; 0 is printed in a special way, always as `0.E`*exp*.

• in conversion style `f`, we switch to style `e` if the exponent is greater or equal to the precision.

• in conversion `g` and `G`, we do not remove trailing zeros from the fractional part of the result; nor a trailing decimal point; 0 is printed in a special way, always as `0.E`*exp*.

3.15.41 printsep(*sep*, {*str*}∗). Outputs its (string) arguments in raw format, ending with a newline. Successive entries are separated by *sep*:

```
? printsep(":", 1,2,3,4)
1:2:3:4
```

3.15.42 printsep1(*sep*, {*str*}∗). Outputs its (string) arguments in raw format, without ending with a newline. Successive entries are separated by *sep*:

```
? printsep1(":", 1,2,3,4);print("|")
1:2:3:4
```

3.15.43 printtex({*str*}∗). Outputs its (string) arguments in TEX format. This output can then be used in a TEX manuscript. The printing is done on the standard output. If you want to print it to a file you should use `writetex` (see there).

Another possibility is to enable the `log` default (see Section 2.12). You could for instance do:

```
default(logfile, "new.tex");
default(log, 1);
printtex(result);
```

3.15.44 quit({*status* = 0}). Exits `gp` and return to the system with exit status `status`, a small integer. A non-zero exit status normally indicates abnormal termination. (Note: the system actually sees only `status` mod 256, see your man pages for `exit(3)` or `wait(2)`).

3.15.45 read({*filename*}). Reads in the file *filename* (subject to string expansion). If *filename* is omitted, re-reads the last file that was fed into `gp`. The return value is the result of the last expression evaluated.

If a GP `binary file` is read using this command (see Section 3.15.60), the file is loaded and the last object in the file is returned.

In case the file you read in contains an `allocatemem` statement (to be generally avoided), you should leave `read` instructions by themselves, and not part of larger instruction sequences.

The library syntax is `GEN gp_read_file(const char *filename)`.

3.15.46 readstr({*filename*}). Reads in the file *filename* and return a vector of GP strings, each component containing one line from the file. If *filename* is omitted, re-reads the last file that was fed into `gp`.

The library syntax is `GEN readstr(const char *filename)`.

3.15.47 readvec({*filename*}). Reads in the file *filename* (subject to string expansion). If *filename* is omitted, re-reads the last file that was fed into `gp`. The return value is a vector whose components are the evaluation of all sequences of instructions contained in the file. For instance, if *file* contains

```
1
2
3
```

then we will get:

```
? \r a
%1 = 1
%2 = 2
%3 = 3
? read(a)
%4 = 3
? readvec(a)
%5 = [1, 2, 3]
```

In general a sequence is just a single line, but as usual braces and `\` may be used to enter multiline sequences.

The library syntax is `GEN gp_readvec_file(const char *filename)`. The underlying library function `GEN gp_readvec_stream(FILE *f)` is usually more flexible.

3.15.48 select($f, A, \{flag = 0\}$). We first describe the default behavior, when *flag* is 0 or omitted. Given a vector or list `A` and a `t_CLOSURE` `f`, `select` returns the elements x of `A` such that $f(x)$ is non-zero. In other words, `f` is seen as a selection function returning a boolean value.

```
? select(x->isprime(x), vector(50,i,i^2+1))
%1 = [2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601]
? select(x->(x<100), %)
%2 = [2, 5, 17, 37]
```

returns the primes of the form $i^2 + 1$ for some $i \leq 50$, then the elements less than 100 in the preceding result. The `select` function also applies to a matrix `A`, seen as a vector of columns, i.e. it selects columns instead of entries, and returns the matrix whose columns are the selected ones.

Remark. For v a `t_VEC`, `t_COL`, `t_LIST` or `t_MAT`, the alternative set-notations

```
[g(x) | x <- v, f(x)]
[x | x <- v, f(x)]
[g(x) | x <- v]
```

are available as shortcuts for

```
apply(g, select(f, Vec(v)))
select(f, Vec(v))
apply(g, Vec(v))
```

respectively:

```
? [ x | x <- vector(50,i,i^2+1), isprime(x) ]
%1 = [2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601]
```

If *flag* $= 1$, this function returns instead the *indices* of the selected elements, and not the elements themselves (indirect selection):

```
? V = vector(50,i,i^2+1);
? select(x->isprime(x), V, 1)
%2 = Vecsmall([1, 2, 4, 6, 10, 14, 16, 20, 24, 26, 36, 40])
? vecextract(V, %)
%3 = [2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601]
```

The following function lists the elements in $(\mathbf{Z}/N\mathbf{Z})^*$:

```
? invertibles(N) = select(x->gcd(x,N) == 1, [1..N])
```

Finally

```
? select(x->x, M)
```

selects the non-0 entries in `M`. If the latter is a `t_MAT`, we extract the matrix of non-0 columns. Note that *removing* entries instead of selecting them just involves replacing the selection function `f` with its negation:

```
? select(x->!isprime(x), vector(50,i,i^2+1))
```

The library syntax is **genselect**(`void *E, long (*fun)(void*,GEN), GEN a`). Also available is `GEN genindexselect(void *E, long (*fun)(void*, GEN), GEN a)`, corresponding to *flag* $= 1$.

3.15.49 self(). Return the calling function or closure as a `t_CLOSURE` object. This is useful for defining anonymous recursive functions.

```
? (n->if(n==0,1,n*self()(n-1)))(5)
%1 = 120
```

The library syntax is `GEN pari_self()`.

3.15.50 setrand(n). Reseeds the random number generator using the seed n. No value is returned. The seed is either a technical array output by `getrand`, or a small positive integer, used to generate deterministically a suitable state array. For instance, running a randomized computation starting by `setrand(1)` twice will generate the exact same output.

The library syntax is `void setrand(GEN n)`.

3.15.51 system(*str*). *str* is a string representing a system command. This command is executed, its output written to the standard output (this won't get into your logfile), and control returns to the PARI system. This simply calls the C `system` command.

The library syntax is `void gpsystem(const char *str)`.

3.15.52 trap($\{e\}, \{rec\}, seq$). This function is obsolete, use `iferr`, which has a nicer and much more powerful interface. For compatibility's sake we now describe the *obsolete* function `trap`.

This function tries to evaluate *seq*, trapping runtime error e, that is effectively preventing it from aborting computations in the usual way; the recovery sequence *rec* is executed if the error occurs and the evaluation of *rec* becomes the result of the command. If e is omitted, all exceptions are trapped. See Section 2.10.2 for an introduction to error recovery under `gp`.

```
? \\ trap division by 0
? inv(x) = trap (e_INV, INFINITY, 1/x)
? inv(2)
%1 = 1/2
? inv(0)
%2 = INFINITY
```

Note that *seq* is effectively evaluated up to the point that produced the error, and the recovery sequence is evaluated starting from that same context, it does not "undo" whatever happened in the other branch (restore the evaluation context):

```
? x = 1; trap (, /* recover: */ x, /* try: */ x = 0; 1/x)
%1 = 0
```

Note. The interface is currently not adequate for trapping individual exceptions. In the current version 2.9.2, the following keywords are recognized, but the name list will be expanded and changed in the future (all library mode errors can be trapped: it's a matter of defining the keywords to `gp`):

`e_ALARM`: alarm time-out

`e_ARCH`: not available on this architecture or operating system

`e_STACK`: the PARI stack overflows

`e_INV`: impossible inverse

`e_IMPL`: not yet implemented

`e_OVERFLOW`: all forms of arithmetic overflow, including length or exponent overflow (when a larger value is supplied than the implementation can handle).

`e_SYNTAX`: syntax error

`e_MISC`: miscellaneous error

`e_TYPE`: wrong type

`e_USER`: user error (from the `error` function)

The library syntax is `GEN trap0(const char *e = NULL, GEN rec = NULL, GEN seq = NULL)`.

3.15.53 type(x). This is useful only under `gp`. Returns the internal type name of the PARI object x as a string. Check out existing type names with the metacommand `\t`. For example `type(1)` will return "`t_INT`".

The library syntax is `GEN type0(GEN x)`. The macro `typ` is usually simpler to use since it returns a `long` that can easily be matched with the symbols `t_*`. The name `type` was avoided since it is a reserved identifier for some compilers.

3.15.54 uninline(). (Experimental) Exit the scope of all current `inline` variables.

3.15.55 version(). Returns the current version number as a `t_VEC` with three integer components (major version number, minor version number and patchlevel); if your sources were obtained through our version control system, this will be followed by further more precise arguments, including e.g. a `git` *commit hash*.

This function is present in all versions of PARI following releases 2.3.4 (stable) and 2.4.3 (testing).

Unless you are working with multiple development versions, you probably only care about the 3 first numeric components. In any case, the `lex` function offers a clever way to check against a particular version number, since it will compare each successive vector entry, numerically or as strings, and will not mind if the vectors it compares have different lengths:

```
if (lex(version(), [2,3,5]) >= 0,
  \\ code to be executed if we are running 2.3.5 or more recent.
,
  \\ compatibility code
);
```

On a number of different machines, `version()` could return either of

```
%1 = [2, 3, 4]    \\ released version, stable branch
%1 = [2, 4, 3]    \\ released version, testing branch
%1 = [2, 6, 1, 15174, ""505ab9b"] \\ development
```

In particular, if you are only working with released versions, the first line of the gp introductory message can be emulated by

```
[M,m,p] = version();
printf("GP/PARI CALCULATOR Version %s.%s.%s", M,m,p);
```

If you *are* working with many development versions of PARI/GP, the 4th and/or 5th components can be profitably included in the name of your logfiles, for instance.

Technical note. For development versions obtained via `git`, the 4th and 5th components are liable to change eventually, but we document their current meaning for completeness. The 4th component counts the number of reachable commits in the branch (analogous to `svn`'s revision number), and the 5th is the `git` commit hash. In particular, `lex` comparison still orders correctly development versions with respect to each others or to released versions (provided we stay within a given branch, e.g. `master`)!

The library syntax is `GEN pari_version()`.

3.15.56 warning({*str*}∗). Outputs the message "user warning" and the argument list (each of them interpreted as a string). If colors are enabled, this warning will be in a different color, making it easy to distinguish.

```
warning(n, " is very large, this might take a while.")
```

3.15.57 whatnow(*key*). If keyword *key* is the name of a function that was present in GP version 1.39.15, outputs the new function name and syntax, if it changed at all. Functions that where introduced since then, then modified are also recognized.

```
? whatnow("mu")
New syntax: mu(n) ===> moebius(n)

moebius(x): Moebius function of x.

? whatnow("sin")
This function did not change
```

When a function was removed and the underlying functionality is not available under a compatible interface, no equivalent is mentioned:

```
? whatnow("buchfu")
This function no longer exists
```

(The closest equivalent would be to set `K = bnfinit(T)` then access `K.fu`.)

3.15.58 write(*filename*, {*str*}∗). Writes (appends) to *filename* the remaining arguments, and appends a newline (same output as `print`).

3.15.59 write1(*filename*, {*str*}∗). Writes (appends) to *filename* the remaining arguments without a trailing newline (same output as `print1`).

3.15.60 writebin(*filename*, {*x*}). Writes (appends) to *filename* the object x in binary format. This format is not human readable, but contains the exact internal structure of x, and is much faster to save/load than a string expression, as would be produced by `write`. The binary file format includes a magic number, so that such a file can be recognized and correctly input by the regular `read` or `\r` function. If saved objects refer to polynomial variables that are not defined in the new session, they will be displayed as `t`n for some integer n (the attached variable number). Installed functions and history objects can not be saved via this function.

If x is omitted, saves all user variables from the session, together with their names. Reading such a "named object" back in a `gp` session will set the corresponding user variable to the saved value. E.g after

```
x = 1; writebin("log")
```

reading `log` into a clean session will set `x` to 1. The relative variables priorities (see Section 2.5.3) of new variables set in this way remain the same (preset variables retain their former priority, but are set to the new value). In particular, reading such a session log into a clean session will restore all variables exactly as they were in the original one.

Just as a regular input file, a binary file can be compressed using `gzip`, provided the file name has the standard `.gz` extension.

In the present implementation, the binary files are architecture dependent and compatibility with future versions of `gp` is not guaranteed. Hence binary files should not be used for long term storage (also, they are larger and harder to compress than text files).

The library syntax is `void gpwritebin(const char *filename, GEN x = NULL)`.

3.15.61 writetex(*filename*, {*str*}∗). As `write`, in TEX format.

3.16 Parallel programming.

These function are only available if PARI was configured using `Configure --mt=...`. Two multithread interfaces are supported:

- POSIX threads
- Message passing interface (MPI)

As a rule, POSIX threads are well-suited for single systems, while MPI is used by most clusters. However the parallel GP interface does not depend on the chosen multithread interface: a properly written GP program will work identically with both.

3.16.1 parapply(f, x). Parallel evaluation of `f` on the elements of `x`. The function `f` must not access global variables or variables declared with local(), and must be free of side effects.

```
parapply(factor,[2^256 + 1, 2^193 - 1])
```

factors $2^{256}+1$ and $2^{193}-1$ in parallel.

```
{
  my(E = ellinit([1,3]), V = vector(12,i,randomprime(2^200)));
  parapply(p->ellcard(E,p), V)
}
```

computes the order of $E(\mathbf{F}_p)$ for 12 random primes of 200 bits.

The library syntax is `GEN parapply(GEN f, GEN x)`.

3.16.2 pareval(x). Parallel evaluation of the elements of `x`, where `x` is a vector of closures. The closures must be of arity 0, must not access global variables or variables declared with `local` and must be free of side effects.

The library syntax is `GEN pareval(GEN x)`.

3.16.3 parfor($i = a, \{b\}, expr1, \{r\}, \{expr2\}$). Evaluates in parallel the expression `expr1` in the formal argument i running from a to b. If b is set to `+oo`, the loop runs indefinitely. If r and `expr2` are present, the expression `expr2` in the formal variables r and i is evaluated with r running through all the different results obtained for `expr1` and i takes the corresponding argument.

The computations of `expr1` are *started* in increasing order of i; otherwise said, the computation for $i = c$ is started after those for $i = 1, \ldots, c-1$ have been started, but before the computation for $i = c+1$ is started. Notice that the order of *completion*, that is, the order in which the different r become available, may be different; `expr2` is evaluated sequentially on each r as it appears.

The following example computes the sum of the squares of the integers from 1 to 10 by computing the squares in parallel and is equivalent to `parsum (i=1, 10, i^2)`:

```
? s=0;
? parfor (i=1, 10, i^2, r, s=s+r)
? s
%3 = 385
```

More precisely, apart from a potentially different order of evaluation due to the parallelism, the line containing `parfor` is equivalent to

```
? my (r); for (i=1, 10, r=i^2; s=s+r)
```

The sequentiality of the evaluation of `expr2` ensures that the variable `s` is not modified concurrently by two different additions, although the order in which the terms are added is non-deterministic.

It is allowed for `expr2` to exit the loop using `break/next/return`. If that happens for $i = c$, then the evaluation of `expr1` and `expr2` is continued for all values $i < c$, and the return value is the one obtained for the smallest i causing an interruption in `expr2` (it may be undefined if this is a `break/next`). In that case, using side-effects in `expr2` may lead to undefined behavior, as the exact number of values of i for which it is executed is non-deterministic. The following example computes `nextprime(1000)` in parallel:

```
? parfor (i=1000, , isprime (i), r, if (r, return (i)))
%1 = 1009
```

3.16.4 parforprime($p = a, \{b\}, expr1, \{r\}, \{expr2\}$). Behaves exactly as `parfor`, but loops only over prime values p. Precisely, the functions evaluates in parallel the expression `expr1` in the formal argument p running through the primes from a to b. If b is set to `+oo`, the loop runs indefinitely. If r and `expr2` are present, the expression `expr2` in the formal variables r and p is evaluated with r running through all the different results obtained for `expr1` and p takes the corresponding argument.

It is allowed fo `expr2` to exit the loop using `break/next/return`; see the remarks in the documentation of `parfor` for details.

3.16.5 parforvec($X = v, expr1, \{j\}, \{expr2\}, \{flag\}$). Evaluates the sequence `expr2` (dependent on X and j) for X as generated by `forvec`, in random order, computed in parallel. Substitute for j the value of `expr1` (dependent on X).

It is allowed fo `expr2` to exit the loop using `break/next/return`, however in that case, `expr2` will still be evaluated for all remaining value of p less than the current one, unless a subsequent `break/next/return` happens.

3.16.6 parselect$(f, A, \{flag = 0\})$. Selects elements of A according to the selection function f, done in parallel. If *flag*is 1, return the indices of those elements (indirect selection) The function `f` must not access global variables or variables declared with local(), and must be free of side effects.

The library syntax is `GEN parselect(GEN f, GEN A, long flag)`.

3.16.7 parsum$(i = a, b, expr, \{x\})$. Sum of expression *expr*, initialized at x, the formal parameter going from a to b, evaluated in parallel in random order. The expression `expr` must not access global variables or variables declared with `local()`, and must be free of side effects.

```
parsum(i=1,1000,ispseudoprime(2^prime(i)-1))
```

returns the numbers of prime numbers among the first 1000 Mersenne numbers.

3.16.8 parvector$(N, i, expr)$. As `vector(N,i,expr)` but the evaluations of `expr` are done in parallel. The expression `expr` must not access global variables or variables declared with `local()`, and must be free of side effects.

```
parvector(10,i,quadclassunit(2^(100+i)+1).no)
```

computes the class numbers in parallel.

3.17 GP defaults.

This section documents the GP defaults, be sure to check out `parisize` and `parisizemax` !

3.17.1 TeXstyle. The bits of this default allow `gp` to use less rigid TeX formatting commands in the logfile. This default is only taken into account when `log` = 3. The bits of `TeXstyle` have the following meaning

2: insert `\right` / `\left` pairs where appropriate.

4: insert discretionary breaks in polynomials, to enhance the probability of a good line break.

The default value is `0`.

3.17.2 breakloop. If true, enables the "break loop" debugging mode, see Section 2.10.3.

The default value is `1` if we are running an interactive `gp` session, and `0` otherwise.

3.17.3 colors. This default is only usable if `gp` is running within certain color-capable terminals. For instance `rxvt`, `color_xterm` and modern versions of `xterm` under X Windows, or standard Linux/DOS text consoles. It causes `gp` to use a small palette of colors for its output. With xterms, the colormap used corresponds to the resources `Xterm*color`n where n ranges from 0 to 15 (see the file `misc/color.dft` for an example). Accepted values for this default are strings "$a_1, \ldots, a_k$" where $k \leq 7$ and each a_i is either

• the keyword `no` (use the default color, usually black on transparent background)

• an integer between 0 and 15 corresponding to the aforementioned colormap

• a triple $[c_0, c_1, c_2]$ where c_0 stands for foreground color, c_1 for background color, and c_2 for attributes (0 is default, 1 is bold, 4 is underline).

The output objects thus affected are respectively error messages, history numbers, prompt, input line, output, help messages, timer (that's seven of them). If $k < 7$, the remaining a_i are assumed to be *no*. For instance

```
default(colors, "9, 5, no, no, 4")
```

typesets error messages in color 9, history numbers in color 5, output in color 4, and does not affect the rest.

A set of default colors for dark (reverse video or PC console) and light backgrounds respectively is activated when `colors` is set to `darkbg`, resp. `lightbg` (or any proper prefix: `d` is recognized as an abbreviation for `darkbg`). A bold variant of `darkbg`, called `boldfg`, is provided if you find the former too pale.

EMACS: In the present version, this default is incompatible with PariEmacs. Changing it will just fail silently (the alternative would be to display escape sequences as is, since Emacs will refuse to interpret them). You must customize color highlighting from the PariEmacs side, see its documentation.

The default value is `""` (no colors).

3.17.4 compatible. Obsolete. This default is now a no-op.

3.17.5 datadir. The name of directory containing the optional data files. For now, this includes the `elldata`, `galdata`, `galpol`, `seadata` packages.

The default value is `/usr/local/share/pari`, or the override specified via `Configure --datadir=`.

3.17.6 debug. Debugging level. If it is non-zero, some extra messages may be printed, according to what is going on (see `\g`).

The default value is `0` (no debugging messages).

3.17.7 debugfiles. File usage debugging level. If it is non-zero, `gp` will print information on file descriptors in use, from PARI's point of view (see `\gf`).

The default value is `0` (no debugging messages).

3.17.8 debugmem. Memory debugging level. If it is non-zero, `gp` will regularly print information on memory usage. If it's greater than 2, it will indicate any important garbage collecting and the function it is taking place in (see `\gm`).

Important Note: As it noticeably slows down the performance, the first functionality (memory usage) is disabled if you're not running a version compiled for debugging (see Appendix A).

The default value is `0` (no debugging messages).

3.17.9 echo. This toggle is either 1 (on) or 0 (off). When `echo` mode is on, each command is reprinted before being executed. This can be useful when reading a file with the `\r` or `read` commands. For example, it is turned on at the beginning of the test files used to check whether `gp` has been built correctly (see `\e`).

The default value is `0` (no echo).

3.17.10 factor_add_primes. This toggle is either 1 (on) or 0 (off). If on, the integer factorization machinery calls `addprimes` on prime factors that were difficult to find (larger than 2^{24}), so they are automatically tried first in other factorizations. If a routine is performing (or has performed) a factorization and is interrupted by an error or via Control-C, this lets you recover the prime factors already found. The downside is that a huge `addprimes` table unrelated to the current computations will slow down arithmetic functions relying on integer factorization; one should then empty the table using `removeprimes`.

The default value is `0`.

3.17.11 factor_proven. This toggle is either 1 (on) or 0 (off). By default, the factors output by the integer factorization machinery are only pseudo-primes, not proven primes. If this toggle is set, a primality proof is done for each factor and all results depending on integer factorization are fully proven. This flag does not affect partial factorization when it is explicitly requested. It also does not affect the private table managed by `addprimes`: its entries are included as is in factorizations, without being tested for primality.

The default value is `0`.

3.17.12 format. Of the form x.n, where x (conversion style) is a letter in {`e`, `f`, `g`}, and n (precision) is an integer; this affects the way real numbers are printed:

• If the conversion style is `e`, real numbers are printed in scientific format, always with an explicit exponent, e.g. `3.3 E-5`.

• In style `f`, real numbers are generally printed in fixed floating point format without exponent, e.g. `0.000033`. A large real number, whose integer part is not well defined (not enough significant digits), is printed in style `e`. For instance `10.^100` known to ten significant digits is always printed in style `e`.

• In style `g`, non-zero real numbers are printed in `f` format, except when their decimal exponent is < -4, in which case they are printed in `e` format. Real zeroes (of arbitrary exponent) are printed in `e` format.

The precision n is the number of significant digits printed for real numbers, except if $n < 0$ where all the significant digits will be printed (initial default 28, or 38 for 64-bit machines). For more powerful formatting possibilities, see `printf` and `Strprintf`.

The default value is `"g.28"` and `"g.38"` on 32-bit and 64-bit machines, respectively.

3.17.13 graphcolormap. A vector of colors, to be used by hi-res graphing routines. Its length is arbitrary, but it must contain at least 3 entries: the first 3 colors are used for background, frame/ticks and axes respectively. All colors in the colormap may be freely used in `plotcolor` calls.

A color is either given as in the default by character strings or by an RGB code. For valid character strings, see the standard `rgb.txt` file in X11 distributions, where we restrict to lowercase letters and remove all whitespace from color names. An RGB code is a vector with 3 integer entries between 0 and 255. For instance `[250, 235, 215]` and `"antiquewhite"` represent the same color. RGB codes are cryptic but often easier to generate.

The default value is [`"white"`, `"black"`, `"blue"`, `"violetred"`, `"red"`, `"green"`, `"grey"`, `"gainsboro"`].

3.17.14 graphcolors. Entries in the `graphcolormap` that will be used to plot multi-curves. The successive curves are drawn in colors

`graphcolormap[graphcolors[1]]`, `graphcolormap[graphcolors[2]]`, ...

cycling when the `graphcolors` list is exhausted.

The default value is `[4,5]`.

3.17.15 help. Name of the external help program to use from within `gp` when extended help is invoked, usually through a `??` or `???` request (see Section 2.13.1), or `M-H` under readline (see Section 2.15).

The default value is the path to the `gphelp` script we install.

3.17.16 histfile. Name of a file where `gp` will keep a history of all *input* commands (results are omitted). If this file exists when the value of `histfile` changes, it is read in and becomes part of the session history. Thus, setting this default in your gprc saves your readline history between sessions. Setting this default to the empty string `""` changes it to `<undefined>`

The default value is `<undefined>` (no history file).

3.17.17 histsize. `gp` keeps a history of the last `histsize` results computed so far, which you can recover using the `%` notation (see Section 2.13.4). When this number is exceeded, the oldest values are erased. Tampering with this default is the only way to get rid of the ones you do not need anymore.

The default value is `5000`.

3.17.18 lines. If set to a positive value, `gp` prints at most that many lines from each result, terminating the last line shown with `[+++]` if further material has been suppressed. The various `print` commands (see Section 3.15) are unaffected, so you can always type `print(%)` or `\a` to view the full result. If the actual screen width cannot be determined, a "line" is assumed to be 80 characters long.

The default value is `0`.

3.17.19 linewrap. If set to a positive value, `gp` wraps every single line after printing that many characters.

The default value is `0` (unset).

3.17.20 log. This can be either 0 (off) or 1, 2, 3 (on, see below for the various modes). When logging mode is turned on, `gp` opens a log file, whose exact name is determined by the `logfile` default. Subsequently, all the commands and results will be written to that file (see `\l`). In case a file with this precise name already existed, it will not be erased: your data will be *appended* at the end.

The specific positive values of `log` have the following meaning

1: plain logfile

2: emit color codes to the logfile (if `colors` is set).

3: write LaTeX output to the logfile (can be further customized using `TeXstyle`).

The default value is `0`.

3.17.21 logfile. Name of the log file to be used when the `log` toggle is on. Environment and time expansion are performed.

The default value is `"pari.log"`.

3.17.22 nbthreads. Number of threads to use for parallel computing. The exact meaning an default depend on the `mt` engine used:

• `single`: not used (always one thread).

• `pthread`: number of threads (unlimited, default: number of core)

• `mpi`: number of MPI process to use (limited to the number allocated by `mpirun`, default: use all allocated process).

3.17.23 new_galois_format. This toggle is either 1 (on) or 0 (off). If on, the `polgalois` command will use a different, more consistent, naming scheme for Galois groups. This default is provided to ensure that scripts can control this behavior and do not break unexpectedly.

The default value is `0`. This value will change to 1 (set) in the next major version.

3.17.24 output. There are three possible values: 0 (= *raw*), 1 (= *prettymatrix*), or 3 (= *external prettyprint*). This means that, independently of the default `format` for reals which we explained above, you can print results in three ways:

• *raw format*, i.e. a format which is equivalent to what you input, including explicit multiplication signs, and everything typed on a line instead of two dimensional boxes. This can have several advantages, for instance it allows you to pick the result with a mouse or an editor, and to paste it somewhere else.

• *prettymatrix format*: this is identical to raw format, except that matrices are printed as boxes instead of horizontally. This is prettier, but takes more space and cannot be used for input. Column vectors are still printed horizontally.

• *external prettyprint*: pipes all `gp` output in TeX format to an external prettyprinter, according to the value of `prettyprinter`. The default script (`tex2mail`) converts its input to readable two-dimensional text.

Independently of the setting of this default, an object can be printed in any of the three formats at any time using the commands `\a` and `\m` and `\B` respectively.

The default value is `1` (*prettymatrix*).

3.17.25 parisize. `gp`, and in fact any program using the PARI library, needs a *stack* in which to do its computations; `parisize` is the stack size, in bytes. It is recommended to increase this default using a `gprc`, to the value you believe PARI should be happy with, given your typical computation. We strongly recommend to also set `parisizemax` to a much larger value, about what you believe your machine can stand: PARI will then try to fit its computations within about `parisize` bytes, but will increase the stack size if needed (up to `parisizemax`). Once the memory intensive computation is over, PARI will restore the stack size to the originally requested `parisize`.

The default value is 4M, resp. 8M on a 32-bit, resp. 64-bit machine.

3.17.26 parisizemax. gp, and in fact any program using the PARI library, needs a *stack* in which to do its computations. If non-zero, `parisizemax` is the maximum size the stack can grow to, in bytes. If zero, the stack will not automatically grow, and will be limited to the value of `parisize`.

We strongly recommend to set `parisizemax` to a non-zero value, about what you believe your machine can stand: PARI will then try to fit its computations within about `parisize` bytes, but will increase the stack size if needed (up to `parisizemax`). Once the memory intensive computation is over, PARI will restore the stack size to the originally requested `parisize`.

The default value is 0.

3.17.27 path. This is a list of directories, separated by colons ':' (semicolons ';' in the DOS world, since colons are preempted for drive names). When asked to read a file whose name is not given by an absolute path (does not start with `/`, `./` or `../`), gp will look for it in these directories, in the order they were written in `path`. Here, as usual, `.` means the current directory, and `..` its immediate parent. Environment expansion is performed.

The default value is `".:~:~/gp"` on UNIX systems, `".;C:\;C:\GP"` on DOS, OS/2 and Windows, and `"."` otherwise.

3.17.28 prettyprinter. The name of an external prettyprinter to use when `output` is 3 (alternate prettyprinter). Note that the default `tex2mail` looks much nicer than the built-in "beautified format" (`output` = 2).

The default value is `"tex2mail -TeX -noindent -ragged -by_par"`.

3.17.29 primelimit. gp precomputes a list of all primes less than `primelimit` at initialization time, and can build fast sieves on demand to quickly iterate over primes up to the *square* of `primelimit`. These are used by many arithmetic functions, usually for trial division purposes. The maximal value is $2^{32} - 2049$ (resp $2^{64} - 2049$) on a 32-bit (resp. 64-bit) machine, but values beyond 10^8, allowing to iterate over primes up to 10^{16}, do not seem useful.

Since almost all arithmetic functions eventually require some table of prime numbers, PARI guarantees that the first 6547 primes, up to and including 65557, are precomputed, even if `primelimit` is 1.

This default is only used on startup: changing it will not recompute a new table.

Deprecated feature. `primelimit` was used in some situations by algebraic number theory functions using the `nf_PARTIALFACT` flag (`nfbasis`, `nfdisc`, `nfinit`, ...): this assumes that all primes $p >$ `primelimit` have a certain property (the equation order is p-maximal). This is never done by default, and must be explicitly set by the user of such functions. Nevertheless, these functions now provide a more flexible interface, and their use of the global default `primelimit` is deprecated.

Deprecated feature. `factor(N, 0)` was used to partially factor integers by removing all prime factors $\leq$ `primelimit`. Don't use this, supply an explicit bound: `factor(N, bound)`, which avoids relying on an unpredictable global variable.

The default value is `500k`.

3.17.30 prompt. A string that will be printed as prompt. Note that most usual escape sequences are available there: `\e` for Esc, `\n` for Newline, ..., `\\` for `\`. Time expansion is performed.

This string is sent through the library function `strftime` (on a Unix system, you can try `man strftime` at your shell prompt). This means that `%` constructs have a special meaning, usually related to the time and date. For instance, `%H` = hour (24-hour clock) and `%M` = minute [00,59] (use `%%` to get a real `%`).

If you use `readline`, escape sequences in your prompt will result in display bugs. If you have a relatively recent `readline` (see the comment at the end of Section 3.17.3), you can brace them with special sequences (`\[` and `\]`), and you will be safe. If these just result in extra spaces in your prompt, then you'll have to get a more recent `readline`. See the file `misc/gprc.dft` for an example.

EMACS: **Caution**: PariEmacs needs to know about the prompt pattern to separate your input from previous `gp` results, without ambiguity. It is not a trivial problem to adapt automatically this regular expression to an arbitrary prompt (which can be self-modifying!). See PariEmacs's documentation.

The default value is `"? "`.

3.17.31 prompt_cont. A string that will be printed to prompt for continuation lines (e.g. in between braces, or after a line-terminating backslash). Everything that applies to `prompt` applies to `prompt_cont` as well.

The default value is `""`.

3.17.32 psfile. Name of the default file where `gp` is to dump its PostScript drawings (these are appended, so that no previous data are lost). Environment and time expansion are performed.

The default value is `"pari.ps"`.

3.17.33 readline. Switches readline line-editing facilities on and off. This may be useful if you are running `gp` in a Sun `cmdtool`, which interacts badly with readline. Of course, until readline is switched on again, advanced editing features like automatic completion and editing history are not available.

The default value is `1`.

3.17.34 realbitprecision. The number of significant bits used to convert exact inputs given to transcendental functions (see Section 3.3), or to create absolute floating point constants (input as `1.0` or `Pi` for instance). Unless you tamper with the `format` default, this is also the number of significant bits used to print a `t_REAL` number; `format` will override this latter behaviour, and allow you to have a large internal precision while outputting few digits for instance.

Note that most PARI's functions currently handle precision on a word basis (by increments of 32 or 64 bits), hence bit precision may be a little larger than the number of bits you expected. For instance to get 10 bits of precision, you need one word of precision which, on a 64-bit machine, correspond to 64 bits. To make things even more confusing, this internal bit accuracy is converted to decimal digits when printing floating point numbers: now 64 bits correspond to 19 printed decimal digits ($19 < \log_{10}(2^{64}) < 20$).

The value returned when typing `default(realbitprecision)` is the internal number of significant bits, not the number of printed decimal digits:

```
? default(realbitprecision, 10)
? \pb
      realbitprecision = 64 significant bits
? default(realbitprecision)
%1 = 64
? \p
      realprecision = 3 significant digits
? default(realprecision)
%2 = 19
```

Note that `realprecision` and `\p` allow to view and manipulate the internal precision in decimal digits.

The default value is `128`, resp. `96`, on a 64-bit, resp .32-bit, machine.

3.17.35 realprecision. The number of significant digits used to convert exact inputs given to transcendental functions (see Section 3.3), or to create absolute floating point constants (input as `1.0` or `Pi` for instance). Unless you tamper with the `format` default, this is also the number of significant digits used to print a `t_REAL` number; `format` will override this latter behaviour, and allow you to have a large internal precision while outputting few digits for instance.

Note that PARI's internal precision works on a word basis (by increments of 32 or 64 bits), hence may be a little larger than the number of decimal digits you expected. For instance to get 2 decimal digits you need one word of precision which, on a 64-bit machine, actually gives you 19 digits ($19 < \log_{10}(2^{64}) < 20$). The value returned when typing `default(realprecision)` is the internal number of significant digits, not the number of printed digits:

```
? default(realprecision, 2)
      realprecision = 19 significant digits (2 digits displayed)
? default(realprecision)
%1 = 19
```

The default value is `38`, resp. `28`, on a 64-bit, resp. 32-bit, machine.

3.17.36 recover. This toggle is either 1 (on) or 0 (off). If you change this to 0, any error becomes fatal and causes the gp interpreter to exit immediately. Can be useful in batch job scripts.

The default value is `1`.

3.17.37 secure. This toggle is either 1 (on) or 0 (off). If on, the `system` and `extern` command are disabled. These two commands are potentially dangerous when you execute foreign scripts since they let `gp` execute arbitrary UNIX commands. `gp` will ask for confirmation before letting you (or a script) unset this toggle.

The default value is `0`.

3.17.38 seriesprecision. Number of significant terms when converting a polynomial or rational function to a power series (see `\ps`).

The default value is `16`.

3.17.39 simplify. This toggle is either 1 (on) or 0 (off). When the PARI library computes something, the type of the result is not always the simplest possible. The only type conversions which the PARI library does automatically are rational numbers to integers (when they are of type `t_FRAC` and equal to integers), and similarly rational functions to polynomials (when they are of type `t_RFRAC` and equal to polynomials). This feature is useful in many cases, and saves time, but can be annoying at times. Hence you can disable this and, whenever you feel like it, use the function `simplify` (see Chapter 3) which allows you to simplify objects to the simplest possible types recursively (see `\y`).

The default value is `1`.

3.17.40 sopath. This is a list of directories, separated by colons ':' (semicolons ';' in the DOS world, since colons are preempted for drive names). When asked to `install` an external symbol from a shared library whose name is not given by an absolute path (does not start with `/`, `./` or `../`), `gp` will look for it in these directories, in the order they were written in `sopath`. Here, as usual, `.` means the current directory, and `..` its immediate parent. Environment expansion is performed.

The default value is `""`, corresponding to an empty list of directories: `install` will use the library name as input (and look in the current directory if the name is not an absolute path).

3.17.41 strictargs. This toggle is either 1 (on) or 0 (off). If on, all arguments to *new* user functions are mandatory unless the function supplies an explicit default value. Otherwise arguments have the default value 0.

In this example,

```
fun(a,b=2)=a+b
```

`a` is mandatory, while `b` is optional. If `strictargs` is on:

```
? fun()
 ***   at top-level: fun()
 ***                 ^-----
 ***   in function fun: a,b=2
 ***                     ^-----
 ***   missing mandatory argument 'a' in user function.
```

This applies to functions defined while `strictargs` is on. Changing `strictargs` does not affect the behavior of previously defined functions.

The default value is `0`.

3.17.42 strictmatch. Obsolete. This toggle is now a no-op.

3.17.43 threadsize. In parallel mode, each thread needs its own private *stack* in which to do its computations, see `parisize`. This value determines the size in bytes of the stacks of each thread, so the total memory allocated will be `parisize` + `nbthreads` × `threadsize`.

If set to 0, the value used is the same as `parisize`.

The default value is 0.

3.17.44 threadsizemax. In parallel mode, each threads needs its own private *stack* in which to do its computations, see `parisize`. This value determines the maximal size in bytes of the stacks of each thread, so the total memory allocated will be between `parisize` + `nbthreads` × `threadsize`. and `parisize` + `nbthreads` × `threadsizemax`.

If set to 0, the value used is the same as `threadsize`.

The default value is 0.

3.17.45 timer. This toggle is either 1 (on) or 0 (off). Every instruction sequence in the gp calculator (anything ended by a newline in your input) is timed, to some accuracy depending on the hardware and operating system. When `timer` is on, each such timing is printed immediately before the output as follows:

```
? factor(2^2^7+1)
time = 108 ms.     \\ this line omitted if 'timer' is 0
%1 =
[     59649589127497217 1]

[5704689200685129054721 1]
```

(See also `#` and `##`.)

The time measured is the user CPU time, *not* including the time for printing the results. If the time is negligible (< 1 ms.), nothing is printed: in particular, no timing should be printed when defining a user function or an alias, or installing a symbol from the library.

The default value is `0` (off).

Appendix A:
Installation Guide for the UNIX Versions

1. Required tools.

Compiling PARI requires an `ANSI C` or a `C++` compiler. If you do not have one, we suggest that you obtain the `gcc/g++` compiler. As for all GNU software mentioned afterwards, you can find the most convenient site to fetch `gcc` at the address

`http://www.gnu.org/order/ftp.html`

(On Mac OS X, this is also provided in the `Xcode` tool suite; or the lightweight "Command-line tools for `Xcode`".) You can certainly compile PARI with a different compiler, but the PARI kernel takes advantage of optimizations provided by `gcc`. This results in at least 20% speedup on most architectures.

Optional libraries and programs. The following programs and libraries are useful in conjunction with `gp`, but not mandatory. In any case, get them before proceeding if you want the functionalities they provide. All of them are free. The download page on our website

`http://pari.math.u-bordeaux.fr/download.html`

contains pointers on how to get these.

• GNU `MP` library. This provides an alternative multiprecision kernel, which is faster than PARI's native one, but unfortunately binary incompatible, so the resulting PARI library SONAME is libpari-gmp.

• GNU `readline` library. This provides line editing under `gp`, an automatic context-dependent completion, and an editable history of commands.

• GNU `emacs` and the `PariEmacs` package. The `gp` calculator can be run in an Emacs buffer, with all the obvious advantages if you are familiar with this editor. Note that `readline` is still useful in this case since it provides a better automatic completion than is provided by Emacs's GP-mode.

• GNU `gzip/gunzip/gzcat` package enables `gp` to read compressed data.

• `perl` provides extended online help (full text from the manual) about functions and concepts. The script handling this online help can be used under `gp` or independently.

2. Compiling the library and the gp calculator.

2.1. Basic configuration. Type

```
./Configure
```

in the toplevel directory. This attempts to configure PARI/GP without outside help. Note that if you want to install the end product in some nonstandard place, you can use the `--prefix` option, as in

```
./Configure --prefix=/an/exotic/directory
```

(the default prefix is `/usr/local`). For example, to build a package for a Linux distribution, you may want to use

```
./Configure --prefix=/usr
```

This phase extracts some files and creates a *build directory*, names `O`*osname*`-`*arch*, where the object files and executables will be built. The *osname* and *arch* components depends on your architecture and operating system, thus you can build PARI/GP for several different machines from the same source tree (the builds are independent and can be done simultaneously).

Decide whether you agree with what `Configure` printed on your screen, in particular the architecture, compiler and optimization flags. Look for messages prepended by `###`, which report genuine problems. Look especially for the `gmp`, `readline` and `X11` libraries, and the `perl` and `gunzip` (or `zcat`) binaries. If anything should have been found and was not, consider that `Configure` failed and follow the instructions in section 3.

The `Configure` run creates a file `config.log` in the build directory, which contains debugging information — in particular, all messages from compilers — that may help diagnose problems. This file is erased and recreated from scratch each time `Configure` is run.

2.2. Advanced configuration. `Configure` accepts many other flags, and you may use any number of them to build quite a complicated configuration command. See `Configure --help` for a complete list. In particular, there are sets of flags related to GNU MP (`--with-gmp*`) and GNU readline library (`--with-readline*`).

Here, we focus on the non-obvious ones:

`--tune`: fine tunes the library for the host used for compilation. This adjusts thresholds by running a large number of comparative tests and creates a file `tune.h` in the build directory, that will be used from now on, overriding the ones in `src/kernel/none/` and `src/kernel/gmp/`. It will take a while: about 30 minutes. Expect a small performance boost, perhaps a 10% speed increase compared to default settings.

If you are using GMP, tune it first, then PARI. Make sure you tune PARI on the machine that will actually run your computations. Do not use a heavily loaded machine for tunings.

You may speed up the compilation by using a parallel make:

```
env MAKE="make -j4" Configure --tune
```

`--graphic=`*lib*: enables a particular graphic library. The default is `X11` on most platforms, but PARI can use `Qt`, `fltk`, `ps`, or `win32` (GDI).

`--time=`*function*: chooses a timing function. The default usually works fine, however you can use a different one that better fits your needs. PARI can use `getrusage`, `clock_gettime`, `times` or

`ftime` as timing functions. (Not all timing functions are available on all platforms.) The three first functions give timings in terms of CPU usage of the current task, approximating the complexity of the algorithm. The last one, `ftime`, gives timings in terms of absolute (wall-clock) time. Moreover, the `clock_gettime` function is more precise, but much slower (at the time of this writing), than `getrusage` or `times`.

`--with-runtime-perl=`*perl*: absolute path to the runtime `perl` binary to be used by the `gphelp` and `tex2mail` scripts. Defaults to the path found by `Configure` on the build host (usually `/usr/bin/perl`). For cross-compiling builds, when the target and build hosts have mismatched configurations; suggested values are

`/usr/bin/env perl`: the first `perl` executable found in user's `PATH`,

`/usr/bin/perl`: `perl`'s standard location.

The remaining options are specific to parallel programming. We provide an *Introduction to parallel GP programming* in the file `doc/parallel.dvi`, and to multi-threaded `libpari` programs in Appendix D. Beware that these options change the library ABI:

`--mt=`*engine*: specify the engine used for parallel computations. Supported value are

• single: (default) no parallellism.

• pthread: use POSIX threads. This is well-suited for multi-core systems. Setting this option also set `--enable-tls`, see below. This option requires the pthread library. For benchmarking, it is often useful to set `--time=ftime` so that GP report wall-clock instead of the sum of the time spent by each thread.

• mpi: use the MPI interface to parallelism. This allows to take advantage of clusters using MPI. This option requires a MPI library. It is usually necessary to set the environment variable `CC` to `mpicc`.

`--enable-tls`: build the thread-safe version of the library. Implied by `--mt=pthread`. This tends to slow down the *shared* library `libpari.so` by about 15%, so you probably want to use the static library `libpari.a` instead.

2.3. Compilation. To compile the `gp` binary and build the documentation, type

```
make all
```

To only compile the `gp` binary, type

```
make gp
```

in the toplevel directory. If your `make` program supports parallel make, you can speed up the process by going to the build directory that `Configure` created and doing a parallel make here, for instance `make -j4` with GNU make. It should even work from the toplevel directory.

2.4. Basic tests.

To test the binary, type `make bench`. This runs a quick series of tests, for a few seconds on modern machines.

In many cases, this will also build a different binary (named `gp-sta` or `gp-dyn`) linked in a slightly different way and run the tests with both. (In exotic configurations, one may pass all the tests while the other fails and we want to check for this.) To test only the default binary, use `make dobench` which starts the bench immediately.

If a `[BUG]` message shows up, something went wrong. The testing utility directs you to files containing the differences between the test output and the expected results. Have a look and decide for yourself if something is amiss. If it looks like a bug in the Pari system, we would appreciate a report, see the last section.

2.5. Cross-compiling.

When cross-compiling, you can set the environment variable `RUNTEST` to a program that is able to run the target binaries, e.g. an emulator. It will be used for both the `Configure` tests and `make bench`.

3. Troubleshooting and fine tuning.

In case the default `Configure` run fails miserably, try

```
./Configure -a
```

(interactive mode) and answer all the questions: there are about 30 of them, and default answers are provided. If you accept all default answers, `Configure` will fail just the same, so be wary. In any case, we would appreciate a bug report (see the last section).

3.1. Installation directories. The precise default destinations are as follows: the `gp` binary, the scripts `gphelp` and `tex2mail` go to `$prefix/bin`. The pari library goes to `$prefix/lib` and include files to `$prefix/include/pari`. Other system-dependent data go to `$prefix/lib/pari`.

Architecture independent files go to various subdirectories of `$share_prefix`, which defaults to `$prefix/share`, and can be specified via the `--share-prefix` argument. Man pages go into `$share_prefix/man`, and other system-independent data under `$share_prefix/pari`: documentation, sample GP scripts and C code, extra packages like `elldata` or `galdata`.

You can also set directly `--bindir` (executables), `--libdir` (library), `--includedir` (include files), `--mandir` (manual pages), `--datadir` (other architecture-independent data), and finally `--sysdatadir` (other architecture-dependent data).

3.2. Environment variables. `Configure` lets the following environment variable override the defaults if set:

`CC`: C compiler.

`DLLD`: Dynamic library linker.

`LD`: Static linker.

For instance, `Configure` may avoid `/bin/cc` on some architectures due to various problems which may have been fixed in your version of the compiler. You can try

```
env CC=cc Configure
```

and compare the benches. Also, if you insist on using a `C++` compiler and run into trouble with a fussy `g++`, try to use `g++ -fpermissive`.

The contents of the following variables are *appended* to the values computed by `Configure`:

`CFLAGS`: Flags for `CC`.

`CPPFLAGS`: Flags for `CC` (preprocessor).

`LDFLAGS`: Flags for `LD`.

The contents of the following variables are *prepended* to the values computed by `Configure`:

`C_INCLUDE_PATH` is prepended to the list of directories searched for include files. Note that adding `-I` flags to `CFLAGS` is not enough since `Configure` sometimes relies on finding the include files and parsing them, and it does not parse `CFLAGS` at this time.

`LIBRARY_PATH` is prepended to the list of directories searched for libraries.

You may disable inlining by adding `-DDISABLE_INLINE` to `CFLAGS`, and prevent the use of the `volatile` keyword with `-DDISABLE_VOLATILE`.

3.3. Debugging/profiling.: If you also want to debug the PARI library,

```
Configure -g
```

creates a directory `O`*xxx*`.dbg` containing a special `Makefile` ensuring that the `gp` and PARI library built there is suitable for debugging. If you want to profile `gp` or the library, using `gprof` for instance,

```
Configure -pg
```

will create an `O`*xxx*`.prf` directory where a suitable version of PARI can be built.

The `gp` binary built above with `make all` or `make gp` is optimized. If you have run `Configure -g` or `-pg` and want to build a special purpose binary, you can `cd` to the `.dbg` or `.prf` directory and type `make gp` there. You can also invoke `make gp.dbg` or `make gp.prf` directly from the toplevel.

3.4. Multiprecision kernel. The kernel can be specified via the

`--kernel=`*fully_qualified_kernel_name*

switch. The PARI kernel consists of two levels: Level 0 (operation on words) and Level 1 (operation on multi-precision integers and reals), which can take the following values.

Level 0: `auto` (as detected), `none` (portable C) or one of the assembler micro-kernels

```
alpha
hppa hppa64
ia64
ix86 x86_64
m68k
ppc ppc64
sparcv7 sparcv8_micro sparcv8_super
```

Level 1: `auto` (as detected), `none` (native code only), or `gmp`

• A fully qualified kernel name is of the form *Level0*`-`*Level1*, the default value being `auto-auto`.

• A *name* not containing a dash '`-`' is an alias for a fully qualified kernel name. An alias stands for *name*`-none`, but `gmp` stands for `auto-gmp`.

3.5. Problems related to readline. `Configure` does not try very hard to find the `readline` library and include files. If they are not in a standard place, it will not find them. You can invoke `Configure` with one of the following arguments:

`--with-readline[=`*prefix to* `lib/libreadline.`*xx and* `include/readline.h]`

`--with-readline-lib=`*path to* `libreadline.`*xx*

`--with-readline-include=`*path to* `readline.h`

Known problems.

• on Linux: Linux distributions have separate `readline` and `readline-devel` packages. You need both of them installed to compile gp with readline support. If only `readline` is installed, `Configure` will complain. `Configure` may also complain about a missing libncurses.so, in which case, you have to install the `ncurses-devel` package (some distributions let you install `readline-devel` without `ncurses-devel`, which is a bug in their package dependency handling).

• on OS X.4 or higher: these systems comes equipped with a fake `readline`, which is not sufficient for our purpose. As a result, gp is built without readline support. Since `readline` is not trivial to install in this environment, a step by step solution can be found in the PARI FAQ, see

`http://pari.math.u-bordeaux.fr/`.

3.6. Testing.

3.6.1. Known problems. if `BUG` shows up in `make bench`

• If when running `gp-dyn`, you get a message of the form

`ld.so: warning: libpari.so.`*xxx* `has older revision than expected` *xxx*

(possibly followed by more errors), you already have a dynamic PARI library installed *and* a broken local configuration. Either remove the old library or unset the `LD_LIBRARY_PATH` environment variable. Try to disable this variable in any case if anything *very* wrong occurs with the `gp-dyn` binary, like an Illegal Instruction on startup. It does not affect `gp-sta`.

• Some implementations of the `diff` utility (on HPUX for instance) output `No differences encountered` or some similar message instead of the expected empty input, thus producing a spurious `[BUG]` message.

3.6.2. Some more testing. [*Optional*]

You can test `gp` in compatibility mode with `make test-compat`. If you want to test the graphic routines, use `make test-ploth`. You will have to click on the mouse button after seeing each image. There will be eight of them, probably shown twice (try to resize at least one of them as a further test).

The `make bench`, `make test-compat` and `make test-ploth` runs all produce a Postscript file `pari.ps` in `O`*xxx* which you can send to a Postscript printer. The output should bear some similarity to the screen images.

3.6.3. Heavy-duty testing. [*Optional*] There are a few extra tests which should be useful only for developers.

`make test-kernel` checks whether the low-level kernel seems to work, and provides simple diagnostics if it does not. Only useful if `make bench` fails horribly, e.g. things like `1+1` do not work.

`make test-all` runs all available test suites. Thorough, but slow. Some of the tests require extra packages (`elldata`, `galdata`, etc.) to be available. If you want to test such an extra package *before* `make install` (which would install it to its final location, where `gp` expects to find it), run

```
env GP_DATA_DIR=$PWD/data make test-all
```

from the PARI toplevel directory, otherwise the test will fail.

`make test-io` tests writing to and reading from files. It requires a working `system()` command (fails on Windows + MingW).

`make test-time` tests absolute and relative timers. This test has a tendency to fail when the machine is heavily loaded or if the granularity of the chosen system timer is bigger than 2ms. Try it a few times before reporting a problem.

`make test-install` tests the GP function `install`. This may not be available on your platform, triggering an error message ("not yet available for this architecture"). The implementation may be broken on your platform triggering an error or a crash when an install'ed function is used.

4. Installation.

When everything looks fine, type

```
make install
```

You may have to do this with superuser privileges, depending on the target directories. (Tip for MacOS X beginners: use `sudo make install`.) In this case, it is advised to type `make all` first to avoid running unnecessary commands as `root`.

Caveat. Install directories are created honouring your `umask` settings: if your umask is too restrictive, e.g. `077`, the installed files will not be world-readable. (Beware that running `sudo` may change your user umask.)

This installs in the directories chosen at `Configure` time the default `gp` executable (probably `gp-dyn`) under the name `gp`, the default PARI library (probably `libpari.so`), the necessary include files, the manual pages, the documentation and help scripts.

To save on disk space, you can manually `gzip` some of the documentation files if you wish: `usersch*.tex` and all `dvi` files (assuming your `xdvi` knows how to deal with compressed files); the online-help system can handle it.

4.1. Static binaries and libraries. By default, if a dynamic library `libpari.so` can be built, the `gp` binary we install is `gp-dyn`, pointing to `libpari.so`. On the other hand, we can build a `gp` binary into which the `libpari` is statically linked (the library code is copied into the binary); that binary is not independent of the machine it was compiled on, and may still refer to other dynamic libraries than `libpari`.

You may want to compile your own programs in the same way, using the static `libpari.a` instead of `libpari.so`. By default this static library `libpari.a` is not created. If you want it as well, use the target `make install-lib-sta`. You can install a statically linked `gp` with the target `make install-bin-sta`. As a rule, programs linked statically (with `libpari.a`) may be slightly faster (about 5% gain, possibly up to 20% when using `pthreads`), but use more disk space and take more time to compile. They are also harder to upgrade: you will have to recompile them all instead of just installing the new dynamic library. On the other hand, there is no risk of breaking them by installing a new pari library.

4.2. Extra packages. The following optional packages endow PARI with some extra capabilities:

• `elldata`: This package contains the elliptic curves in John Cremona's database. It is needed by the functions `ellidentify`, `ellsearch`, `forell` and can be used by `ellinit` to initialize a curve given by its standard code.

• `galdata`: The default `polgalois` function can only compute Galois groups of polynomials of degree less or equal to 7. Install this package if you want to handle polynomials of degree bigger than 7 (and less than 11).

• `seadata`: This package contains the database of modular polynomials extracted from the ECHIDNA databases and computed by David R. Kohel. It is used to speed up the functions `ellap`, `ellcard` and `ellgroup` for primes larger than 10^{20}.

• `galpol`: This package contains the GALPOL database of polynomials defining Galois extensions of the rationals, accessed by `galoisgetpol`.

To install package *pack*, you need to fetch the separate archive: *pack*`.tgz` which you can download from the `pari` server. Copy the archive in the PARI toplevel directory, then extract its contents; these will go to `data/`*pack*`/`. Typing `make install` installs all such packages.

4.3. The `GPRC` file. Copy the file `misc/gprc.dft` (or `gprc.dos` if you are using `GP.EXE`) to `$HOME/.gprc`. Modify it to your liking. For instance, if you are not using an ANSI terminal, remove control characters from the `prompt` variable. You can also enable colors.

If desired, read `$datadir/misc/gpalias` from the `gprc` file, which provides some common shortcuts to lengthy names; fix the path in gprc first. (Unless you tampered with this via Configure, `datadir` is `$prefix/share/pari`.) If you have superuser privileges and want to provide system-wide defaults, copy your customized `.gprc` file to `/etc/gprc`.

In older versions, `gphelp` was hidden in pari lib directory and was not meant to be used from the shell prompt, but not anymore. If gp complains it cannot find `gphelp`, check whether your `.gprc` (or the system-wide `gprc`) does contain explicit paths. If so, correct them according to the current `misc/gprc.dft`.

5. Getting Started.

5.1. Printable Documentation. Building gp with `make all` also builds its documentation. You can also type directly `make doc`. In any case, you need a working (plain) TEX installation.

After that, the `doc` directory contains various `dvi` files: `libpari.dvi` (manual for the PARI library), `users.dvi` (manual for the `gp` calculator), `tutorial.dvi` (a tutorial), and `refcard.dvi` (a reference card for GP). You can send these files to your favorite printer in the usual way, probably via `dvips`. The reference card is also provided as a `PostScript` document, which may be easier to print than its `dvi` equivalent (it is in Landscape orientation and assumes A4 paper size).

If `pdftex` is part of your TEX setup, you can produce these documents in PDF format, which may be more convenient for online browsing (the manual is complete with hyperlinks); type

```
make docpdf
```

All these documents are available online from PARI home page (see the last section).

5.2. C programming. Once all libraries and include files are installed, you can link your C programs to the PARI library. A sample makefile `examples/Makefile` is provided to illustrate the use of the various libraries. Type `make all` in the `examples` directory to see how they perform on the `extgcd.c` program, which is commented in the manual.

This should produce a statically linked binary `extgcd-sta` (standalone), a dynamically linked binary `extgcd-dyn` (loads libpari at runtime) and a shared library `libextgcd`, which can be used from `gp` to `install` your new `extgcd` command.

The standalone binary should be bulletproof, but the other two may fail for various reasons. If when running `extgcd-dyn`, you get a message of the form "DLL not found", then stick to statically linked binaries or look at your system documentation to see how to indicate at linking time where the required DLLs may be found! (E.g. on Windows, you will need to move `libpari.dll` somewhere in your `PATH`.)

5.3. GP scripts. Several complete sample GP programs are also given in the `examples` directory, for example Shanks's SQUFOF factoring method, the Pollard rho factoring method, the Lucas-Lehmer primality test for Mersenne numbers and a simple general class group and fundamental unit algorithm. See the file `examples/EXPLAIN` for some explanations.

5.4. The PARI Community. PARI's home page at the address

`http://pari.math.u-bordeaux.fr/`

maintains an archive of mailing lists dedicated to PARI, documentation (including Frequently Asked Questions), a download area and our Bug Tracking System (BTS). Bug reports should be submitted online to the BTS, which may be accessed from the navigation bar on the home page or directly at

`http://pari.math.u-bordeaux.fr/Bugs/`

Further information can be found at that address but, to report a configuration problem, make sure to include the relevant `*.dif` files in the `0`*xxx* directory and the file `pari.cfg`.

There are a number of mailing lists devoted to PARI/GP, and most feedback should be directed there. Instructions and archives can be consulted at

`http://pari.math.u-bordeaux1.fr/lists-index.html`

The most important are:

• `pari-announce` (*read-only*): to announce major version changes. You cannot write to this one, but you should probably subscribe.

• `pari-dev`: for everything related to the development of PARI, including suggestions, technical questions or patch submissions. Bug reports can be discussed here, but as a rule it is better to submit them directly to the BTS.

• `pari-users`: for everything else.

You may send an email to the last two without being subscribed. To subscribe, send an message respectively to

```
pari-announce-request@pari.math.u-bordeaux.fr
   pari-users-request@pari.math.u-bordeaux.fr
     pari-dev-request@pari.math.u-bordeaux.fr
```

with the word `subscribe` in the `Subject:`. You can also write to us at the address

```
pari@math.u-bordeaux.fr
```

but we cannot promise you will get an individual answer.

If you have used PARI in the preparation of a paper, please cite it in the following form (BibTeX format):

```
@preamble{\usepackage{url}}
@manual{PARI2,
    organization = "{The PARI~Group}",
    title        = "{PARI/GP version 2.9.2}",
    year         = 2017,
```

```
    address      = "Bordeaux",
    note         = "available from \url{http://pari.math.u-bordeaux.fr/}"
}
```

In any case, if you like this software, we would be indebted if you could send us an email message giving us some information about yourself and what you use PARI for.

Good luck and enjoy!

SomeWord refers to PARI-GP concepts.
`SomeWord` is a PARI-GP keyword.
SomeWord is a generic index entry.

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